The gamma and the beta function

As mentioned in the book [1], see page 6, the integral representation (1.1.18) is often taken as a definition for the gamma function $\Gamma(z)$. The advantage of this alternative definition is that we might avoid the use of infinite products (see appendix A).

Definition 1.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \quad \operatorname{Re} z > 0. \tag{1}$$

From this definition it is clear that $\Gamma(z)$ is analytic for $\operatorname{Re} z > 0$. By using integration by parts we find that

$$\begin{aligned} \Gamma(z+1) &= \int_0^\infty e^{-t} t^z\, dt = -\int_0^\infty t^z\, de^{-t} = -e^{-t}t^z \Big|_0^\infty + \int_0^\infty e^{-t}\, dt^z \\ &= z\int_0^\infty e^{-t} t^{z-1}\, dt = z\Gamma(z), \quad \operatorname{Re} z > 0. \end{aligned}$$

Hence we have

Theorem 1.

$$\Gamma(z+1) = z\Gamma(z), \quad \operatorname{Re} z > 0. \tag{2}$$

Further we have

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = -e^{-t}\Big|_0^\infty = 1. \tag{3}$$

Combining (2) and (3), this leads to

$$\Gamma(n+1) = n!, \quad n = 0, 1, 2, \ldots. \tag{4}$$

The functional relation (2) can be used to find an analytic continuation of the gamma function for $\operatorname{Re} z \leq 0$. For $\operatorname{Re} z > 0$ the gamma function $\Gamma(z)$ is defined by (1). The functional relation (2) also holds for $\operatorname{Re} z > 0$.

Let $-1 < \operatorname{Re} z \leq 0$, then we have $\operatorname{Re}(z+1) > 0$. Hence, $\Gamma(z+1)$ is defined by the integral representation (1). Now we define

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \quad -1 < \operatorname{Re} z \leq 0, \quad z \neq 0.$$

Then the gamma function $\Gamma(z)$ is analytic for $\operatorname{Re} z > -1$ except $z = 0$. For $z = 0$ we have

$$\lim_{z\to 0} z\Gamma(z) = \lim_{z\to 0} \Gamma(z+1) = \Gamma(1) = 1.$$

This implies that $\Gamma(z)$ has a single pole at $z = 0$ with residue 1.

This process can be repeated for $-2 < \operatorname{Re} z \leq -1$, $-3 < \operatorname{Re} z \leq -2$, etcetera. Then the gamma function turns out to be an analytic function on $\mathbb{C}$ except for single poles at $z = 0, -1, -2, \ldots$. The residue at $z = -n$ equals

$$\begin{aligned} \lim_{z\to -n} (z+n)\Gamma(z) &= \lim_{z\to -n} (z+n)\frac{\Gamma(z+1)}{z} = \lim_{z\to -n} (z+n)\frac{1}{z}\frac{1}{z+1}\cdots\frac{1}{z+n-1}\frac{\Gamma(z+n+1)}{z+n} \\ &= \frac{\Gamma(1)}{(-n)(-n+1)\cdots(-1)} = \frac{(-1)^n}{n!}, \quad n = 0, 1, 2, \ldots. \end{aligned}$$

As indicated in the book [1], see page 8, the limit formula (1.1.5) can be obtained from the integral representation (1) by using induction as follows. We first prove that

$$\int_0^1 (1-t)^n t^{z-1}\,dt = \frac{n!}{(z)_{n+1}} \tag{5}$$

for $\operatorname{Re} z > 0$ and $n = 0, 1, 2, \ldots$. Here the shifted factorial $(a)_k$ is defined by

Definition 2.

$$(a)_k = a(a+1)\cdots(a+k-1), \quad k = 1, 2, 3, \ldots \quad \text{and} \quad (a)_0 = 1. \tag{6}$$

In order to prove (5) by induction we first take $n = 0$ to obtain for $\operatorname{Re} z > 0$

$$\int_0^1 t^{z-1}\,dt = \left.\frac{t^z}{z}\right|_0^1 = \frac{1}{z} = \frac{0!}{(z)_1}.$$

Now we assume that (5) holds for $n = k$. Then we have

$$\begin{aligned}\int_0^1 (1-t)^{k+1} t^{z-1}\,dt &= \int_0^1 (1-t)(1-t)^k t^{z-1}\,dt = \int_0^1 (1-t)^k t^{z-1}\,dt - \int_0^1 (1-t)^k t^z\,dt \\ &= \frac{k!}{(z)_{k+1}} - \frac{k!}{(z+1)_{k+1}} = \frac{k!}{(z)_{k+2}}(z+k+1-z) = \frac{(k+1)!}{(z)_{k+2}},\end{aligned}$$

which is (5) for $n = k+1$. This proves that (5) holds for all $n = 0, 1, 2, \ldots$.

Now we set $t = u/n$ into (5) to find that

$$\frac{1}{n^z}\int_0^n \left(1-\frac{u}{n}\right)^n u^{z-1}\,du = \frac{n!}{(z)_{n+1}} \quad \Longrightarrow \quad \int_0^n \left(1-\frac{u}{n}\right)^n u^{z-1}\,du = \frac{n!\,n^z}{(z)_{n+1}}.$$

Since we have

$$\lim_{n\to\infty}\left(1-\frac{u}{n}\right)^n = e^{-u},$$

we conclude that

$$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1}\,du = \lim_{n\to\infty}\frac{n!\,n^z}{(z)_{n+1}}.$$

The beta function $B(u,v)$ is also defined by means of an integral:

Definition 3.

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}\,dt, \quad \operatorname{Re} u > 0, \quad \operatorname{Re} v > 0. \tag{7}$$

This integral is often called the beta integral. From the definition we easily obtain the symmetry

$$B(u,v) = B(v,u), \tag{8}$$

since we have by using the substitution $t = 1-s$

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}\,dt = -\int_1^0 (1-s)^{u-1}s^{v-1}\,ds = \int_0^1 s^{v-1}(1-s)^{u-1}\,ds = B(v,u).$$

The connection between the beta function and the gamma function is given by the following theorem:

Theorem 2.

$$B(u,v)=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},\quad \operatorname{Re}u>0,\quad \operatorname{Re}v>0. \tag{9}$$

In order to prove this theorem we use the definition (1) to obtain

$$\Gamma(u)\Gamma(v)=\int_0^\infty e^{-t}t^{u-1}\,dt\int_0^\infty e^{-s}s^{v-1}\,ds=\int_0^\infty\int_0^\infty e^{-(t+s)}t^{u-1}s^{v-1}\,dt\,ds.$$

Now we apply the change of variables $t=xy$ and $s=x(1-y)$ to this double integral. Note that $t+s=x$ and that $0<t<\infty$ and $0<s<\infty$ imply that $0<x<\infty$ and $0<y<1$. The Jacobian of this transformation is

$$\frac{\partial(t,s)}{\partial(x,y)}=\begin{vmatrix} y & x \\ 1-y & -x\end{vmatrix}=-xy-x+xy=-x.$$

Since $x>0$ we conclude that $dt\,ds=\left|\frac{\partial(t,s)}{\partial(x,y)}\right|\,dx\,dy=x\,dx\,dy$. Hence we have

$$\begin{aligned}\Gamma(u)\Gamma(v) &= \int_0^1\int_0^\infty e^{-x}x^{u-1}y^{u-1}x^{v-1}(1-y)^{v-1}x\,dx\,dy\\ &= \int_0^\infty e^{-x}x^{u+v-1}\,dx\int_0^1 y^{u-1}(1-y)^{v-1}\,dy=\Gamma(u+v)B(u,v).\end{aligned}$$

This proves (9).

There exist many useful forms of the beta integral which can be obtained by an appropriate change of variables. For instance, if we set $t=s/(s+1)$ into (7) we obtain

$$\begin{aligned}B(u,v) &= \int_0^1 t^{u-1}(1-t)^{v-1}\,dt=\int_0^\infty s^{u-1}(s+1)^{-u+1}(s+1)^{-v+1}(s+1)^{-2}\,ds\\ &= \int_0^\infty\frac{s^{u-1}}{(s+1)^{u+v}}\,ds,\quad \operatorname{Re}u>0,\quad \operatorname{Re}v>0.\end{aligned}$$

This proves

Theorem 3.

$$B(u,v)=\int_0^\infty\frac{s^{u-1}}{(s+1)^{u+v}}\,ds,\quad \operatorname{Re}u>0,\quad \operatorname{Re}v>0. \tag{10}$$

If we set $t=\cos^2\varphi$ into (7) we find that

$$\begin{aligned}B(u,v) &= \int_0^1 t^{u-1}(1-t)^{v-1}\,dt=-2\int_{\pi/2}^0(\cos\varphi)^{2u-2}(\sin\varphi)^{2v-2}\cos\varphi\sin\varphi\,d\varphi\\ &= 2\int_0^{\pi/2}(\cos\varphi)^{2u-1}(\sin\varphi)^{2v-1}\,d\varphi,\quad \operatorname{Re}u>0,\quad \operatorname{Re}v>0.\end{aligned}$$

Hence we have

Theorem 4.

$$B(u,v)=2\int_0^{\pi/2}(\cos\varphi)^{2u-1}(\sin\varphi)^{2v-1}\,d\varphi,\quad \operatorname{Re}u>0,\quad \operatorname{Re}v>0. \tag{11}$$

Finally, the substitution $t = (s-a)/(b-a)$ in (7) leads to

$$\begin{aligned} B(u,v) &= \int_0^1 t^{u-1}(1-t)^{v-1}\,dt \\ &= \int_a^b (s-a)^{u-1}(b-a)^{-u+1}(b-s)^{v-1}(b-a)^{-v+1}(b-a)^{-1}\,ds \\ &= (b-a)^{-u-v+1}\int_a^b (s-a)^{u-1}(b-s)^{v-1}\,ds, \quad \operatorname{Re} u > 0, \quad \operatorname{Re} v > 0. \end{aligned}$$

Hence we have

Theorem 5.

$$\int_a^b (s-a)^{u-1}(b-s)^{v-1}\,ds = (b-a)^{u+v-1}B(u,v), \quad \operatorname{Re} u > 0, \quad \operatorname{Re} v > 0. \tag{12}$$

The special case $a = -1$ and $b = 1$ is of special interest as we will see later:

$$\int_{-1}^1 (1+s)^{u-1}(1-s)^{v-1}\,ds = 2^{u+v-1}B(u,v), \quad \operatorname{Re} u > 0, \quad \operatorname{Re} v > 0.$$

The different forms for the beta function have a lot of consequences. For instance, if we set $u = v = 1/2$ in (9) we find that

$$B(1/2,1/2) = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)} = \{\Gamma(1/2)\}^2.$$

On the other hand, we have by using (11)

$$B(1/2,1/2) = 2\int_0^{\pi/2} d\varphi = 2\cdot\frac{\pi}{2} = \pi.$$

This implies that

$$\Gamma(1/2) = \sqrt{\pi}. \tag{13}$$

By using the transformation $x^2 = t$ we now easily obtain the value of the normal integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_0^{\infty} e^{-x^2}\,dx = \int_0^{\infty} e^{-t}t^{-1/2}\,dt = \Gamma(1/2) = \sqrt{\pi}. \tag{14}$$

The combination of (9) and (11) can be used to compute integrals such as

$$\int_0^{\pi/2} (\cos\varphi)^5(\sin\varphi)^7\,d\varphi = \frac{1}{2}\cdot B(3,4) = \frac{1}{2}\cdot\frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = \frac{1}{2}\cdot\frac{2!\,3!}{6!} = \frac{1}{2}\cdot\frac{2}{4\cdot5\cdot6} = \frac{1}{120},$$

$$\begin{aligned} \int_0^{\pi/2} (\cos\varphi)^7(\sin\varphi)^4\,d\varphi &= \frac{1}{2}\cdot B(4,5/2) = \frac{1}{2}\cdot\frac{\Gamma(4)\Gamma(5/2)}{\Gamma(13/2)} = \frac{1}{2}\cdot\frac{3!}{\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\cdot\frac{11}{2}} \\ &= \frac{1}{2}\cdot\frac{6\cdot2^4}{5\cdot7\cdot9\cdot11} = \frac{16}{1155} \end{aligned}$$

and

$$\int_0^{\pi/2} (\cos\varphi)^4 (\sin\varphi)^6 \, d\varphi = \frac{1}{2} \cdot B(5/2, 7/2) = \frac{1}{2} \cdot \frac{\Gamma(5/2)\Gamma(7/2)}{\Gamma(6)}$$
$$= \frac{1}{2} \cdot \frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma(1/2) \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma(1/2)}{5!} = \frac{5 \cdot 3^2 \cdot \pi}{2^6 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{3\,\pi}{2^9} = \frac{3\,\pi}{512}.$$

Another important consequence of (9) and (11) is Legendre's duplication formula for the gamma function:

Theorem 6.

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z), \quad \operatorname{Re} z > 0. \tag{15}$$

In order to prove this we use (11) and the transformation $2\varphi = \tau$ to find that

$$\begin{aligned} B(z,z) &= 2\int_0^{\pi/2} (\cos\varphi)^{2z-1}(\sin\varphi)^{2z-1}\, d\varphi = 2 \cdot 2^{1-2z} \int_0^{\pi/2} (\sin 2\varphi)^{2z-1}\, d\varphi \\ &= 2^{1-2z}\int_0^{\pi} (\sin\tau)^{2z-1}\, d\tau = 2^{1-2z} \cdot 2 \int_0^{\pi/2} (\sin\tau)^{2z-1}\, d\tau = 2^{1-2z} \cdot B(z, 1/2). \end{aligned}$$

Now we apply (9) to obtain

$$\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = B(z,z) = 2^{1-2z} \cdot B(z,1/2) = 2^{1-2z} \cdot \frac{\Gamma(z)\Gamma(1/2)}{\Gamma(z+1/2)}, \quad \operatorname{Re} z > 0.$$

Finally, by using (13), this implies that

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z), \quad \operatorname{Re} z > 0.$$

This proves the theorem.

Legendre's duplication formula can be generalized to Gauss's multiplication formula:

Theorem 7.

$$\Gamma(z)\prod_{k=1}^{n-1}\Gamma(z+k/n) = n^{1/2-nz}(2\pi)^{(n-1)/2}\Gamma(nz), \quad n \in \{1,2,3,\ldots\}. \tag{16}$$

The case $n = 1$ is trivial and the case $n = 2$ is Legendre's duplication formula.

Another property of the gamma function is given by Euler's reflection formula:

Theorem 8.

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \neq 0, \pm 1, \pm 2, \ldots. \tag{17}$$

This can be shown by using contour integration in the complex plane as follows. First we restrict to real values of z, say $z = x$ with $0 < x < 1$. By using (9) and (10) we have

$$\Gamma(x)\Gamma(1-x) = B(x, 1-x) = \int_0^\infty \frac{t^{x-1}}{t+1}\, dt.$$

In order to compute this integral we consider the contour integral

$$\int_{\mathcal{C}} \frac{z^{x-1}}{1-z}\, dz,$$

where the contour $\mathcal{C}$ consists of two circles about the origin of radii R and ϵ respectively, which are joined along the negative real axis from $-R$ to $-\epsilon$. Move along the outer circle with radius R in the positive (counterclockwise) direction and along the inner circle with radius ϵ in the negative (clockwise) direction. By the residue theorem we have

$$\int_{\mathcal{C}} \frac{z^{x-1}}{1-z}\,dz = -2\pi i,$$

when z^{x-1} has its principal value. This implies that

$$-2\pi i = \int_{\mathcal{C}_1} \frac{z^{x-1}}{1-z}\,dz + \int_{\mathcal{C}_2} \frac{z^{x-1}}{1-z}\,dz + \int_{\mathcal{C}_3} \frac{z^{x-1}}{1-z}\,dz + \int_{\mathcal{C}_4} \frac{z^{x-1}}{1-z}\,dz,$$

where $\mathcal{C}_1$ denotes the outer circle with radius R, $\mathcal{C}_2$ denotes the line segment from $-R$ to $-\epsilon$, $\mathcal{C}_3$ denotes the inner circle with radius ϵ and $\mathcal{C}_4$ denotes the line segment from $-\epsilon$ to $-R$. Then we have by writing $z = Re^{i\theta}$ for the outer circle

$$\int_{\mathcal{C}_1} \frac{z^{x-1}}{1-z}\,dz = \int_{-\pi}^{\pi} \frac{R^{x-1}e^{i(x-1)\theta}}{1-Re^{i\theta}}\,d\left(Re^{i\theta}\right) = \int_{-\pi}^{\pi} \frac{iR^x e^{ix\theta}}{1-Re^{i\theta}}\,d\theta.$$

In the same way we have by writing $z = \epsilon e^{i\theta}$ for the inner circle

$$\int_{\mathcal{C}_3} \frac{z^{x-1}}{1-z}\,dz = \int_{\pi}^{-\pi} \frac{i\epsilon^x e^{ix\theta}}{1-\epsilon e^{i\theta}}\,d\theta.$$

For the line segment from $-R$ to $-\epsilon$ we have by writing $z = -t = te^{\pi i}$

$$\int_{\mathcal{C}_2} \frac{z^{x-1}}{1-z}\,dz = \int_{R}^{\epsilon} \frac{t^{x-1}e^{i(x-1)\pi}}{1+t}\,d\left(te^{\pi i}\right) = \int_{R}^{\epsilon} \frac{t^{x-1}e^{ix\pi}}{1+t}\,dt.$$

In the same way we have by writing $z = -t = te^{-\pi i}$

$$\int_{\mathcal{C}_4} \frac{z^{x-1}}{1-z}\,dz = \int_{\epsilon}^{R} \frac{t^{x-1}e^{-ix\pi}}{1+t}\,dt.$$

Since $0 < x < 1$ we have

$$\lim_{R\to\infty} \int_{-\pi}^{\pi} \frac{iR^x e^{ix\theta}}{1-Re^{i\theta}}\,d\theta = 0 \quad \text{and} \quad \lim_{\epsilon\downarrow 0} \int_{\pi}^{-\pi} \frac{i\epsilon^x e^{ix\theta}}{1-\epsilon e^{i\theta}}\,d\theta = 0.$$

Hence we have

$$-2\pi i = \int_{\infty}^{0} \frac{t^{x-1}e^{ix\pi}}{1+t}\,dt + \int_{0}^{\infty} \frac{t^{x-1}e^{-ix\pi}}{1+t}\,dt,$$

or

$$-2\pi i = \left(e^{-ix\pi} - e^{ix\pi}\right) \int_0^\infty \frac{t^{x-1}}{1+t}\,dt \quad \Longrightarrow \quad \int_0^\infty \frac{t^{x-1}}{1+t}\,dt = \frac{2\pi i}{e^{ix\pi} - e^{-ix\pi}} = \frac{\pi}{\sin \pi x}.$$

This proves the theorem for real values of z, say $z = x$ with $0 < x < 1$. The full result follows by analytic continuation. Alternatively, the result can be obtained as follows. If (17) holds for real values of z with $0 < z < 1$, then it holds for all complex z with $0 < \operatorname{Re} z < 1$ by analyticity. Then it also holds for $\mathrm{Re}\, z = 0$ with $z \neq 0$ by continuity. Finally, the full result follows for z shifted by integers using (2) and $\sin(z+\pi) = -\sin z$. Note that (17) holds for all complex values of z with $z \neq 0, -1, -2, \ldots$. Instead of (17) we may write

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{\sin \pi z}{\pi}, \tag{18}$$

which holds for all $z \in \mathbb{C}$.

Now we will prove an asymptotic formula which is due to Stirling. First we define

Definition 4. *Two functions f and g of a real variable x are called asymptotically equal, notation*

$$f \sim g \quad \text{for} \quad x \to \infty, \quad \text{if} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

Now we have Stirling's formula:

Theorem 9.

$$\Gamma(x+1) \sim x^{x+1/2} e^{-x} \sqrt{2\pi}, \quad x \to \infty. \tag{19}$$

Here x denotes a real variable. This can be proved as follows. Consider

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x \, dt,$$

where $x \in \mathbb{R}$. Then we obtain by using the transformation $t = x(1+u)$

$$\begin{aligned} \Gamma(x+1) &= \int_{-1}^\infty e^{-x(1+u)} x^x (1+u)^x x \, du = x^{x+1} e^{-x} \int_{-1}^\infty e^{-xu} (1+u)^x \, du \\ &= x^{x+1} e^{-x} \int_{-1}^\infty e^{x(-u+\ln(1+u))} \, du. \end{aligned}$$

The function $f(u) = -u + \ln(1+u)$ equals zero for $u = 0$. For other values of u we have $f(u) < 0$. This implies that the integrand of the last integral equals 1 at $u = 0$ and that this integrand becomes very small for large values of x at other values of u. So for large values of x we only have to deal with the integrand near $u = 0$. Note that we have

$$f(u) = -u + \ln(1+u) = -\frac{1}{2}u^2 + \mathcal{O}(u^3) \quad \text{for} \quad u \to 0.$$

This implies that

$$\int_{-1}^\infty e^{x(-u+\ln(1+u))} \, du \sim \int_{-\infty}^\infty e^{-xu^2/2} \, du \quad \text{for} \quad x \to \infty.$$

If we set $u = t\sqrt{2/x}$ we have by using the normal integral (14)

$$\int_{-\infty}^\infty e^{-xu^2/2} \, du = x^{-1/2} \sqrt{2} \int_{-\infty}^\infty e^{-t^2} \, dt = x^{-1/2} \sqrt{2\pi}.$$

Hence we have

$$\Gamma(x+1) \sim x^{x+1/2} e^{-x} \sqrt{2\pi}, \quad x \to \infty,$$

which proves the theorem.

Note that Stirling's formula implies that

$$n! \sim n^n e^{-n}\sqrt{2\pi n} \quad \text{for} \quad n \to \infty$$

and that

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \sim n^{a-b} \quad \text{for} \quad n \to \infty.$$

The theorem can be extended for z in the complex plane:

Theorem 10. *For $\delta > 0$ we have*

$$\Gamma(z+1) \sim z^{z+1/2}e^{-z}\sqrt{2\pi} \quad \textit{for} \quad |z| \to \infty \quad \textit{with} \quad |\arg z| \leq \pi - \delta. \tag{20}$$

Stirling's asymptotic formula can be used to give an alternative proof for Euler's reflection formula (17) for the gamma function. Consider the function

$$f(z) = \Gamma(z)\Gamma(1-z)\sin \pi z.$$

Then we have

$$f(z+1) = \Gamma(z+1)\Gamma(-z)\sin \pi(z+1) = z\Gamma(z) \cdot \frac{\Gamma(1-z)}{-z} \cdot -\sin \pi z = f(z).$$

Hence, f is periodic with period 1. Further we have

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \Gamma(z)\Gamma(1-z)\sin \pi z = \lim_{z\to 0} \Gamma(z+1)\Gamma(1-z)\frac{\sin \pi z}{z} = \pi, \tag{21}$$

which implies that f has no poles. Hence, f is analytic and periodic with period 1. Now we want to apply Liouville's theorem for entire functions, id est functions which are analytic on the whole complex plane:

Theorem 11. *Every bounded entire function is constant.*

Therefore, we want to show that f is bounded. Since f is periodic with period 1 we consider $0 \leq \operatorname{Re} z \leq 1$, say $z = x + iy$ with x and y real and $0 \leq x \leq 1$. Then we have

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i} \sim -\frac{1}{2i}e^{-i\pi z} = \frac{i}{2}e^{-i\pi z} \quad \text{for} \quad y \to \infty.$$

Now we apply Stirling's formula to obtain

$$f(z) = \Gamma(z)\Gamma(1-z)\sin \pi z \sim z^{z-1/2}e^{-z}\sqrt{2\pi}\,(-z)^{-z+1/2}e^{z}\sqrt{2\pi}\,\frac{i}{2}e^{-i\pi z}.$$

For $y > 0$ we have $-z/z = e^{-\pi i}$. Hence, $f(z) \sim \pi$ for $y \to \infty$. This implies that f is bounded. So, Liouville's theorem implies that f is constant. By using (21) we conclude that $f(z) = \pi$, which proves Euler's reflection formula (17) or (18).

Stirling's formula can also be used to give an alternative proof for Legendre's duplication formula (15). Consider the function

$$g(z) = 2^{2z-1}\frac{\Gamma(z)\Gamma(z+1/2)}{\Gamma(1/2)\Gamma(2z)}.$$

Then we have by using (2)

$$g(z+1) = 2^{2z+1}\frac{\Gamma(z+1)\Gamma(z+3/2)}{\Gamma(1/2)\Gamma(2z+2)} = 2^{2z+1}\frac{z\Gamma(z)(z+1/2)\Gamma(z+1/2)}{\Gamma(1/2)(2z+1)2z\Gamma(2z)} = g(z).$$

Further we have by using (13) and Stirling's asymptotic formula (20)

$$g(z) \sim 2^{2z-1}\frac{z^{z-1/2}e^{-z}\sqrt{2\pi}\,z^z e^{-z}\sqrt{2\pi}}{\sqrt{\pi}\,2^{2z-1/2}z^{2z-1/2}e^{-2z}\sqrt{2\pi}} = 1.$$

This implies that

$$\lim_{n\to\infty} g(z+n) = 1,$$

also for integer values for n. On the other hand we have $g(z+n) = g(z)$ for integer values of n. This implies that $g(z) = 1$ for all z. This proves (15).

Finally we have the digamma function $\psi(z)$ which is related to the gamma function. This function $\psi(z)$ is defined as follows.

Definition 5.

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz}\ln\Gamma(z), \quad z \neq 0, -1, -2, \ldots. \tag{22}$$

A property of this digamma function that is easily proved by using (2) is given by the following theorem:

Theorem 12.

$$\psi(z+1) = \psi(z) + \frac{1}{z}. \tag{23}$$

By using (2) we have

$$\psi(z+1) = \frac{d}{dz}\ln\Gamma(z+1) = \frac{d}{dz}\ln(z\Gamma(z)) = \frac{d}{dz}\ln z + \frac{d}{dz}\ln\Gamma(z) = \frac{1}{z} + \psi(z).$$

This proves the theorem. Iteration of (23) easily leads to

Theorem 13.

$$\psi(z+n) = \psi(z) + \frac{1}{z} + \frac{1}{z+1} + \ldots + \frac{1}{z+n-1}, \quad n = 1, 2, 3, \ldots. \tag{24}$$

Another property of the digamma function is given by

Theorem 14.

$$\psi(z) - \psi(1-z) = -\frac{\pi}{\tan\pi z}, \quad z \neq 0, \pm1, \pm2, \ldots. \tag{25}$$

The proof of this theorem is based on (17). We have

$$\begin{aligned}\psi(z) - \psi(1-z) &= \frac{d}{dz}\ln\Gamma(z) + \frac{d}{dz}\ln\Gamma(1-z) = \frac{d}{dz}\ln(\Gamma(z)\Gamma(1-z)) \\ &= \frac{d}{dz}\ln\frac{\pi}{\sin\pi z} = \frac{\sin\pi z}{\pi}\cdot\frac{-\pi^2\cos\pi z}{(\sin\pi z)^2} = -\frac{\pi}{\tan\pi z}.\end{aligned}$$

References

[1] G.E. Andrews, R. Askey and R. Roy, *Special Functions.* Encyclopedia of Mathematics and its Applications **71**, Cambridge University Press, 1999.