Orthogonal polynomials

We start with

Definition 1. *A sequence of polynomials* $\{p_n(x)\}_{n=0}^{\infty}$ *with degree*$[p_n(x)] = n$ *for each* n *is called orthogonal with respect to the weight function* $w(x)$ *on the interval* (a,b) *with* $a < b$ *if*

$$\int_a^b w(x)p_m(x)p_n(x)\,dx = h_n\,\delta_{mn} \quad \textit{with} \quad \delta_{mn} := \begin{cases} 0, & m \neq n \\ 1, & m = n. \end{cases}$$

The weight function $w(x)$ *should be continuous and positive on* (a,b) *such that the moments*

$$\mu_n := \int_a^b w(x)\,x^n\,dx, \quad n = 0,1,2,\ldots$$

exist. Then the integral

$$\langle f,\, g\rangle := \int_a^b w(x)f(x)g(x)\,dx$$

denotes an inner product of the polynomials f *and* g*. The interval* (a,b) *is called the interval of orthogonality. This interval needs not to be finite.*

If $h_n = 1$ for each $n \in \{0,1,2,\ldots\}$ the sequence of polynomials is called orthonormal, and if

$$p_n(x) = k_n x^n + \text{ lower order terms} \quad \text{with} \quad k_n = 1$$

for each $n \in \{0,1,2,\ldots\}$ the polynomials are called monic.

Example. As an example we take $w(x) = 1$ and $(a,b) = (0,1)$. Using the Gram-Schmidt process the orthogonal polynomials can be constructed as follows. Start with the sequence $\{1, x, x^2, \ldots\}$. Choose $p_0(x) = 1$. Then we have

$$p_1(x) = x - \frac{\langle x,\, p_0(x)\rangle}{\langle p_0(x),\, p_0(x)\rangle}\,p_0(x) = x - \frac{\langle x,\, 1\rangle}{\langle 1,\, 1\rangle} = x - \frac{1}{2},$$

since

$$\langle 1,\, 1\rangle = \int_0^1 dx = 1 \quad \text{and} \quad \langle x,\, 1\rangle = \int_0^1 x\,dx = \frac{1}{2}.$$

Further we have

$$\begin{aligned} p_2(x) &= x^2 - \frac{\langle x^2,\, p_0(x)\rangle}{\langle p_0(x),\, p_0(x)\rangle}\,p_0(x) - \frac{\langle x^2,\, p_1(x)\rangle}{\langle p_1(x),\, p_1(x)\rangle}\,p_1(x) \\ &= x^2 - \frac{\langle x^2,\, 1\rangle}{\langle 1,\, 1\rangle} - \frac{\langle x^2,\, x-\frac{1}{2}\rangle}{\langle x-\frac{1}{2},\, x-\frac{1}{2}\rangle}\left(x - \frac{1}{2}\right) = x^2 - \frac{1}{3} - \left(x - \frac{1}{2}\right) = x^2 - x + \frac{1}{6}, \end{aligned}$$

since

$$\langle x^2,\, 1\rangle = \int_0^1 x^2\,dx = \frac{1}{3}, \quad \langle x^2,\, x - \tfrac{1}{2}\rangle = \int_0^1 x^2\left(x - \frac{1}{2}\right)dx = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

and

$$\langle x - \tfrac{1}{2},\, x - \tfrac{1}{2}\rangle = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \int_0^1 \left(x^2 - x + \frac{1}{4}\right)dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}.$$

The polynomials $p_0(x) = 1$, $p_1(x) = x - \frac{1}{2}$ and $p_2(x) = x^2 - x + \frac{1}{6}$ are the first three monic orthogonal polynomials on the interval $(0,1)$ with respect to the weight function $w(x) = 1$.

Repeating this process we obtain

$$p_3(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}, \quad p_4(x) = x^4 - 2x^3 + \frac{9}{7}x^2 - \frac{2}{7}x + \frac{1}{70},$$
$$p_5(x) = x^5 - \frac{5}{2}x^4 + \frac{20}{9}x^3 - \frac{5}{6}x^2 + \frac{5}{42}x - \frac{1}{252},$$

and so on.

The orthonormal polynomials would be $q_0(x) = p_0(x)/\sqrt{h_0} = 1$,

$$q_1(x) = p_1(x)/\sqrt{h_1} = 2\sqrt{3}(x - 1/2), \quad q_2(x) = \frac{p_2(x)}{\sqrt{h_2}} = 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right),$$
$$q_3(x) = \frac{p_3(x)}{\sqrt{h_3}} = 20\sqrt{7}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}\right),$$

etcetera.

All sequences of orthogonal polynomials satisfy a three term recurrence relation:

Theorem 1. *A sequence of orthogonal polynomials* $\{p_n(x)\}_{n=0}^{\infty}$ *satisfies*

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) + C_n p_{n-1}(x), \quad n = 1, 2, 3, \ldots,$$

where

$$A_n = \frac{k_{n+1}}{k_n}, \quad n = 0, 1, 2, \ldots \quad \text{and} \quad C_n = -\frac{A_n}{A_{n-1}} \cdot \frac{h_n}{h_{n-1}}, \quad n = 1, 2, 3, \ldots.$$

Proof. Since degree $[p_n(x)] = n$ for each $n \in \{0, 1, 2, \ldots\}$ the sequence of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ is linearly independent. Let $A_n = k_{n+1}/k_n$. Then $p_{n+1}(x) - A_n x p_n(x)$ is a polynomial of degree $\leq n$. Hence

$$p_{n+1}(x) - A_n x p_n(x) = \sum_{k=0}^{n} c_k p_k(x).$$

The orthogonality property now gives

$$\langle p_{n+1}(x) - A_n x p_n(x),\, p_k(x)\rangle = \sum_{m=0}^{n} c_m \langle p_m(x),\, p_k(x)\rangle = c_k \langle p_k(x),\, p_k(x)\rangle = c_k\, h_k.$$

This implies

$$\begin{aligned} h_k\, c_k &= \langle p_{n+1}(x) - A_n x p_n(x),\, p_k(x)\rangle \\ &= \langle p_{n+1}(x),\, p_k(x)\rangle - A_n \langle x p_n(x),\, p_k(x)\rangle = -A_n \langle p_n(x),\, x p_k(x)\rangle. \end{aligned}$$

For $k < n-1$ we have degree $[x p_k(x)] < n$ which implies that $\langle p_n(x),\, x p_k(x)\rangle = 0$. Hence: $c_k = 0$ for $k < n-1$. This proves that the polynomials satisfy the three term recurrence relation

$$p_{n+1}(x) - A_n x p_n(x) = c_n p_n(x) + c_{n-1} p_{n-1}(x), \quad n = 1, 2, 3, \ldots.$$

Further we have

$$h_{n-1}\, c_{n-1} = -A_n \langle p_n(x),\, x p_{n-1}(x)\rangle = -A_n\, \frac{k_{n-1}}{k_n}\, h_n \quad \Longrightarrow \quad c_{n-1} = -\frac{A_n}{A_{n-1}} \cdot \frac{h_n}{h_{n-1}}.$$

This proves the theorem.

Note that the three term recurrence relation for a sequence of monic ($k_n = 1$) orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ has the form

$$p_{n+1}(x) = xp_n(x) + B_n p_n(x) + C_n p_{n-1}(x) \quad \text{with} \quad C_n = -\frac{h_n}{h_{n-1}}, \quad n = 1, 2, 3, \ldots.$$

A consequence of the three term recurrence relation is

Theorem 2. *A sequence of orthogonal polynomials* $\{p_n(x)\}_{n=0}^{\infty}$ *satisfies*

$$\sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{h_k} = \frac{k_n}{h_n\, k_{n+1}} \cdot \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{x-y}, \quad n = 0, 1, 2, \ldots \tag{1}$$

and

$$\sum_{k=0}^{n} \frac{\{p_k(x)\}^2}{h_k} = \frac{k_n}{h_n\, k_{n+1}} \cdot \left(p'_{n+1}(x)p_n(x) - p_{n+1}(x)p'_n(x)\right), \quad n = 0, 1, 2, \ldots. \tag{2}$$

Formula (1) is called the Christoffel-Darboux formula and (2) its confluent form.

Proof. The three term recurrence relation implies that

$$p_{n+1}(x)p_n(y) = (A_n x + B_n)p_n(x)p_n(y) + C_n p_{n-1}(x)p_n(y)$$

and

$$p_{n+1}(y)p_n(x) = (A_n y + B_n)p_n(y)p_n(x) + C_n p_{n-1}(y)p_n(x).$$

Subtraction gives

$$p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x) = A_n(x-y)p_n(x)p_n(y) + C_n\left[p_{n-1}(x)p_n(y) - p_{n-1}(y)p_n(x)\right].$$

This leads to the telescoping sum

$$\begin{aligned}
(x-y)\sum_{k=1}^{n} \frac{p_k(x)p_k(y)}{h_k} &= \sum_{k=1}^{n} \frac{p_{k+1}(x)p_k(y) - p_{k+1}(y)p_k(x)}{A_k\, h_k} \\
&\qquad - \sum_{k=1}^{n} \frac{p_k(x)p_{k-1}(y) - p_k(y)p_{k-1}(x)}{A_{k-1}\, h_{k-1}} \\
&= \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{A_n\, h_n} - \frac{k_0^2(x-y)}{h_0}.
\end{aligned}$$

This implies that

$$\begin{aligned}
(x-y)\sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{h_k} &= \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{A_n\, h_n} \\
&= \frac{k_n}{h_n\, k_{n+1}} \left(p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)\right),
\end{aligned}$$

which proves (1). The confluent form (2) then follows by taking the limit $y \to x$:

$$\begin{aligned}
&\lim_{y\to x} \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{x-y} \\
&\quad = \lim_{y\to x} \frac{p_n(x)\left(p_{n+1}(x) - p_{n+1}(y)\right) - p_{n+1}(x)\left(p_n(x) - p_n(y)\right)}{x-y} \\
&\quad = p_n(x)p'_{n+1}(x) - p_{n+1}(x)p'_n(x).
\end{aligned}$$

Zeros

Theorem 3. *If* $\{p_n(x)\}_{n=0}^{\infty}$ *is a sequence of orthogonal polynomials on the interval* (a,b) *with respect to the weight function* $w(x)$*, then the polynomial* $p_n(x)$ *has exactly* n *real simple zeros in the interval* (a,b)*.*

Proof. Since degree$[p_n(x)] = n$ the polynomial has at most n real zeros. Suppose that $p_n(x)$ has $m \leq n$ distinct real zeros $x_1, x_2, \ldots, x_m$ in (a,b) of odd order (or multiplicity). Then the polynomial

$$p_n(x)(x - x_1)(x - x_2) \cdots (x - x_m)$$

does not change sign on the interval (a,b). This implies that

$$\int_a^b w(x) p_n(x)(x - x_1)(x - x_2) \cdots (x - x_m)\, dx \neq 0.$$

By orthogonality this integral equals zero if $m < n$. Hence: $m = n$, which implies that $p_n(x)$ has n distinct real zeros of odd order in (a,b). This proves that all n zeros are distinct and simple (have order or multiplicity equal to one).

Theorem 4. *If* $\{p_n(x)\}_{n=0}^{\infty}$ *is a sequence of orthogonal polynomials on the interval* (a,b) *with respect to the weight function* $w(x)$*, then the zeros of* $p_n(x)$ *and* $p_{n+1}(x)$ *separate each other.*

Proof. This follows from the confluent form (2) of the Christoffel-Darboux formula. Note that

$$h_n = \int_a^b w(x) \{p_n(x)\}^2 \, dx > 0, \quad n = 0, 1, 2, \ldots.$$

This implies that

$$\frac{k_n}{h_n \, k_{n+1}} \cdot \left(p'_{n+1}(x) p_n(x) - p_{n+1}(x) p'_n(x)\right) = \sum_{k=0}^{n} \frac{\{p_k(x)\}^2}{h_k} > 0.$$

Hence

$$\frac{k_n}{k_{n+1}} \cdot \left(p'_{n+1}(x) p_n(x) - p_{n+1}(x) p'_n(x)\right) > 0.$$

Now suppose that $x_{n,k}$ and $x_{n,k+1}$ are two consecutive zeros of $p_n(x)$ with $x_{n,k} < x_{n,k+1}$. Since all n zeros of $p_n(x)$ are real and simple $p'_n(x_{n,k})$ and $p'_n(x_{n,k+1})$ should have opposite signs. Hence we have

$$p_n(x_{n,k}) = 0 = p_n(x_{n,k+1}) \quad \text{and} \quad p'_n(x_{n,k}) \cdot p'_n(x_{n,k+1}) < 0.$$

This implies that $p_{n+1}(x_{n,k}) \cdot p_{n+1}(x_{n,k+1})$ should be negative too. Then the continuity of $p_{n+1}(x)$ implies that there should be at least one zero of $p_{n+1}(x)$ between $x_{n,k}$ and $x_{n,k+1}$. However, this holds for each two consecutive zeros of $p_n(x)$. This proves the theorem.

Moreover, if $\{x_{n,k}\}_{k=1}^{n}$ and $\{x_{n+1,k}\}_{k=1}^{n+1}$ denote the consecutive zeros of $p_n(x)$ and $p_{n+1}(x)$ respectively, then we have

$$a < x_{n+1,1} < x_{n,1} < x_{n+1,2} < x_{n,2} < \cdots < x_{n+1,n} < x_{n,n} < x_{n+1,n+1} < b.$$

Gauss quadrature

If f is a continuous function on (a,b) and $x_1 < x_2 < \cdots < x_n$ are n distinct points in (a,b), then there exists exactly one polynomial P with degree $\leq n-1$ such that $P(x_i) = f(x_i)$ for all $i = 1, 2, \ldots, n$. This polynomial P can easily be found by using Lagrange interpolation as follows. Define

$$p(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$

and consider

$$P(x) = \sum_{i=1}^{n} f(x_i) \frac{p(x)}{(x - x_i)p'(x_i)} = \sum_{i=1}^{n} f(x_i) \frac{(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}.$$

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of orthogonal polynomials on the interval (a,b) with respect to the weight function $w(x)$. Then for $x_1 < x_2 < \cdots < x_n$ we take the n distinct real zeros of the polynomial $p_n(x)$. If f is a polynomial of degree $\leq 2n-1$, then $f(x) - P(x)$ is a polynomial of degree $\leq 2n-1$ with at least the zeros $x_1 < x_2 < \cdots < x_n$. Now we define

$$f(x) = P(x) + r(x)p_n(x),$$

where $r(x)$ is a polynomial of degree $\leq n-1$. This can also be written as

$$f(x) = \sum_{i=1}^{n} f(x_i) \frac{p_n(x)}{(x - x_i)p_n'(x_i)} + r(x)p_n(x).$$

This implies that

$$\int_a^b w(x)f(x)\,dx = \sum_{i=1}^{n} f(x_i) \int_a^b \frac{w(x)p_n(x)}{(x - x_i)p_n'(x_i)}\,dx + \int_a^b w(x)r(x)p_n(x)\,dx.$$

Since degree$[r(x)] \leq n-1$ the orthogonality property implies that the latter integral equals zero. This implies that

$$\int_a^b w(x)f(x)\,dx = \sum_{i=1}^{n} \lambda_{n,i} f(x_i) \quad \text{with} \quad \lambda_{n,i} := \int_a^b \frac{w(x)p_n(x)}{(x - x_i)p_n'(x_i)}\,dx, \quad i = 1, 2, \ldots, n.$$

This is the Gauss quadrature formula. This gives the value of the integral in the case that f is a polynomial of degree $\leq 2n-1$ if the value of $f(x_i)$ is known for the n zeros $x_1 < x_2 < \cdots < x_n$ of the polynomial $p_n(x)$.
If f is not a polynomial of degree $\leq 2n-1$ this leads to an approximation of the integral:

$$\int_a^b w(x)f(x)\,dx \approx \sum_{i=1}^{n} \lambda_{n,i} f(x_i) \quad \text{with} \quad \lambda_{n,i} := \int_a^b \frac{w(x)p_n(x)}{(x - x_i)p_n'(x_i)}\,dx, \quad i = 1, 2, \ldots, n.$$

The coefficients $\{\lambda_{n,i}\}_{i=1}^{n}$ are called Christoffel numbers. Note that these numbers do not depend on the function f. These Christoffel numbers are all positive. This can be shown as follows. We have

$$\lambda_{n,i} = \int_a^b w(x)\ell_{n,i}(x)\,dx \quad \text{with} \quad \ell_{n,i}(x) := \frac{p_n(x)}{(x - x_i)p_n'(x_i)}, \quad i = 1, 2, \ldots, n.$$

Then $\ell_{n,i}^2(x) - \ell_{n,i}(x)$ is a polynomial of degree $\leq 2n-2$ which vanishes at the zeros $\{x_{n,k}\}_{k=1}^n$ of $p_n(x)$. Hence

$$\ell_{n,i}^2(x) - \ell_{n,i}(x) = p_n(x)q(x) \quad \text{for some polynomial } q \text{ of degree } \leq n-2.$$

This implies that

$$\int_a^b w(x)\left(\ell_{n,i}^2(x) - \ell_{n,i}(x)\right)\,dx = \int_a^b w(x)p_n(x)q(x)\,dx = 0$$

by orthogonality. Hence we have

$$\lambda_{n,i} = \int_a^b w(x)\ell_{n,i}(x)\,dx = \int_a^b w(x)\left\{\ell_{n,i}(x)\right\}^2\,dx > 0.$$

Now we can also prove

Theorem 5. *Let $\{p_n(x)\}_{n=0}^\infty$ be a sequence of orthogonal polynomials on the interval (a,b) with respect to the weight function $w(x)$ and let $m < n$. Then we have: between any two zeros of $p_m(x)$ there is at least one zero of $p_n(x)$.*

Proof. Suppose that $x_{m,k}$ and $x_{m,k+1}$ are two consecutive zeros of $p_m(x)$ and that there is no zero of $p_n(x)$ in $(x_{m,k}, x_{m,k+1})$. Then consider the polynomial

$$g(x) = \frac{p_m(x)}{(x - x_{m,k})(x - x_{m,k+1})}.$$

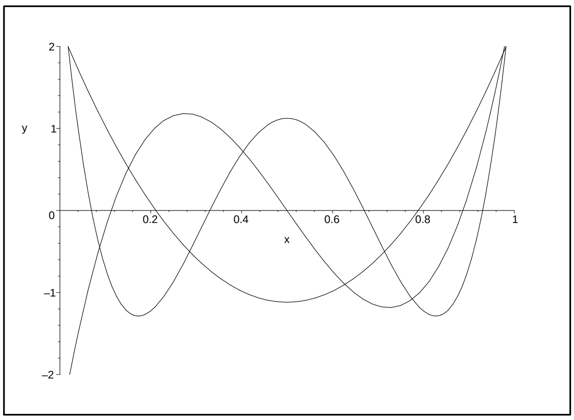
Then we have

$$g(x)p_m(x) \geq 0 \quad \text{for} \quad x \notin (x_{m,k}, x_{m,k+1}).$$

Now the Gauss quadrature formula gives

$$\int_a^b w(x)g(x)p_m(x)\,dx = \sum_{i=1}^n \lambda_{n,i}g(x_{n,i})p_m(x_{n,i}),$$

where $\{x_{n,i}\}_{i=1}^n$ are the zeros of $p_n(x)$. Since there are no zeros of $p_n(x)$ in $(x_{m,k}, x_{m,k+1})$ we conclude that $g(x_{n,i})p_m(x_{n,i}) \geq 0$ for all $i = 1, 2, \ldots, n$. Further we have $\lambda_{n,i} > 0$ for all $i = 1, 2, \ldots, n$ which implies that the sum at the right-hand side cannot vanish. However, the integral at the left-hand side is zero by orthogonality. This contradiction proves that there should be at least one zero of $p_n(x)$ between the two consecutive zeros of $p_m(x)$.



The polynomials $q_2(x)$, $q_3(x)$ and $q_4(x)$

Classical orthogonal polynomials

The classical orthogonal polynomials are named after Hermite, Laguerre and Jacobi:

name	$p_n(x)$	$w(x)$	(a,b)
Hermite	$H_n(x)$	e^{-x^2}	$(-\infty,\infty)$
Laguerre	$L_n^{(\alpha)}(x)$	$e^{-x}x^{\alpha}$	$(0,\infty)$
Jacobi	$P_n^{(\alpha,\beta)}(x)$	$(1-x)^{\alpha}(1+x)^{\beta}$	$(-1,1)$
Legendre	$P_n(x)$	1	$(-1,1)$

The Hermite polynomials are orthogonal on the interval $(-\infty,\infty)$ with respect to the normal distribution $w(x)=e^{-x^2}$, the Laguerre polynomials are orthogonal on the interval $(0,\infty)$ with respect to the gamma distribution $w(x)=e^{-x}x^{\alpha}$ and the Jacobi polynomials are orthogonal on the interval $(-1,1)$ with respect to the beta distribution $w(x)=(1-x)^{\alpha}(1+x)^{\beta}$.

The Legendre polynomials form a special case ($\alpha=\beta=0$) of the Jacobi polynomials.

These classical orthogonal polynomials satisfy an orthogonality relation, a three term recurrence relation, a second order linear differential equation and a so-called Rodrigues formula. Moreover, for each family of classical orthogonal polynomials we have a generating function.

In the sequel we will often use the Kronecker delta which is defined by

$$\delta_{mn} := \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

for $m,n \in \{0,1,2,\ldots\}$ and the notation

$$D = \frac{d}{dx} \tag{3}$$

for the differentiation operator. Then we have Leibniz' rule

$$D^n\left[f(x)g(x)\right] = \sum_{k=0}^{n} \binom{n}{k} D^k f(x) D^{n-k} g(x), \quad n = 0,1,2,\ldots \tag{4}$$

which is a generalization of the product rule. The proof is by mathematical induction and by use of Pascal's triangle identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}, \quad k = 1,2,\ldots,n.$$

Hermite

The Hermite polynomials are orthogonal on the interval $(-\infty, \infty)$ with respect to the normal distribution $w(x) = e^{-x^2}$. They can be defined by means of their Rodrigues formula:

$$H_n(x) = \frac{(-1)^n}{w(x)} D^n w(x) = (-1)^n e^{x^2} D^n e^{-x^2}, \quad n = 0, 1, 2, \ldots, \tag{5}$$

where the differentiation operator D is defined by (3). Since $D^{n+1} = D\, D^n$, we obtain

$$\begin{aligned} D^{n+1} w(x) &= D\left[D^n w(x)\right] = (-1)^n D\left[w(x) H_n(x)\right] = (-1)^n \left[w'(x) H_n(x) + w(x) H_n'(x)\right] \\ &= (-1)^{n+1} w(x) \left[2x H_n(x) - H_n'(x)\right], \quad n = 0, 1, 2, \ldots, \end{aligned}$$

which implies that

$$H_{n+1}(x) = 2x H_n(x) - H_n'(x), \quad n = 0, 1, 2, \ldots. \tag{6}$$

The definition (5) implies that $H_0(x) = 1$. Then (6) implies by induction that $H_n(x)$ is a polynomial of degree n. Further we have that $H_{2n}(x)$ is even and $H_{2n+1}(x)$ is odd and that the leading coefficient of the polynomial $H_n(x)$ equals $k_n = 2^n$.

The Hermite polynomials satisfy the orthogonality relation

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = 2^n\, n!\, \delta_{mn}, \quad m, n \in \{0, 1, 2, \ldots\}. \tag{7}$$

To prove this we use the definition (5) to obtain

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = (-1)^n \int_{-\infty}^{\infty} H_m(x) D^n e^{-x^2}\, dx.$$

Now we use integration by parts n times to conclude that the integral vanishes for $m < n$. For $m = n$ we have using integration by parts

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_n(x)\, dx &= (-1)^n \int_{-\infty}^{\infty} H_n(x) D^n e^{-x^2}\, dx = \int_{-\infty}^{\infty} D^n H_n(x) \cdot e^{-x^2}\, dx \\ &= k_n\, n! \int_{-\infty}^{\infty} e^{-x^2}\, dx = 2^n\, n!\, \sqrt{\pi}. \end{aligned}$$

This proves the orthogonality relation (7).

In order to find the three term recurrence relation we start with

$$w(x) = e^{-x^2} \quad \Longrightarrow \quad w'(x) = -2x w(x).$$

Then we have by using Leibniz' rule (4)

$$D^{n+1} w(x) = D^n w'(x) = D^n\left[-2x w(x)\right] = -2x D^n w(x) - 2n D^{n-1} w(x),$$

which implies that

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x), \quad n = 1, 2, 3, \ldots. \tag{8}$$

Combining (6) and (8) we find that

$$H_n'(x) = 2nH_{n-1}(x), \quad n = 1, 2, 3, \ldots. \tag{9}$$

Differentiation of (6) gives

$$H_{n+1}'(x) = 2xH_n'(x) + 2H_n(x) - H_n''(x), \quad n = 0, 1, 2, \ldots.$$

Now we use (9) to conclude that

$$2(n+1)H_n(x) = 2xH_n'(x) + 2H_n(x) - H_n''(x), \quad n = 0, 1, 2, \ldots,$$

which implies that $H_n(x)$ satisfies the second order linear differential equation

$$y''(x) - 2xy'(x) + 2ny(x) = 0, \quad n \in \{0, 1, 2, \ldots\}.$$

Finally we will prove the generating function

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!}\, t^n. \tag{10}$$

We start with

$$f(t) = e^{-(x-t)^2} = e^{-x^2} \cdot e^{2xt-t^2}.$$

The Taylor series for $f(t)$ is

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, t^n$$

with, by using the substitution $x - t = u$,

$$f^{(n)}(0) = \left[\frac{d^n}{dt^n}\, e^{-(x-t)^2}\right]_{t=0} = (-1)^n \left[\frac{d^n}{du^n}\, e^{-u^2}\right]_{u=x} = (-1)^n D^n e^{-x^2} = e^{-x^2} H_n(x)$$

for $n = 0, 1, 2, \ldots$. Hence we have

$$e^{-x^2} \cdot e^{2xt-t^2} = e^{-(x-t)^2} = f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, t^n = e^{-x^2} \sum_{n=0}^{\infty} \frac{H_n(x)}{n!}\, t^n.$$

This proves the generating function (10).

Laguerre

The Laguerre polynomials are orthogonal on the interval $(0,\infty)$ with respect to the gamma distribution $w(x)=e^{-x}x^{\alpha}$. They can be defined by means of their Rodrigues formula:

$$L_n^{(\alpha)}(x)=\frac{1}{n!}\frac{1}{w(x)}\,D^n\left[w(x)\,x^n\right]=\frac{1}{n!}\,e^x\,x^{-\alpha}\,D^n\left[e^{-x}x^{n+\alpha}\right],\quad n=0,1,2,\ldots. \tag{11}$$

By using Leibniz' rule (4) we have

$$\begin{aligned}D^n\left[e^{-x}x^{n+\alpha}\right] &= \sum_{k=0}^{n}\binom{n}{k}D^k e^{-x}D^{n-k}x^{n+\alpha}\\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k e^{-x}(n+\alpha)(n+\alpha-1)\cdots(\alpha+k+1)x^{\alpha+k}\\ &= e^{-x}\,x^{\alpha}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)}\,x^k.\end{aligned}$$

Hence we have

$$L_n^{(\alpha)}(x)=\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{x^k}{k!},\quad n=0,1,2,\ldots, \tag{12}$$

where

$$\binom{n+\alpha}{n-k}=\frac{\Gamma(n+\alpha+1)}{(n-k)!\,\Gamma(k+\alpha+1)}=\frac{(k+\alpha+1)_{n-k}}{(n-k)!},\quad k=0,1,2,\ldots,n.$$

This proves that $L_n^{(\alpha)}(x)$ is a polynomial of degree n. Since

$$(-1)^k\binom{n+\alpha}{n-k}=\frac{(-1)^k}{(n-k)!}\,\frac{(\alpha+1)_n}{(\alpha+1)_k}=\frac{(\alpha+1)_n}{n!}\,\frac{(-n)_k}{(\alpha+1)_k},\quad k=0,1,2,\ldots,n$$

we also have for $n=0,1,2,\ldots$

$$L_n^{(\alpha)}(x)=\frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(\alpha+1)_k}\,\frac{x^k}{k!}=\binom{n+\alpha}{n}\,{}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix};x\right). \tag{13}$$

Note that we have

$$L_n^{(\alpha)}(0)=\binom{n+\alpha}{n}=\frac{(\alpha+1)_n}{n!},\quad n=0,1,2,\ldots$$

and that the leading coefficient of the polynomial $L_n^{(\alpha)}(x)$ equals

$$k_n=\frac{(-1)^n}{n!},\quad n=0,1,2,\ldots.$$

Further we have

$$\begin{aligned}\frac{d}{dx}L_n^{(\alpha)}(x) &= \frac{d}{dx}\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{x^k}{k!}=\sum_{k=1}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{x^{k-1}}{(k-1)!}\\ &= \sum_{k=0}^{n-1}(-1)^{k-1}\binom{n+\alpha}{n-k-1}\frac{x^k}{k!}=-L_{n-1}^{(\alpha+1)}(x),\quad n=1,2,3,\ldots. \end{aligned} \tag{14}$$

Now we can prove the orthogonality relation

$$\int_0^\infty e^{-x}x^\alpha L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)\,dx = \frac{\Gamma(n+\alpha+1)}{n!}\,\delta_{mn}, \quad \alpha > -1 \tag{15}$$

for $m, n \in \{0, 1, 2, \ldots\}$. First of all, the integral converges if the moments

$$\mu_n = \int_0^\infty e^{-x}x^{n+\alpha}\,dx$$

exists for all $n \in \{0, 1, 2, \ldots\}$. This leads to the condition $\alpha > -1$. Note that $\mu_n = \Gamma(n+\alpha+1)$. Now we use (11) to obtain

$$\int_0^\infty e^{-x}x^\alpha L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)\,dx = \frac{1}{n!}\int_0^\infty L_m^{(\alpha)}(x)D^n\left[e^{-x}x^{n+\alpha}\right]\,dx.$$

We apply integration by parts to obtain

$$\int_0^\infty L_m^{(\alpha)}(x)D^n\left[e^{-x}x^{n+\alpha}\right]\,dx = (-1)^n\int_0^\infty D^n L_m^{(\alpha)}(x)e^{-x}x^{n+\alpha}\,dx$$

which equals zero for $m < n$. For $m = n$ we find

$$\int_0^\infty D^n L_n^{(\alpha)}(x)e^{-x}x^{n+\alpha}\,dx = k_n\,n!\int_0^\infty e^{-x}x^{n+\alpha}\,dx = (-1)^n\,\Gamma(n+\alpha+1).$$

This proves the orthogonality relation (15).

The Laguerre polynomials can also be defined by their generating function

$$(1-t)^{-\alpha-1}\exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^\infty L_n^{(\alpha)}(x)t^n. \tag{16}$$

The proof is straightforward. Start with (13) and change the order of summation to obtain

$$\begin{aligned}
\sum_{n=0}^\infty L_n^{(\alpha)}(x)t^n &= \sum_{n=0}^\infty \frac{(\alpha+1)_n}{n!}\,t^n\sum_{k=0}^n \frac{(-n)_k}{(\alpha+1)_k}\,\frac{x^k}{k!} = \sum_{k=0}^\infty\sum_{n=k}^\infty \frac{(\alpha+1)_n}{(\alpha+1)_k}\,\frac{(-1)^k x^k t^n}{k!\,(n-k)!}\\
&= \sum_{k=0}^\infty\sum_{n=0}^\infty \frac{(\alpha+1)_{n+k}}{(\alpha+1)_k}\,\frac{(-1)^k x^k t^{n+k}}{k!\,n!} = \sum_{k=0}^\infty \frac{(-xt)^k}{k!}\sum_{n=0}^\infty \frac{(\alpha+k+1)_n}{n!}\,t^n\\
&= \sum_{k=0}^\infty \frac{(-xt)^k}{k!}\,(1-t)^{-\alpha-k-1} = (1-t)^{-\alpha-1}\sum_{k=0}^\infty \frac{1}{k!}\left(-\frac{xt}{1-t}\right)^k.
\end{aligned}$$

This proves (16).

If we define

$$F(x,t) := (1-t)^{-\alpha-1}\exp\left(-\frac{xt}{1-t}\right),$$

then we easily obtain

$$\frac{\partial}{\partial x}F(x,t) = -t(1-t)^{-\alpha-2}\exp\left(-\frac{xt}{1-t}\right) \quad\Longrightarrow\quad (1-t)\frac{\partial}{\partial x}F(x,t) + tF(x,t) = 0$$

and

$$\begin{aligned}\frac{\partial}{\partial t}F(x,t) &= \left\{(\alpha+1)(1-t)^{-\alpha-2}+(1-t)^{-\alpha-1}\cdot\frac{-x(1-t)-xt}{(1-t)^2}\right\}\exp\left(-\frac{xt}{1-t}\right)\\ &= \left\{\alpha+1-\frac{x}{1-t}\right\}(1-t)^{-\alpha-2}\exp\left(-\frac{xt}{1-t}\right),\end{aligned}$$

which implies that

$$(1-t)^2\frac{\partial}{\partial t}F(x,t)+[x-(\alpha+1)(1-t)]\,F(x,t)=0.$$

The first relation leads to

$$(1-t)\sum_{n=0}^{\infty}\frac{d}{dx}L_n^{(\alpha)}(x)t^n+t\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)t^n=0$$

or equivalently

$$\sum_{n=0}^{\infty}\frac{d}{dx}L_n^{(\alpha)}(x)t^n-\sum_{n=0}^{\infty}\frac{d}{dx}L_n^{(\alpha)}(x)t^{n+1}+\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)t^{n+1}=0.$$

Hence we have

$$\frac{d}{dx}L_{n+1}^{(\alpha)}(x)-\frac{d}{dx}L_n^{(\alpha)}(x)+L_n^{(\alpha)}(x)=0,\quad n=0,1,2,\ldots \tag{17}$$

or equivalently, by using (14),

$$\frac{d}{dx}L_n^{(\alpha)}(x)=L_n^{(\alpha)}(x)-L_n^{(\alpha+1)}(x),\quad n=0,1,2,\ldots.$$

The second relation leads to

$$(1-t)^2\sum_{n=1}^{\infty}nL_n^{(\alpha)}(x)t^{n-1}+[x-(\alpha+1)(1-t)]\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)t^n=0$$

or equivalently

$$\begin{aligned}&\sum_{n=1}^{\infty}nL_n^{(\alpha)}(x)t^{n-1}-2\sum_{n=1}^{\infty}nL_n^{(\alpha)}(x)t^n+\sum_{n=1}^{\infty}nL_n^{(\alpha)}(x)t^{n+1}\\ &\qquad+x\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)t^n-(\alpha+1)\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)t^n+(\alpha+1)\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)t^{n+1}=0.\end{aligned}$$

Equating the coefficients of equal powers of t we obtain the three term recurrence relation

$$(n+1)L_{n+1}^{(\alpha)}(x)+(x-2n-\alpha-1)L_n^{(\alpha)}(x)+(n+\alpha)L_{n-1}^{(\alpha)}(x)=0,\quad n=1,2,3,\ldots. \tag{18}$$

Note that this can also be written as

$$xL_n^{(\alpha)}(x)+(n+1)\left[L_{n+1}^{(\alpha)}(x)-L_n^{(\alpha)}(x)\right]-(n+\alpha)\left[L_n^{(\alpha)}(x)-L_{n-1}^{(\alpha)}(x)\right]=0,\quad n=1,2,3,\ldots.$$

Now we differentiate and use (17) to obtain

$$x\frac{d}{dx}L_n^{(\alpha)}(x) + L_n^{(\alpha)}(x) - (n+1)L_n^{(\alpha)}(x) + (n+\alpha)L_{n-1}^{(\alpha)}(x) = 0, \quad n = 1, 2, 3, \ldots.$$

This implies that

$$x\frac{d}{dx}L_n^{(\alpha)}(x) = nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x), \quad n = 1, 2, 3, \ldots. \tag{19}$$

Now we differentiate (19) and use (17) and (19) to find

$$\begin{aligned} x\frac{d^2}{dx^2}L_n^{(\alpha)}(x) + \frac{d}{dx}L_n^{(\alpha)}(x) &= n\frac{d}{dx}L_n^{(\alpha)}(x) - (n+\alpha)\frac{d}{dx}L_{n-1}^{(\alpha)}(x) \\ &= (n+\alpha)\left[\frac{d}{dx}L_n^{(\alpha)}(x) - \frac{d}{dx}L_{n-1}^{(\alpha)}(x)\right] - \alpha\,\frac{d}{dx}L_n^{(\alpha)}(x) \\ &= -(n+\alpha)L_{n-1}^{(\alpha)}(x) - \alpha\,\frac{d}{dx}L_n^{(\alpha)}(x) \\ &= x\frac{d}{dx}L_n^{(\alpha)}(x) - nL_n^{(\alpha)}(x) - \alpha\,\frac{d}{dx}L_n^{(\alpha)}(x). \end{aligned}$$

This proves that the polynomial $L_n^{(\alpha)}(x)$ satisfies the second order linear differential equation

$$xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0, \quad n \in \{0, 1, 2, \ldots\}.$$

Finally, we use the generating function (16) to prove the addition formula

$$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{n} L_k^{(\alpha)}(x)L_{n-k}^{(\beta)}(y), \quad n = 0, 1, 2, \ldots. \tag{20}$$

The generating function (16) implies that

$$\begin{aligned} \sum_{n=0}^{\infty} L_n^{(\alpha+\beta+1)}(x+y)t^n &= (1-t)^{-\alpha-\beta-2}\exp\left(-\frac{(x+y)t}{1-t}\right) \\ &= (1-t)^{-\alpha-1}\exp\left(-\frac{xt}{1-t}\right)\cdot(1-t)^{-\beta-1}\exp\left(-\frac{yt}{1-t}\right) \\ &= \sum_{k=0}^{\infty} L_k^{(\alpha)}(x)t^k \cdot \sum_{m=0}^{\infty} L_m^{(\beta)}(y)t^m = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} L_k^{(\alpha)}(x)L_{n-k}^{(\beta)}(y)\right)t^n. \end{aligned}$$

This proves (20).

Jacobi

The Jacobi polynomials are orthogonal on the interval $(-1,1)$ with respect to the beta distribution $w(x)=(1-x)^{\alpha}(1+x)^{\beta}$. They can be defined by means of their Rodrigues formula:

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(-1)^n}{2^n\,n!}\,\frac{1}{w(x)}\,D^n\left[w(x)\,(1-x^2)^n\right] \\ &= \frac{(-1)^n}{2^n\,n!}\,(1-x)^{-\alpha}(1+x)^{-\beta}\,D^n\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right] \end{aligned} \tag{21}$$

for $n=0,1,2,\ldots$. By using Leibniz' rule (4) we have

$$\begin{aligned} D^n\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right] &= \sum_{k=0}^{n}\binom{n}{k}D^k(1-x)^{n+\alpha}D^{n-k}(1+x)^{n+\beta} \\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k(n+\alpha)(n+\alpha-1)\cdots(n+\alpha-k+1)(1-x)^{n+\alpha-k} \\ &\qquad \times (n+\beta)(n+\beta-1)\cdots(\beta+k+1)(1+x)^{\beta+k} \\ &= n!\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(1-x)^{n+\alpha-k}(1+x)^{\beta+k}, \quad n=0,1,2,\ldots. \end{aligned}$$

This implies that

$$P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(1-x)^{n-k}(1+x)^k, \quad n=0,1,2,\ldots. \tag{22}$$

This shows that $P_n^{(\alpha,\beta)}(x)$ is a polynomial of degree n. Note that we have the symmetry

$$P_n^{(\alpha,\beta)}(-x)=(-1)^nP_n^{(\beta,\alpha)}(x), \quad n=0,1,2,\ldots \tag{23}$$

and

$$P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n} \quad \text{and} \quad P_n^{(\alpha,\beta)}(-1)=(-1)^n\binom{n+\beta}{n}, \quad n=0,1,2,\ldots.$$

In order to find a hypergeometric representation we write for $x\neq 1$

$$P_n^{(\alpha,\beta)}(x)=\left(\frac{x-1}{2}\right)^n\sum_{k=0}^{n}\binom{n+\alpha}{n}\binom{n+\beta}{n-k}\left(\frac{x+1}{x-1}\right)^k, \quad n=0,1,2,\ldots.$$

Now we have for $x\neq 1$

$$\left(\frac{x+1}{x-1}\right)^k=\left(1+\frac{2}{x-1}\right)^k=\sum_{i=0}^{k}\binom{k}{i}\left(\frac{2}{x-1}\right)^i, \quad k=0,1,2,\ldots.$$

Now we obtain by changing the order of summations for $x\neq 1$ and $n=0,1,2,\ldots$

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \left(\frac{x-1}{2}\right)^n\sum_{i=0}^{n}\sum_{k=i}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}\binom{k}{i}\left(\frac{2}{x-1}\right)^i \\ &= \left(\frac{x-1}{2}\right)^n\sum_{i=0}^{n}\sum_{k=0}^{n-i}\binom{n+\alpha}{i+k}\binom{n+\beta}{n-i-k}\binom{i+k}{i}\left(\frac{2}{x-1}\right)^i. \end{aligned}$$

Now we reverse the order in the first sum to find for $x \neq 1$ and $n = 0, 1, 2, \ldots$

$$\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= \left(\frac{x-1}{2}\right)^n \sum_{i=0}^n \sum_{k=0}^n \binom{n+\alpha}{n-i+k}\binom{n+\beta}{i-k}\binom{n-i+k}{n-i}\left(\frac{2}{x-1}\right)^{n-i} \\
&= \sum_{i=0}^n \sum_{k=0}^n \binom{n+\alpha}{n-i+k}\binom{n+\beta}{i-k}\binom{n-i+k}{n-i}\left(\frac{x-1}{2}\right)^{i} \\
&= \sum_{i=0}^n \sum_{k=0}^n \frac{\Gamma(n+\alpha+1)}{(n-i+k)!\,\Gamma(i-k+\alpha+1)} \\
&\qquad\qquad \times \frac{\Gamma(n+\beta+1)}{(i-k)!\,\Gamma(n-i+k+\beta+1)}\,\frac{(n-i+k)!}{(n-i)!\,k!}\left(\frac{x-1}{2}\right)^{i} \\
&= \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!}\sum_{i=0}^n \frac{(-n)_i}{\Gamma(i+\alpha+1)\,i!\,\Gamma(n-i+\beta+1)}\left(\frac{1-x}{2}\right)^{i} \\
&\qquad\qquad \times \sum_{k=0}^n \frac{(-i)_k(-i-\alpha-1)_k}{(n-i+\beta+1)_k\,k!}.
\end{aligned}$$

Since $i \in \{0, 1, 2, \ldots, n\}$ we have by using the Chu-Vandermonde summation formula

$$\sum_{k=0}^n \frac{(-i)_k(-i-\alpha-1)_k}{(n-i+\beta+1)_k\,k!} = {}_2F_1\left(\begin{matrix}-i,\, -i-\alpha-1\\ n-i+\beta+1\end{matrix}\,;\,1\right) = \frac{(n+\alpha+\beta+1)_i}{(n-i+\beta+1)_i}.$$

Hence we have by using $\Gamma(n-i+\beta+1)\,(n-i+\beta+1)_i = \Gamma(n+\beta+1)$

$$\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= \frac{\Gamma(n+\alpha+1)}{n!}\sum_{i=0}^n \frac{(-n)_i(n+\alpha+\beta+1)_i}{\Gamma(i+\alpha+1)\,i!}\left(\frac{1-x}{2}\right)^{i} \\
&= \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}\sum_{i=0}^n \frac{(-n)_i(n+\alpha+\beta+1)_i}{(\alpha+1)_i\,i!}\left(\frac{1-x}{2}\right)^{i}, \quad n = 0, 1, 2, \ldots.
\end{aligned}$$

This proves the hypergeometric representation

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}\,{}_2F_1\left(\begin{matrix}-n,\, n+\alpha+\beta+1\\ \alpha+1\end{matrix}\,;\,\frac{1-x}{2}\right), \quad n = 0, 1, 2, \ldots. \tag{24}$$

Note that this result also holds for $x = 1$. In view of the symmetry (23) we also have

$$P_n^{(\alpha,\beta)}(x) = (-1)^n\binom{n+\beta}{n}\,{}_2F_1\left(\begin{matrix}-n,\, n+\alpha+\beta+1\\ \beta+1\end{matrix}\,;\,\frac{1+x}{2}\right), \quad n = 0, 1, 2, \ldots.$$

Note that the hypergeometric representation implies that

$$\begin{aligned}
\frac{d}{dx}P_n^{(\alpha,\beta)}(x) &= \binom{n+\alpha}{n}\frac{(-n)(n+\alpha+\beta+1)}{\alpha+1}\cdot\left(-\frac{1}{2}\right) \\
&\qquad \times {}_2F_1\left(\begin{matrix}-n+1,\, n+\alpha+\beta+2\\ \alpha+2\end{matrix}\,;\,\frac{1-x}{2}\right) \\
&= \frac{n+\alpha+\beta+1}{2}\binom{n+\alpha}{n-1}\,{}_2F_1\left(\begin{matrix}-n+1,\, n+\alpha+\beta+2\\ \alpha+2\end{matrix}\,;\,\frac{1-x}{2}\right) \\
&= \frac{n+\alpha+\beta+1}{2}\,P_{n-1}^{(\alpha+1,\beta+1)}(x), \quad n = 1, 2, 3, \ldots.
\end{aligned}$$

Another consequence of the hypergeometric representation is that the leading coefficient of the polynomial $P_n^{(\alpha,\beta)}(x)$ equals

$$k_n = \binom{n+\alpha}{n} \frac{(-n)_n(n+\alpha+\beta+1)_n}{(\alpha+1)_n\, n!} \frac{(-1)^n}{2^n} = \frac{(n+\alpha+\beta+1)_n}{2^n\, n!}, \quad n = 0, 1, 2, \ldots.$$

Now it can be shown that the Jacobi polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\, dx = \frac{2^{\alpha+\beta+1}\, \Gamma(n+\alpha+1)\, \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\, \Gamma(n+\alpha+\beta+1)\, n!} \delta_{mn}$$

for $\alpha > -1$, $\beta > -1$ and $m, n \in \{0, 1, 2, \ldots\}$. This can be shown by using the definition (21) and integration by parts. The value of the integral in the case $m = n$ can be computed by using the leading coefficient and then writing the integral in terms of a beta integral:

$$\begin{aligned}
&\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta \left\{P_n^{(\alpha,\beta)}(x)\right\}^2 dx \\
&\quad = \frac{(-1)^n}{2^n\, n!} \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) D^n \left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right] dx \\
&\quad = \frac{1}{2^n\, n!} \int_{-1}^{1} D^n P_n^{(\alpha,\beta)}(x) (1-x)^{n+\alpha}(1+x)^{n+\beta}\, dx \\
&\quad = \frac{(n+\alpha+\beta+1)_n}{2^{2n}\, n!} \int_{-1}^{1} (1-x)^{n+\alpha}(1+x)^{n+\beta}\, dx \\
&\quad = \frac{\Gamma(2n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)\, 2^{2n}\, n!} \int_{-1}^{1} (1-x)^{n+\alpha}(1+x)^{n+\beta}\, dx, \quad n = 0, 1, 2, \ldots
\end{aligned}$$

and by using the substitution $1 - x = 2t$

$$\begin{aligned}
&\int_{-1}^{1} (1-x)^{n+\alpha}(1+x)^{n+\beta}\, dx = \int_{0}^{1} (2t)^{n+\alpha}(2-2t)^{n+\beta}\, 2\, dt \\
&\quad = 2^{2n+\alpha+\beta+1} \int_{0}^{1} t^{n+\alpha}(1-t)^{n+\beta}\, dt = 2^{2n+\alpha+\beta+1}\, B(n+\alpha+1, n+\beta+1) \\
&\quad = 2^{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)} \\
&\quad = 2^{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\, \Gamma(2n+\alpha+\beta+1)}, \quad n = 0, 1, 2, \ldots.
\end{aligned}$$

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy the second order linear differential equation

$$(1-x^2)y''(x) + [\beta - \alpha - (\alpha+\beta+2)x]\, y'(x) + n(n+\alpha+\beta+1)y(x) = 0, \quad n = 0, 1, 2, \ldots.$$

A generating function for the Jacobi polynomials is given by

$$\frac{2^{\alpha+\beta}}{R(1+R-t)^\alpha(1+R+t)^\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) t^n, \quad R = \sqrt{1 - 2xt + t^2}.$$

Legendre

The Legendre polynomials are orthogonal on the interval $(-1,1)$ with respect to the weight function $w(x)=1$. They can be defined by means of their Rodrigues formula:

$$P_n(x)=\frac{(-1)^n}{2^n\,n!}\,\frac{1}{w(x)}\,D^n\left[w(x)\,(1-x^2)^n\right]=\frac{(-1)^n}{2^n\,n!}\,D^n\left[(1-x^2)^n\right],\quad n=0,1,2,\ldots. \tag{25}$$

This is the special case $\alpha=\beta=0$ of the Jacobi polynomials:

$$P_n(x)=P_n^{(0,0)}(x)={}_2F_1\left(\begin{matrix}-n,\,n+1\\1\end{matrix};\frac{1-x}{2}\right),\quad n=0,1,2,\ldots. \tag{26}$$

Further we have

$$P_n(-x)=(-1)^nP_n(x),\quad P_n(1)=1\quad\text{and}\quad P_n(-1)=(-1)^n,\quad n=0,1,2,\ldots.$$

The leading coefficient of the polynomial $P_n(x)$ equals

$$k_n=\frac{(-n)_n\,(n+1)_n}{(1)_n\,n!}\,\frac{(-1)^n}{2^n}=\frac{(2n)!}{2^n\,(n!)^2},\quad n=0,1,2,\ldots.$$

The orthogonality relation is

$$\int_{-1}^1 P_m(x)P_n(x)\,dx=\frac{2}{2n+1}\,\delta_{mn},\quad m,n\in\{0,1,2,\ldots\}. \tag{27}$$

This can be shown by using the Rodrigues formula (25) and integration by parts

$$\int_{-1}^1 P_m(x)P_n(x)\,dx=\frac{(-1)^n}{2^n\,n!}\int_{-1}^1 P_m(x)D^n\left[(1-x^2)^n\right]\,dx=\frac{1}{2^n\,n!}\int_{-1}^1 D^nP_m(x)\,(1-x^2)^n\,dx,$$

which vanishes for $m<n$. For $m=n$ we have

$$\int_{-1}^1 D^nP_n(x)\,(1-x^2)^n\,dx=k_n\,n!\int_{-1}^1(1-x^2)^n\,dx=\frac{(2n)!}{2^n\,n!}\int_{-1}^1(1-x^2)^n\,dx.$$

Finally we have by using the substitution $1-x=2t$ for $n=0,1,2,\ldots$

$$\begin{aligned}\int_{-1}^1(1-x^2)^n\,dx &= \int_{-1}^1(1-x)^n(1+x)^n\,dx=\int_0^1(2t)^n(2-2t)^n\,2\,dx\\ &= 2^{2n+1}\,B(n+1,n+1)=2^{2n+1}\,\frac{\Gamma(n+1)\,\Gamma(n+1)}{\Gamma(2n+2)}=\frac{2^{2n+1}\,(n!)^2}{(2n+1)!}.\end{aligned}$$

Hence we have

$$\int_{-1}^1\left\{P_n(x)\right\}^2\,dx=\frac{(2n)!}{2^{2n}\,(n!)^2}\,\frac{2^{2n+1}\,(n!)^2}{(2n+1)!}=\frac{2}{2n+1},\quad n=0,1,2,\ldots.$$

This proves the orthogonality relation (27).

In order to find a generating function for the Legendre polynomials we use the hypergeometric representation (26) to find

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n(x)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-n)_k\,(n+1)_k}{(1)_k\,k!}\left(\frac{1-x}{2}\right)^k t^n \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{(-n)_k\,(n+1)_k}{k!\,k!}\left(\frac{1-x}{2}\right)^k t^n \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-n-k)_k\,(n+k+1)_k}{k!\,k!}\left(\frac{1-x}{2}\right)^k t^{n+k} \\
&= \sum_{k=0}^{\infty} \frac{(2k)!}{k!\,k!}\left(\frac{x-1}{2}\right)^k t^k \sum_{n=0}^{\infty}\frac{(2k+1)_n}{n!}\,t^n \\
&= \sum_{k=0}^{\infty} \frac{(2k)!}{k!\,k!}\left(\frac{x-1}{2}\right)^k t^k (1-t)^{-2k-1} \\
&= \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!}\,[2(x-1)t]^k\,(1-t)^{-2k-1} = (1-t)^{-1}\left[1-\frac{2(x-1)t}{(1-t)^2}\right]^{-1/2} \\
&= \left[(1-t)^2-2(x-1)t\right]^{-1/2} = (1-2xt+t^2)^{-1/2} = \frac{1}{\sqrt{1-2xt+t^2}}.
\end{aligned}$$

This proves the generating function

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n. \tag{28}$$

If we define $F(x,t) = (1-2xt+t^2)^{-1/2}$, then we have

$$\frac{\partial}{\partial t}F(x,t) = -\frac{1}{2}(1-2xt+t^2)^{-3/2}\,(-2x+2t) = \frac{x-t}{(1-2xt+t^2)^{3/2}}.$$

This implies that

$$(1-2xt+t^2)\frac{\partial}{\partial t}F(x,t) = (x-t)F(x,t).$$

Now we use (28) to obtain

$$(1-2xt+t^2)\sum_{n=1}^{\infty} nP_n(x)t^{n-1} = (x-t)\sum_{n=0}^{\infty} P_n(x)t^n.$$

This can also be written as

$$\sum_{n=1}^{\infty} nP_n(x)t^{n-1} - 2x\sum_{n=1}^{\infty} nP_n(x)t^n + \sum_{n=1}^{\infty} nP_n(x)t^{n+1} = x\sum_{n=0}^{\infty} P_n(x)t^n - \sum_{n=0}^{\infty} P_n(x)t^{n+1}$$

or equivalently

$$\sum_{n=0}^{\infty}(n+1)P_{n+1}(x)t^n - x\sum_{n=0}^{\infty}(2n+1)P_n(x)t^n + \sum_{n=0}^{\infty}(n+1)P_n(x)t^{n+1} = 0.$$

This leads to $P_1(x) = xP_0(x)$ and the three term recurrence relation

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \quad n = 1,2,3,\ldots. \tag{29}$$

Chebyshev

For $x \in [-1,1]$ the Chebyshev polynomials $T_n(x)$ of the first kind and the Chebyshev polynomials $U_n(x)$ of the second kind can be defined by

$$T_n(x) = \cos(n\theta) \quad \text{and} \quad U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta, \quad n = 0, 1, 2, \ldots. \tag{30}$$

The orthogonality property is given by

$$\int_{-1}^{1} (1-x^2)^{-1/2} T_m(x) T_n(x)\, dx = \int_0^{\pi} \cos(m\theta)\, \cos(n\theta)\, d\theta = 0, \quad m \neq n$$

and

$$\int_{-1}^{1} (1-x^2)^{1/2} U_m(x) U_n(x)\, dx = \int_0^{\pi} \sin(m+1)\theta\, \sin(n+1)\theta\, d\theta = 0, \quad m \neq n.$$

Both families of orthogonal polynomials satisfy the three term recurrence relation

$$P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x), \quad n = 1, 2, 3, \ldots,$$

since we have

$$T_{n+1}(x) + T_{n-1}(x) = \cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos(n\theta) = 2xT_n(x)$$

and

$$U_{n+1}(x) + U_{n-1}(x) = \frac{\sin(n+2)\theta + \sin(n\theta)}{\sin\theta} = \frac{2\,\cos\theta\,\sin(n+1)\theta}{\sin\theta} = 2xU_n(x).$$

Note that

$$T_0(x) = U_0(x) = 1, \quad T_1(x) = x \quad \text{and} \quad U_1(x) = 2x.$$

We also have

$$T_n(x) = U_n(x) - xU_{n-1}(x), \quad n = 1, 2, 3, \ldots,$$

since

$$U_n(x) - xU_{n-1}(x) = \frac{\sin(n+1)\theta - \cos\theta\,\sin(n\theta)}{\sin\theta} = \frac{\sin\theta\,\cos(n\theta)}{\sin\theta} = \cos(n\theta) = T_n(x).$$

In order to find a generating function for the Chebyshev polynomials $T_n(x)$ of the first kind, we multiply the recurrence relation by t^{n+1} and take the sum to obtain

$$\sum_{n=1}^{\infty} T_{n+1}(x)t^{n+1} = 2x\sum_{n=1}^{\infty} T_n(x)t^{n+1} - \sum_{n=1}^{\infty} T_{n-1}(x)t^{n+1}.$$

If we define

$$F(x,t) = \sum_{n=0}^{\infty} T_n(x)t^n, \quad |t| < 1,$$

then we have

$$F(x,t) - T_1(x)t - T_0(x) = 2xt\,[F(x,t) - T_0(x)] - t^2F(x,t).$$

This implies that

$$(1-2xt+t^2)F(x,t) = T_0(x) + T_1(x)t - 2xtT_0(x) = 1 + xt - 2xt = 1 - xt.$$

Hence we have the generating function

$$\sum_{n=0}^{\infty} T_n(x)t^n = F(x,t) = \frac{1-xt}{1-2xt+t^2}, \quad |t| < 1.$$

In the same way we have for the Chebyshev polynomials $U_n(x)$ of the second kind:

$$G(x,t) = \sum_{n=0}^{\infty} U_n(x)t^n, \quad |t| < 1$$

where

$$(1-2xt+t^2)G(x,t) = U_0(x) + U_1(x)t - 2xtU_0(x) = 1 + 2xt - 2xt = 1.$$

Hence we have

$$\sum_{n=0}^{\infty} U_n(x)t^n = G(x,t) = \frac{1}{1-2xt+t^2}, \quad |t| < 1.$$

This can be used, for instance, to prove that

$$\sum_{k=0}^{n} T_k(x)x^{n-k} = U_n(x), \quad n = 0, 1, 2, \ldots.$$

In fact, we have for $|t| < 1$

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} T_k(x)x^{n-k}\right)t^n &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} T_k(x)x^{n-k}t^n = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} T_k(x)x^n t^{n+k} \\
&= \sum_{k=0}^{\infty} T_k(x)t^k \cdot \sum_{n=0}^{\infty}(xt)^n = \frac{1-xt}{1-2xt+t^2}\cdot\frac{1}{1-xt} \\
&= \frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n.
\end{aligned}$$

In a similar way we can prove that

$$\sum_{k=0}^{n} P_k(x)P_{n-k}(x) = U_n(x), \quad n = 0, 1, 2, \ldots,$$

where $P_n(x)$ denotes the Legendre polynomial. In fact, we have for $|t| < 1$

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} P_k(x)P_{n-k}(x)\right)t^n &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} P_k(x)P_{n-k}(x)t^n = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} P_k(x)P_n(x)t^{n+k} \\
&= \sum_{k=0}^{\infty} P_k(x)t^k \cdot \sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\sqrt{1-2xt+t^2}}\cdot\frac{1}{\sqrt{1-2xt+t^2}} \\
&= \frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n.
\end{aligned}$$