Asymptotic expansions and Watson's lemma

Let z be a complex variable with $\alpha \leq \arg(z) \leq \beta$ and let

$$\sum_{n=0}^{\infty} \frac{a_n}{z^n} = \sum_{n=0}^{\infty} a_n z^{-n} \tag{1}$$

be a formal power series that may be convergent or divergent.

Definition 1. *The series (1) is called an asymptotic expansion, or an asymptotic power series, of a function f for $|z| \to \infty$ and $\alpha \leq \arg(z) \leq \beta$ if for each $n \in \{1, 2, 3, \ldots\}$*

$$f(z) = \sum_{k=0}^{n-1} a_k z^{-k} + R_n(z),$$

where

$$R_n(z) = \mathcal{O}\left(z^{-n}\right) \quad \textit{for} \quad |z| \to \infty \quad \textit{and} \quad \alpha \leq \arg(z) \leq \beta.$$

Notation:

$$f(z) \sim a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots \quad \text{for} \quad |z| \to \infty \quad \text{and} \quad \alpha \leq \arg(z) \leq \beta.$$

Now we have:

Theorem 1. *A function f has an asymptotic expansion of the form (1) for $|z| \to \infty$ and $\alpha \leq \arg(z) \leq \beta$ if and only if for each $n \in \{1, 2, 3, \ldots\}$*

$$z^n \left[f(z) - \sum_{k=0}^{n-1} a_k z^{-k} \right] \to a_n \quad \textit{for} \quad |z| \to \infty \quad \textit{and} \quad \alpha \leq \arg(z) \leq \beta.$$

A consequence of this theorem is that a function f has at most one asymptotic expansion of the form (1) for $\alpha \leq \arg(z) \leq \beta$. In a different unbounded region $\alpha' \leq \arg(z) \leq \beta'$ the asymptotic expansion may be different. On the other hand, two different functions may have the same asymptotic expansion in some region. For instance, if for some $\delta > 0$

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^{-n} \quad \text{for} \quad |z| \to \infty \quad \text{and} \quad |\arg(z)| \leq \frac{\pi}{2} - \delta < \frac{\pi}{2},$$

then $f(z) + e^{-z}$ has the same asymptotic expansion.

Some examples:

1. The exponential integral E_1 is defined by

$$E_1(x) = \int_x^{\infty} \frac{e^{-t}}{t} \, dt = \int_x^{\infty} e^{-t} t^{-1} \, dt, \quad x > 0.$$

Integration by parts leads to

$$\begin{aligned}
E_1(x) &= -e^{-t}t^{-1}\Big|_x^\infty - \int_x^\infty e^{-t}t^{-2}\,dt = -e^{-t}t^{-1}\Big|_x^\infty + e^{-t}t^{-2}\Big|_x^\infty + 2\int_x^\infty e^{-t}t^{-3}\,dt \\
&= e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \ldots + (-1)^{n-1}\frac{(n-1)!}{x^n}\right] + (-1)^n n! \int_x^\infty e^{-t}t^{-n-1}\,dt.
\end{aligned}$$

Note that for fixed n

$$\left|(-1)^n n! \int_x^\infty e^{-t}t^{-n-1}\,dt\right| = n! \int_x^\infty e^{-t}t^{-n-1}\,dt < \frac{n!}{x^{n+1}} \int_x^\infty e^{-t}\,dt = \frac{n!\,e^{-x}}{x^{n+1}}.$$

This tends to zero for $x \to \infty$. In fact, this is $\mathcal{O}(x^{-n-1})$ for $x \to \infty$. Hence we have

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \sim e^{-x}\sum_{n=0}^\infty (-1)^n \frac{n!}{x^{n+1}} \quad \text{for} \quad x \to \infty.$$

Note that this asymptotic series diverges for all $x > 0$. However, if a fixed number of terms is taken, then for x large enough a good approximation of $E_1(x)$ is obtained.

2. For $\operatorname{Re} a > 0$ the incomplete gamma function is defined by

$$\Gamma(a,x) = \int_x^\infty e^{-t}t^{a-1}\,dt = \Gamma(a) - \int_0^x e^{-t}t^{a-1}\,dt = \Gamma(a) - \gamma(a,x), \quad x > 0.$$

Similarly as in the previous example, integration by parts leads to

$$\Gamma(a,x) \sim e^{-x}x^a \sum_{n=0}^\infty \frac{(a-1)(a-2)\cdots(a-n+1)}{x^{n+1}} \quad \text{for} \quad x \to \infty.$$

3. The complementary error function is defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt, \quad x > 0.$$

Since we have

$$\int_x^\infty e^{-t^2}\,dt = -\int_x^\infty \frac{1}{2t}\,de^{-t^2},$$

successive integration by parts gives for $x > 0$

$$\begin{aligned}
\operatorname{erfc}(x) &= \frac{2}{\sqrt{\pi}}\left[\frac{e^{-x^2}}{2x} - \int_x^\infty \frac{e^{-t^2}}{2t^2}\,dt\right] = \frac{2}{\sqrt{\pi}}\left[\frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{4x^3} + \int_x^\infty \frac{e^{-t^2}}{4t^3}\,dt\right] \\
&= \frac{2}{\sqrt{\pi}}\frac{e^{-x^2}}{2x}\left[\sum_{k=0}^n (-1)^k \frac{1\cdot 3\cdots(2k-1)}{(2x^2)^k} + R_n(x)\right],
\end{aligned}$$

where

$$R_n(x) = (-1)^{n+1}\frac{1\cdot 3\cdots(2n+1)}{2^{n+1}}\,2xe^{x^2}\int_x^\infty \frac{e^{-t^2}}{t^{2n+2}}\,dt.$$

Then we have

$$|R_n(x)| \le \frac{1\cdot 3\cdots(2n+1)}{(2x^2)^{n+1}}.$$

Hence we have

$$\operatorname{erfc}(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdots (2n-1)}{(2x^2)^n} \quad \text{for} \quad x \to \infty.$$

Now we will prove **Watson's lemma**:

Theorem 2. *Let f be a complex valued function of a real variable t such that*

1. *f is continuous on $(0, \infty)$,*
2. $$f(t) \sim \sum_{n=0}^{\infty} a_n t^{\lambda_n - 1} \quad \textit{for} \quad t \downarrow 0 \tag{2}$$

   with

   $$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

   and
3. *for some fixed $c > 0$*
   $$f(t) = \mathcal{O}\left(e^{ct}\right) \quad \textit{for} \quad t \to \infty. \tag{3}$$

Then we have

$$F(z) = \int_0^{\infty} e^{-zt} f(t)\, dt \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\lambda_n)}{z^{\lambda_n}} \quad \textit{for} \quad |z| \to \infty \quad \textit{and} \quad |\arg(z)| \leq \frac{\pi}{2} - \delta < \frac{\pi}{2}$$

for some δ such that $0 < \delta < \pi/2$.

Proof. From the theory of Laplace transforms it is well known that

$$F(z) = \int_0^{\infty} e^{-zt} f(t)\, dt,$$

the Laplace transform of f, exists for $\operatorname{Re} z > c$ if f satisfies the three conditions mentioned in the theorem. That is: the integral converges for $\operatorname{Re} z > c$.

Note that (2) implies that

$$\left| f(t) - \sum_{n=0}^{N-1} a_n t^{\lambda_n - 1} \right| \leq M\, t^{\lambda_N - 1} \quad \text{for} \quad t \downarrow 0,$$

where $M > 0$ is some constant. Together with (3) this implies that

$$\left| f(t) - \sum_{n=0}^{N-1} a_n t^{\lambda_n - 1} \right| \leq K\, e^{ct} t^{\lambda_N - 1} \quad \text{for} \quad t > 0,$$

where $K > 0$ is some constant. Hence we have

$$\left| \int_0^{\infty} e^{-zt} f(t)\, dt - \sum_{n=0}^{N-1} a_n \int_0^{\infty} e^{-zt} t^{\lambda_n - 1}\, dt \right| \leq K \int_0^{\infty} e^{-(\operatorname{Re} z - c)t} t^{\lambda_N - 1}\, dt.$$

Note that we have for $\operatorname{Re} z > 0$

$$\int_0^\infty e^{-zt} t^{\lambda_n - 1}\, dt = \frac{1}{z^{\lambda_n}} \int_0^\infty e^{-\tau} \tau^{\lambda_n - 1}\, d\tau = \frac{\Gamma(\lambda_n)}{z^{\lambda_n}}.$$

Hence we have

$$\left| F(z) - \sum_{n=0}^{N-1} a_n \frac{\Gamma(\lambda_n)}{z^{\lambda_n}} \right| \leq K \frac{\Gamma(\lambda_N)}{(\operatorname{Re} z - c)^{\lambda_N}} = K \frac{\Gamma(\lambda_N)}{|z|^{\lambda_N}} \left(\frac{|z|}{\operatorname{Re} z - c} \right)^{\lambda_N}.$$

Since $|\arg(z)| \leq \frac{\pi}{2} - \delta < \frac{\pi}{2}$, we have $\operatorname{Re} z \geq |z| \sin \delta$ which implies that $\operatorname{Re} z - c \geq \frac{1}{2}|z| \sin \delta$ for $|z|$ large enough. This implies that we have

$$F(z) - \sum_{n=0}^{N-1} a_n \frac{\Gamma(\lambda_n)}{z^{\lambda_n}} = \mathcal{O}\left(z^{-\lambda_N} \right),$$

which proves Watson's lemma.

Some examples of the application of Watson's lemma:

1. Consider the function $f(t) = 1/(1+t)$. Then we have: f is continuous on $(0, \infty)$ and

   $$f(t) = \frac{1}{1+t} = \sum_{n=0}^\infty (-1)^n t^n, \quad |t| < 1.$$

   Now Watson's lemma implies that

   $$F(z) = \int_0^\infty e^{-zt} f(t)\, dt = \int_0^\infty \frac{e^{-zt}}{1+t}\, dt \sim \sum_{n=0}^\infty (-1)^n \frac{\Gamma(n+1)}{z^{n+1}} = \sum_{n=0}^\infty (-1)^n \frac{n!}{z^{n+1}}$$

   for $|z| \to \infty$ and $|\arg(z)| \leq \frac{\pi}{2} - \delta < \frac{\pi}{2}$.

2. Consider the function $f(t) = 1/\sqrt{1+t^2}$. Then we have: f is continuous on $(0, \infty)$ and

   $$f(t) = \frac{1}{\sqrt{1+t^2}} = (1+t^2)^{-\frac{1}{2}} = \sum_{n=0}^\infty (-1)^n \frac{\left(\frac{1}{2}\right)_n}{n!} t^{2n}, \quad |t| < 1.$$

   Now Watson's lemma implies that

   $$F(z) = \int_0^\infty e^{-zt} f(t)\, dt = \int_0^\infty \frac{e^{-zt}}{\sqrt{1+t^2}}\, dt \sim \sum_{n=0}^\infty (-1)^n \frac{\left(\frac{1}{2}\right)_n}{n!} \cdot \frac{(2n)!}{z^{2n+1}}$$

   for $|z| \to \infty$ and $|\arg(z)| \leq \frac{\pi}{2} - \delta < \frac{\pi}{2}$. Since we have

   $$\left(\frac{1}{2}\right)_n = \frac{1}{2} \cdot \frac{3}{2} \cdots \left(\frac{1}{2} + n - 1 \right) = \frac{1 \cdot 3 \cdots (2n-1)}{2^n} \cdot \frac{2 \cdot 4 \cdots (2n)}{2^n\, n!} = \frac{(2n)!}{2^{2n}\, n!},$$

   this can also be written as

   $$\int_0^\infty \frac{e^{-zt}}{\sqrt{1+t^2}}\, dt \sim \sum_{n=0}^\infty (-1)^n \left(\frac{(2n)!}{2^n\, n!} \right)^2 \frac{1}{z^{2n+1}}$$

   for $|z| \to \infty$ and $|\arg(z)| \leq \frac{\pi}{2} - \delta < \frac{\pi}{2}$.

3. Consider the function $f(t)=\ln(1+t^2)$. Then we have: f is continuous on $(0,\infty)$ and

$$f(t)=\ln(1+t^2)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}t^{2n+2},\quad |t|<1.$$

Now Watson's lemma implies that

$$F(z)=\int_0^{\infty}e^{-zt}f(t)\,dt=\int_0^{\infty}e^{-zt}\ln(1+t^2)\,dt\sim\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\cdot\frac{(2n+2)!}{z^{2n+3}}$$

for $|z|\to\infty$ and $|\arg(z)|\leq\frac{\pi}{2}-\delta<\frac{\pi}{2}$. Note that this can also be written as

$$\int_0^{\infty}e^{-zt}\ln(1+t^2)\,dt\sim 2\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!}{z^{2n+3}}$$

for $|z|\to\infty$ and $|\arg(z)|\leq\frac{\pi}{2}-\delta<\frac{\pi}{2}$.

4. For $\operatorname{Re}c>\operatorname{Re}a>0$ the confluent hypergeometric function can be written as

$${}_1F_1\left({a \atop c};z\right)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{z\tau}\tau^{a-1}(1-\tau)^{c-a-1}\,d\tau.$$

Now we use the substitution $\tau=1-t$ to find that

$${}_1F_1\left({a \atop c};z\right)=\frac{\Gamma(c)\,e^z}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{-zt}(1-t)^{a-1}t^{c-a-1}\,dt=\int_0^{\infty}e^{-zt}f(t)\,dt,$$

where

$$f(t)=\frac{\Gamma(c)\,e^z}{\Gamma(a)\Gamma(c-a)}(1-t)^{a-1}t^{c-a-1}\quad\text{for}\quad 0<t<1$$

and $f(t)=0$ for $t\geq 1$. Note that we have

$$f(t)=\frac{\Gamma(c)\,e^z}{\Gamma(a)\Gamma(c-a)}\sum_{n=0}^{\infty}(-1)^n\binom{a-1}{n}t^{n+c-a-1}\quad\text{for}\quad t\downarrow 0.$$

Now Watson's lemma implies that

$${}_1F_1\left({a \atop c};z\right)\sim\frac{\Gamma(c)\,e^z}{\Gamma(a)\Gamma(c-a)}\sum_{n=0}^{\infty}(-1)^n\binom{a-1}{n}\frac{\Gamma(n+c-a)}{z^{n+c-a}}$$

for $|z|\to\infty$ and $|\arg(z)|\leq\frac{\pi}{2}-\delta<\frac{\pi}{2}$. Since we have

$$\frac{\Gamma(n+c-a)}{\Gamma(c-a)}=(c-a)_n\quad\text{and}\quad(-1)^n\binom{a-1}{n}=\frac{(1-a)_n}{n!},$$

this can also be written as

$$\begin{aligned}{}_1F_1\left({a \atop c};z\right)&\sim\frac{\Gamma(c)}{\Gamma(a)}\,e^z\,z^{a-c}\sum_{n=0}^{\infty}\frac{(1-a)_n(c-a)_n}{n!}\frac{1}{z^n}\\&=\frac{\Gamma(c)}{\Gamma(a)}\,e^z\,z^{a-c}{}_2F_0\left({1-a,\;c-a \atop -};\frac{1}{z}\right)\end{aligned}$$

for $|z|\to\infty$ and $|\arg(z)|\leq\frac{\pi}{2}-\delta<\frac{\pi}{2}$.

Finally, we derive two asymptotic expansions for the Bessel function.

For $\operatorname{Re}\nu > -1/2$ the Bessel function of the first kind of order ν can be written as

$$J_\nu(z) = \frac{1}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\left(\frac{z}{2}\right)^\nu \int_{-1}^{1} e^{izt}(1-t^2)^{\nu-\frac{1}{2}}\,dt.$$

Now we use the substitution $t = 2\tau - 1$ to obtain for $\operatorname{Re}\nu > -1/2$

$$J_\nu(z) = \frac{2^{2\nu}e^{-iz}}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\left(\frac{z}{2}\right)^\nu \int_{0}^{1} e^{2iz\tau}\tau^{\nu-\frac{1}{2}}(1-\tau)^{\nu-\frac{1}{2}}\,d\tau.$$

For $\operatorname{Re}c > \operatorname{Re}a > 0$ we have

$${}_1F_1\left(\begin{matrix}a\\c\end{matrix};z\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{c-a-1}\,dt,$$

which implies that for $\operatorname{Re}\nu > -1/2$ we have

$$J_\nu(z) = \frac{2^{2\nu}e^{-iz}}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\left(\frac{z}{2}\right)^\nu \frac{\Gamma(\nu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{\Gamma(2\nu+1)}\,{}_1F_1\left(\begin{matrix}\nu+\frac{1}{2}\\2\nu+1\end{matrix};2iz\right).$$

Now we use the asymptotic expansion for the confluent hypergeometric function to obtain

$$J_\nu(z) \sim \frac{2^{2\nu}e^{-iz}}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\left(\frac{z}{2}\right)^\nu e^{2iz}\sum_{n=0}^{\infty}(-1)^n\binom{\nu-\frac{1}{2}}{n}\frac{\Gamma(n+\nu+\frac{1}{2})}{(2iz)^{n+\nu+\frac{1}{2}}}$$

for $|z| \to \infty$ and $|\arg(2iz)| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$. Now we use

$$\frac{1}{\Gamma(\nu+\frac{1}{2})}\binom{\nu-\frac{1}{2}}{n} = \frac{1}{\Gamma(\nu+\frac{1}{2})}\cdot\frac{\Gamma(\nu+\frac{1}{2})}{n!\,\Gamma(\nu-n+\frac{1}{2})} = \frac{1}{\Gamma(\nu-n+\frac{1}{2})\,n!}$$

and the fact that $i = e^{\pi i/2}$ to obtain

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi z}}\,e^{i\left(z-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)}\sum_{n=0}^{\infty}\frac{\Gamma(\nu+n+\frac{1}{2})}{\Gamma(\nu-n+\frac{1}{2})\,n!}\left(\frac{i}{2z}\right)^n$$

for $|z| \to \infty$ and $|\arg(iz)| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$ or equivalently $-\pi < -\pi + \delta \le \arg(z) \le -\delta < 0$.

In the same way the substitution $t = 1 - 2\tau$ leads to

$$J_\nu(z) \sim \frac{2^{2\nu}e^{iz}}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\left(\frac{z}{2}\right)^\nu e^{-2iz}\sum_{n=0}^{\infty}(-1)^n\binom{\nu-\frac{1}{2}}{n}\frac{\Gamma(n+\nu+\frac{1}{2})}{(-2iz)^{n+\nu+\frac{1}{2}}}$$

for $|z| \to \infty$ and $|\arg(-2iz)| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$. In that case we use the fact that $-i = e^{-\pi/2}$ to obtain

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi z}}\,e^{-i\left(z-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)}\sum_{n=0}^{\infty}\frac{\Gamma(\nu+n+\frac{1}{2})}{\Gamma(\nu-n+\frac{1}{2})\,n!}\left(\frac{1}{2iz}\right)^n$$

for $|z| \to \infty$ and $|\arg(-iz)| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$ or equivalently $0 < \delta \le \arg(z) \le \pi - \delta < \pi$.