NUMERICAL LINEAR ALGEBRA
ACADEMIC YEAR 2008-2009

Theoretical assignment 4

1. The Power method can be used to approximate the largest eigenvalue λ_1. In this exercise two methods are given to estimate the eigenvalue λ_2 if λ_1 and eigenvector x_1 are known.
   (a) Take $q_0 = (A - \lambda_1 I)q$, where q is an arbitrary vector. Show that the Power method applied to this starting vector leads to an approximation of λ_2 (Annihilation Technique).
   (b) Show that if the Power method is applied to the matrix
   $$B = A - \frac{\lambda_1}{x_1^T x_1} x_1 x_1^T$$
   one gets an approximation of λ_2. What is the amount of work per iteration using B (Hotelling Deflation)?

2. Suppose that A is symmetric and positive definite.
   (a) Show that one can write $A = D_A - L_A - L_A^T$ where D_A is diagonal with $d_{ii} > 0$ for each $1 \leq i \leq n$ and L_A is strictly lower triangular. Further show that $D_A - L_A$ is nonsingular.
   (b) Let $T_g = (D_A - L_A)^{-1} L_A^T$ and $P = A - T_g^T A T_g$. Show that P is symmetric.
   (c) Show that T_g can also be written as $T_g = I - (D_A - L_A)^{-1} A$.
   (d) Let $Q = (D_A - L_A)^{-1} A$. Show that $T_g = I - Q$ and
   $$P = Q^T (AQ^{-1} - A + (Q^T)^{-1} A) Q.$$
   (e) Show that $P = Q^T D_A Q$ and P is symmetric and positive definite.
   (f) Let λ be an eigenvalue of T_g with eigenvector x. Use part (b) to show that $x^T P x > 0$ implies that $|\lambda| < 1$.
   (g) Show that the Gauss Seidel method converges.

3. Consider the QR algorithm. Show by induction that without shifts, $(Q_0 Q_1 \cdots Q_k)(R_k \cdots R_1 R_0)$ is exactly the QR factorisation of A^{k+1}. This identity connects QR to the power method and leads to an explanation of its convergence; if $|\lambda_1| > |\lambda_2| > \cdots > \lambda_n|$, then these eigenvalues will gradually appear in descending order on the main diagonal of A_k.

4. Consider the basic iterative method

$$Mx_{k+1} = Rx_k + b.$$

The calculation of the residual is an expensive operation. We would therefore like to use a cheaper termination criterion, preferably one on basis of the true error instead of on the residual. In the next assignment we derive such a criterion.

(a) Show that the spectral radius of $G = M^{-1}R$ approximately satisfies

$$\rho(G) \approx \frac{\|x_{k+1} - x_k\|}{\|x_k - x_{k-1}\|} \ .$$

(b) Show that if $\rho(M^{-1}R)$ is known, an estimate for the error is given by $\|x - x_k\|_2 \leq \frac{\rho(G)}{1-\rho(G)}\|x_k - x_{k-1}\|_2$.
Hint: first bound $\|x_j - x_k\|_2$ in terms of $\rho(G)$ and $\|x_k - x_{k-1}\|_2$. Then take the limit $x = \lim_{j\to\infty} x_j$.