Assignment PhD-course on iterative methods for linear systems of equations

The assignment is designed to give a work load of 30-35 hours. The report has to be made individually. It should be written as a paper, meaning that it can be read without consulting the assignment. It should be handed in (preferably in electronic form) to Per Christian Hansen (pch@imm.dtu.dk) no later than December 1, 2008.

In the assignment you will investigate the iterative solution of the linear system $Ax = b$, in which A is a block matrix of the following form

$$A = \begin{pmatrix} F & B^T \\ B & -C \end{pmatrix} . \tag{1}$$

Here, F is symmetric and positive definite, and C is symmetric positive semi-definite. C may be zero. Such block matrices arise in many applications. Systems with a block matrix (1) are known as KKT systems or saddle-point systems. In this assignment both F and C are diagonal matrices.

Saddle-point systems are notoriously difficult to solve by iterative methods. One approach is to precondition them using a preconditioner based on the block LU factorization

$$\begin{pmatrix} F & B^T \\ B & -C \end{pmatrix} = \begin{pmatrix} I & O^T \\ BF^{-1} & I \end{pmatrix} \begin{pmatrix} F & B^T \\ O & -M_S \end{pmatrix} .$$

Here $M_S = BF^{-1}B^T + C$ is the Schur complement. Taking as right preconditioner the matrix

$$P = \begin{pmatrix} F & B^T \\ O & -M_S \end{pmatrix} \tag{2}$$

yields for the right-preconditioned matrix

$$AP^{-1} = \begin{pmatrix} I & O^T \\ BF^{-1} & I \end{pmatrix} .$$

Assignment 1. Let v be any nonzero vector.

- Compute $AP^{-1}v$ and $(AP^{-1})^2v$. Show that v can be expressed as a linear combination of $AP^{-1}v$ and $(AP^{-1})^2v$.
- Explain why this implies that GMRES applied to a right-preconditioned system saddle point system with (2) as preconditioner must find the exact solution in at most two iterations.

The preconditioner that we have defined above is *nonsymmetric* while the system matrix (1) is symmetric. This may seem somewhat unnatural. A *symmetric* preconditioner for saddle point systems is the block diagonal matrix

$$P = \begin{pmatrix} F & O^T \\ O & M_S \end{pmatrix} \tag{3}$$

Assignment 2.

- Assume that $C = 0$. Show that in this case the preconditioned matrix AP^{-1} has three distinct eigenvalues: 1, $\frac{1}{2} + \frac{1}{2}\sqrt{5}$, and $\frac{1}{2} - \frac{1}{2}\sqrt{5}$.
  Hint: solve the generalised eigenvalue problem

  $$\begin{pmatrix} F & B^T \\ B & O \end{pmatrix} \begin{pmatrix} x_t \\ x_b \end{pmatrix} = \lambda \begin{pmatrix} F & O^T \\ O & M_S \end{pmatrix} \begin{pmatrix} x_t \\ x_b \end{pmatrix}$$

  by first eliminating x_t.
- Assume again that $C = 0$. Explain why GMRES applied to a saddle-point system, preconditioned with (3) must find the exact solution in at most three iterations.

In the next assignments you have to perform several numerical experiments on saddle-point systems from the test-set `Schenk_IBMNA`, which is part of the Tim Davis' matrix collection at the University of Florida (see course web page). In all the assignments you are allowed to use build in MATLAB routines. Use of other routines (e.g. from the 'templates' library, or a routine that you have written yourself) is also allowed, as long as you indicate which routines you have used. Use 10^{-6} as the tolerance and 1000 for the maximum number of iterations in all experiments.

Assignment 3. Download the matrix `c-18.mtx` and right-hand side `c-18_b.mtx` from the Tim Davis' collection. Check that the matrix `c-18.mtx` has the above block structure. What is the size of the matrix? Explain why MINRES is a natural choice for solving the system. Try to solve this system with MINRES without preconditioning. Conclusion?

Now we are going to combine MINRES with preconditioning.

Assignment 4. Solve the problem with preconditioned MINRES, taking (3) as the preconditioner. Discuss the convergence. (Warning: MATLAB uses left preconditioning in MINRES, but gives the residual norms for the *unpreconditioned* system. Therefore, residual norms may go up in your convergence curve.)

For large systems it is not possible to compute the Schur-complement matrix. A simple idea is to approximate the Schur complement by its main diagonal. The resulting preconditioning matrix then becomes

$$P = \begin{pmatrix} F & O^T \\ O & diag(M_S) \end{pmatrix} \tag{4}$$

which is a diagonal matrix.

Assignment 5. Solve the problem with preconditioned MINRES, taking (4) as the preconditioner. Also solve the system with the following methods (preconditioned with (4)): Bi-CGSTAB, IDR(1) and IDR(4). Plot the convergence curves of the four methods in one figure. Put on the x-axis the number of matrix-vector multiplications and on the y-axis the residual norm divided by the norm of the right-hand side vector. Which method is fastest? Is this what you would expect?

Following the same reasoning as above we can also use

$$P = \begin{pmatrix} F & B^T \\ O & -diag(M_S) \end{pmatrix} \tag{5}$$

as preconditioner. This matrix is nonsymmetric, but a Krylov solver for nonsymmetric systems combined with this preconditioner might be more efficient than MINRES combined with the symmetric preconditioner. We will investigate this in the next assignment.

Assignment 6. Solve the problem preconditioned with (5) with the following methods: Bi-CGSTAB, IDR(1), IDR(4), Bi-CG and QMR. Plot the convergence curves of the five methods in one figure. Put on the x-axis the number of matrix-vector multiplications and on the y-axis the residual norm divided by the norm of the right-hand side vector. Note that Bi-CG and QMR need two matrix-vector multiplications per iteration. Discuss the convergence. Which method is fastest?

Assignment 7. Select at least two other test problems from the same test set. Repeat assignments 5 and 6. Draw and discuss conclusions.