PhD-course on Iterative Methods for Linear Systems of Equations

Theoretical assignments day 2

1. - Show that CG cannot break down if A is SPD, unless $r_k = 0$ (which is called happy break).
   - In which step can CG break down if A is not SPD? Why?

2. Let A be the SPD system matrix and M a SPD preconditioner.
   - Show that $M^{-1}A$ is self-adjoint in the M-inner product.
   - The CG method can be applied to the preconditioned system $M^{-1}Ax = M^{-1}b$ by working with the M-inner product. Let $z_k = M^{-1}r_k$. Show that this leads to the following algorithm (igonoring the initialisation step):

$$
\begin{aligned}
\alpha_k &= \tfrac{(z_k, z_k)_M}{(M^{-1}Ap_k, p_k)_M} \\
x_{k+1} &= x_k + \alpha_k p_k \\
r_{k+1} &= r_k - \alpha_k A p_k \\
z_{k+1} &= M^{-1} r_{k+1} \\
\beta_k &= \tfrac{(z_{k+1}, z_{k+1})_M}{(z_k, z_k)_M} \\
p_{k+1} &= z_{k+1} + \beta_k p_k
\end{aligned}
$$

   - Show that $(z_k, z_k)_M = (r_k, z_k)_2$ and that $(M^{-1}Ap_k, p_k)_M = (Ap_k, p_k)_2$.
   - Show that the above algorithm is equivalent the following algorithm (which is expressed in the standard inner product):

$$
\begin{aligned}
\alpha_k &= \tfrac{(r_k, z_k)_2}{(Ap_k, p_k)_2} \\
x_{k+1} &= x_k + \alpha_k p_k \\
r_{k+1} &= r_k - \alpha_k A p_k \\
z_{k+1} &= M^{-1} r_{k+1} \\
\beta_k &= \tfrac{(r_{k+1}, z_{k+1})_2}{(r_k, z_k)_2} \\
p_{k+1} &= z_{k+1} + \beta_k p_k
\end{aligned}
$$

3. - Show that the Lanczos vectors satisfy $Aq_k = \beta_k q_{k-1} + \alpha_k q_k + \beta_{k+1} q_{k+1}$.
   - Derive the Lanczos algorithm by making use of the fact that the Lanczos vectors form an orthonormal basis of the Krylov subspace.

4.
   - Prove that CG applied to the Normal Equations minimizes the residual $r_k = b - Ax_k$.
   - Write down the CGLS algorithm. The algorithm should:
     - Contain a recursion for $r_k = b - Ax_k$, and for $s_k = A^T b - A^T Ax_k$.
     - Avoid explicit computations with $A^T A$.

5. Assuming that $\kappa_2(A) = 100$:
   - Give an upper bound on the number of CG iterations to satisfy $\frac{\|x-x_k\|_A}{\|x-x_0\|_A} < 10^{-6}$.
   - Same question for CGLS.