Energy and momentum of light in dielectric media

Sjoerd Stallinga*
Philips Research Laboratories, Professor Holstlaan 4, 5611 AA Eindhoven, The Netherlands
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The conservation of energy, linear momentum, and angular momentum of the electromagnetic field in linear dielectric media with arbitrary dispersion and absorption is studied in the framework of an auxiliary field approach in which the electric and magnetic fields are complemented by a material field. This material field depends on a continuous variable ω, and describes harmonic motions of the charges with eigenfrequency ω. It carries an electric dipole moment and couples as such to the electric field. The equations of motion of the model are equivalent to Maxwell's equations in an arbitrary dispersive and absorbing dielectric and imply that several quantities are conserved. These quantities may be interpreted as the energy, momentum, and angular momentum of the total system, and can be viewed as the sum of the corresponding quantities of the field and matter subsystems. The total momentum turns out to be equal to the Minkowski momentum plus a dispersive contribution. The total energy and total momentum of a wave packet both travel with the group velocity, while the ratio of total momentum and total energy is given by the phase velocity.

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I. INTRODUCTION

The linear momentum of light in dielectric media is a complicated concept, as evidenced by the variety of views on the subject that can be found in the literature. Most of the work focuses on the Abraham and Minkowski forms for the electromagnetic momentum (see [1] for a review). Different approaches to the problem can be found in the papers by Gordon [2], Nelson [3], Garrison and Chiao [4], Loudon and co-workers [5,6], Obukhov and Hehl [7], and Mansuripur [8]. This list of references is far from comprehensive, but gives a fair view of the different approaches.

Several aspects of the momentum concept are very subtle and do not lend themselves to easy understanding. In this paper, two of these aspects are studied in some detail. The first is the role of dispersion and dissipation. The dynamics of dissipative systems cannot be described by the theoretical tools that are frequently used for conservative systems, in particular the canonical framework based on the use of Lagrangians and Hamiltonians. For that reason it is not clear how to define momentum, a conserved quantity, for dissipative systems. The second aspect concerns the difference between uniformity of space and homogeneity of matter. The invariance for translations of the total system gives rise to conservation of momentum, the invariance for material displacements of the dielectric gives rise to conservation of pseudomomentum. Depending on the experimental circumstances one or the other, or even a combination of both types of momenta seems useful. The difficulty in describing dissipative systems can be overcome, at least in some cases, by making the system larger. Additional degrees of freedom that interact with the dissipative system can be introduced so that the total system is conservative. It is the goal of this paper to find such an enlarged system description, investigate the attendant conservation laws for the enlarged system, and interpret the physical meaning of these conserved quantities.

The starting point of this paper is an auxiliary field model for the description of electromagnetic fields in linear dielectric media with arbitrary dispersion and absorption introduced by Tip [9,10]. A similar model has later been proposed by Figotin and Schenker [11]. The basic variables of the theory are the electric field **E** and magnetic induction **B** and an auxiliary field **F** representing the material degrees of freedom interacting with the electromagnetic field. The material field **F** effectively describes the harmonic motions of the charges inside the dielectric. A difference between the electromagnetic fields **E** and **B** and the material field **F** is that the former depend on position **r** and time t only, whereas the latter depends on a third continuous variable ω as well. This third variable can be interpreted as the (angular) eigenfrequency of the harmonic material motions. The electromagnetic and material fields interact through a dipole coupling. The coupling is proportional to a function $\hat{\sigma}(\omega)$ which turns out to be (the Fourier transform of) the conductivity, which for a dielectric may be defined as $\hat{\varepsilon}_i(\omega)\omega$, where $\hat{\varepsilon}_i(\omega)$ is the imaginary part of (the Fourier transform of) the dielectric function. The strength of the model is that the equations of motion are formally equivalent to the set of equations consisting of Maxwell's equations and the constitutive relation between the dielectric displacement **D** and **E** for an arbitrary dispersive and absorbing medium. The equations of motion can be derived from the standard variational principle based upon the action being the integral over space and time of the Lagrangian density.

The canonical framework defined by the Lagrangian density implies the existence of several conserved quantities, which may be interpreted as the energy, momentum, and angular momentum of the total system, the total system consisting of the electromagnetic field and the material system. The conservation laws will be derived from the equations of motion of the model. Alternative proofs based on Noether's theorem are possible but will not be presented. For each conserved quantity a density ρ and a flow **v** may be defined

*Electronic address: sjoerd.stallinga@philips.com

satisfying a transport equation of the form $\partial_t\rho+\nabla\cdot\mathbf{v}=0$ (with obvious generalization to conserved quantities with a vectorial character). Balance equations for two subsystems, for example the "field" and "matter" subsystems, have the form

$$\partial_t\rho_1+\nabla\cdot\mathbf{v}_1=-Q, \tag{1}$$

$$\partial_t\rho_2+\nabla\cdot\mathbf{v}_2=Q, \tag{2}$$

where $\rho=\rho_1+\rho_2$ and $\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2$, and where Q represents the dissipation of field energy, momentum or angular momentum from subsystem 1 to 2. It turns out that the exchange of energy, momentum, and angular momentum between field and material parts is such that the dissipation integrated over the duration of the interaction is always positive. This irreversibility is related to the coupling of the electromagnetic degrees of freedom to a continuum of harmonic oscillators, rather than to a finite number of degrees of freedom.

The split of the conservation laws into balance equations for the field and material subsystems is to some extent arbitrary, and various definitions will do. As a consequence the dissipation of energy, momentum, and angular momentum of the field to matter are also ambiguous. A key point of interpretation is thus how to relate these quantities to the absorbed heat, force, and torque on the medium that are actually observed in experiment. It may therefore be the case that different experimental circumstances require the application of different descriptions of momenta and forces. The answer to the Abraham-Minkowksi debate in this view is not a definition of "the" momentum of light in dielectric media but rather a prescription of when to use which type of momentum. An attempt is made in this paper to find out for which physical situation the total field-plus-matter momentum of the auxiliary field model is a useful quantity.

The main shortcoming of the auxiliary field model is that it does not take into account deformation or displacement of the material medium. It is assumed that the position of each material point is kept fixed throughout the interaction with the electromagnetic field. This implies that the distinction between the space-fixed coordinate frame and the coordinate frame fixed to the material points is lost so that a clear identification as to which quantity is momentum and which quantity is pseudomomentum cannot be made. This indistinguishability of uniformity of space and homogeneity of matter has the consequence that only one meaningful momentumlike-conserved quantity exists within the model. This total system momentum corresponds to what is called pseudomomentum by Gordon [2], wave momentum (the sum of momentum and pseudomomentum) by Nelson [3] and canonical momentum by Garrison and Chiao [4].

The paper is organized as follows. In Sec. II the equations of motion are derived and shown to be equivalent to Maxwell's equations in general linear dielectrics. The conservation laws are treated in Sec. III, and Sec. IV focuses on the energy and momentum of a one-dimensional wave packet. The paper is concluded in Sec. V with a discussion of the obtained results and an outlook on possibilities for future explorations.

Concerning the notation, it is mentioned that in the following the dependence of $\mathbf{E}$ and $\mathbf{B}$ on position $\mathbf{r}$ and time t and the dependence of $\mathbf{F}$ on (angular) eigenfrequency ω, position $\mathbf{r}$, and time t is suppressed, except when this compact notation can give rise to ambiguity. Vector notation is used if convenient and the tensor notation in all other cases. The partial derivative with respect to time is denoted by ∂_t, the partial derivative with respect to the spatial coordinates by ∂_α, where $\alpha=x,y,z$, and the Einstein summation convention is used. Partial derivatives only apply to the quantity directly following the derivative unless brackets indicate otherwise. The tensor $\delta_{\alpha\beta}$ is the Kronecker tensor ($\delta_{\alpha\beta}=1$ if $\alpha=\beta$ and 0 otherwise), and the tensor $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita tensor ($\epsilon_{\alpha\beta\gamma}=1$ for $\alpha\beta\gamma$ even permutations of xyz, -1 for odd permutations, and 0 otherwise).

II. EQUATIONS OF MOTION

The action is the integral over time and space of the Lagrangian density

$$I=\int dt\int d^3r\mathcal{L}, \tag{3}$$

where the Lagrangian density is the sum of an electromagnetic contribution, a contribution from the material field, and an interaction contribution

$$\mathcal{L}=\frac{\varepsilon_0}{2}\mathbf{E}^2-\frac{1}{2\mu_0}\mathbf{B}^2+\frac{\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)[(\partial_t\mathbf{F})^2-\omega^2\mathbf{F}^2+2\mathbf{F}\cdot\mathbf{E}]. \tag{4}$$

The function $\hat{\sigma}(\omega)$ is positive for all nonzero ω and defined for negative angular frequencies by $\hat{\sigma}(\omega)=\hat{\sigma}(-\omega)$. The absence of free charges and currents implies that $\hat{\sigma}(\omega)\to 0$ if $\omega\to 0$. It may be defined for complex ω by analytical continuation and is assumed to have no poles in the upper half complex plane (in view of causality). The electromagnetic part of the Lagrangian density is just the vacuum electromagnetic Lagrangian density, the material part describes a continuous set of harmonic oscillators, and the interaction term describes the interaction of the electric field with a continuous set of electric dipoles. The polarization $\mathbf{P}$ is thus entirely defined in terms of the material field $\mathbf{F}$

$$\mathbf{P}=\frac{2\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)\mathbf{F}. \tag{5}$$

The dielectric displacement $\mathbf{D}$ and magnetic field $\mathbf{H}$ are then defined by

$$\mathbf{D}=\frac{\partial\mathcal{L}}{\partial\mathbf{E}}=\varepsilon_0\mathbf{E}+\frac{2\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)\mathbf{F}, \tag{6}$$

$$\mathbf{H}=-\frac{\partial\mathcal{L}}{\partial\mathbf{B}}=\frac{1}{\mu_0}\mathbf{B}. \tag{7}$$

The scalar potential Φ and vector potential $\mathbf{A}$ are introduced via

$$\mathbf{E}=-\nabla\Phi-\partial_t\mathbf{A}, \tag{8}$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{9}$$

which solves the two homogeneous Maxwell equations

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = \mathbf{0}, \tag{10}$$

$$\nabla \cdot \mathbf{B} = \mathbf{0}. \tag{11}$$

The Euler-Lagrange equations for the potentials are the two "inhomogeneous" Maxwell equations where we use quotes because in the present context there are no free charges and currents so that these equations are in fact homogeneous as well

$$\nabla \cdot \mathbf{D} = \mathbf{0}, \tag{12}$$

$$\nabla \times \mathbf{H} - \partial_t \mathbf{D} = \mathbf{0}, \tag{13}$$

which can be demonstrated with textbook manipulations [12,13].

The Euler-Lagrange equation for the material field $\mathbf{F}$ is the equation of a driven harmonic oscillator

$$\partial_t^2 \mathbf{F} + \omega^2 \mathbf{F} = \mathbf{E}. \tag{14}$$

The inhomogeneous solution of this equation is (with dependence on ω and t explicit)

$$\mathbf{F}(\omega,t) = \int_{-\infty}^{\infty} dt' \mathcal{G}(\omega,t-t')\mathbf{E}(t'), \tag{15}$$

where $\mathcal{G}(\omega,t)$ is a Green's function of the harmonic oscillator equation. The homogeneous solution is not present in this classical theory. However, in the quantum theory it must be taken into account. There it describes a noise polarization, a quantity which can even be interpreted as the basic ingredient of the quantum theory on which all other fields depend [14–17]. In the classical theory it turns out that the solution is causal provided the Green's function is chosen to be the retarded Green's function

$$\mathcal{G}(\omega,t) = \theta(t)\frac{\sin(\omega t)}{\omega}, \tag{16}$$

where $\theta(t)$ is the step function [$\theta(t)=1$ if $t>0$, $\theta(t)=1/2$ if $t=0$, $\theta(t)=0$ if $t<0$]. The retarded Green's function has a Fourier representation

$$\mathcal{G}(\omega,t) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{1}{\omega^2 - (\omega' + i\gamma)^2} \exp(-i\omega' t), \tag{17}$$

where γ is a positive infinitesimal quantity. The resulting expression for the material field $\mathbf{F}$ leads to a dielectric displacement

$$\mathbf{D}(t) = \varepsilon_0 \int_{-\infty}^{\infty} dt' \varepsilon(t-t')\mathbf{E}(t'), \tag{18}$$

with the dielectric function

$$\begin{aligned}\varepsilon(t) &= \delta(t) + \frac{2}{\pi}\int_0^{\infty} d\omega \hat{\sigma}(\omega)\theta(t)\frac{\sin(\omega t)}{\omega} \\ &= \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\left[1 + \frac{2}{\pi}\int_0^{\infty} d\omega \frac{\hat{\sigma}(\omega)}{\omega^2 - (\omega' + i\gamma)^2}\right]\exp(-i\omega' t).\end{aligned} \tag{19}$$

It follows that the Fourier transform of the dielectric function is given by

$$\begin{aligned}\hat{\varepsilon}(\omega) &= 1 + \frac{2}{\pi}\int_0^{\infty} d\omega' \frac{\hat{\sigma}(\omega')}{\omega'^2 - (\omega + i\gamma)^2} \\ &= 1 + \frac{1}{\pi}\int_0^{\infty} d\omega' \frac{\hat{\sigma}(\omega')}{\omega'}\left(\frac{1}{\omega' - \omega - i\gamma} - \frac{1}{\omega' + \omega + i\gamma}\right) \\ &= 1 + \frac{1}{\pi}\int_{-\infty}^{\infty} d\omega' \frac{\hat{\sigma}(\omega')}{\omega'}\frac{1}{\omega' - \omega - i\gamma},\end{aligned} \tag{20}$$

where it has been used that $\hat{\sigma}(\omega)=\hat{\sigma}(-\omega)$. Using that

$$\frac{1}{\omega - i\gamma} = \mathrm{P}\frac{1}{\omega} + i\pi\delta(\omega), \tag{21}$$

where the capital "P" indicates the principal value, it follows that

$$\begin{aligned}\hat{\varepsilon}(\omega) &= 1 + \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty} d\omega' \frac{\hat{\sigma}(\omega')}{\omega'(\omega' - \omega)} + \frac{i\hat{\sigma}(\omega)}{\omega} \\ &= 1 + \frac{2}{\pi}\mathrm{P}\int_0^{\infty} d\omega' \frac{\hat{\sigma}(\omega')}{\omega'^2 - \omega^2} + \frac{i\hat{\sigma}(\omega)}{\omega}.\end{aligned} \tag{22}$$

By construction, this function satisfies the Kramers-Kronig relations as well as the symmetry relation $\hat{\varepsilon}(\omega)=\hat{\varepsilon}(-\omega)^*$. As a consequence, the dielectric function in the time domain is real [$\varepsilon(t)=\varepsilon(t)^*$] and causal [$\varepsilon(t)=0$ if $t\leqslant 0$]. This proves that the constitutive relation for media with arbitrary dispersion and absorption is properly described by the present model. As a consequence, the equations of motion for the electromagnetic field in such media are formally equivalent to the Euler-Lagrange equations for the proposed Lagrangian density.

III. CONSERVATION LAWS

A. Energy

The transport and dissipation of electromagnetic energy is described by the energy balance equation

$$\partial_t u^{EM} + \nabla \cdot \mathbf{S} = -W, \tag{23}$$

where the electromagnetic field energy density u^{EM}, energy flux density $\mathbf{S}$ (Poynting vector), and the density of the rate of work on the material subsystem W are defined by

$$u^{EM} = \frac{\varepsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2, \tag{24}$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \tag{25}$$

$$W = \partial_t \mathbf{P} \cdot \mathbf{E}. \tag{26}$$

This energy balance equation follows directly from Maxwell's equations [12].

It appears that the rate of work can be written as the time derivative of a quantity that may be interpreted as the energy of the material subsystem

$$\begin{aligned}\partial_t \mathbf{P} \cdot \mathbf{E} &= \frac{\varepsilon_0}{\pi}\int_0^\infty d\omega \hat{\sigma}(\omega) \partial_t \mathbf{F} \cdot \mathbf{E} \\ &= \frac{\varepsilon_0}{\pi}\int_0^\infty d\omega \hat{\sigma}(\omega) \partial_t \mathbf{F} \cdot (\partial_t^2 \mathbf{F} + \omega^2 \mathbf{F}) \\ &= \partial_t \left[\frac{\varepsilon_0}{\pi}\int_0^\infty d\omega \hat{\sigma}(\omega) [(\partial_t \mathbf{F})^2 + \omega^2 \mathbf{F}^2] \right]. \end{aligned} \tag{27}$$

Here, the equation of motion of the material field $\mathbf{F}$, Eq. (14), is used to eliminate $\mathbf{E}$ in favor of $\mathbf{F}$. The energy of the material subsystem thus follows as

$$u^{MT} = \frac{\varepsilon_0}{\pi}\int_0^\infty d\omega \hat{\sigma}(\omega) [(\partial_t \mathbf{F})^2 + \omega^2 \mathbf{F}^2]. \tag{28}$$

Conservation of energy of the total system is expressed by

$$\partial_t u + \nabla \cdot \mathbf{S} = 0, \tag{29}$$

where the total energy density is given by

$$\begin{aligned} u &= u^{EM} + u^{MT} \\ &= \frac{\varepsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2 + \frac{\varepsilon_0}{\pi}\int_0^\infty d\omega \hat{\sigma}(\omega) [(\partial_t \mathbf{F})^2 + \omega^2 \mathbf{F}^2]. \end{aligned} \tag{30}$$

The total energy is the sum of squares, and therefore always positive, which guarantees thermodynamic stability. The same energy conservation law has been found previously by Tip [10], and by Glasgow, Ware, and Peatross by deduction from Maxwell's equations and the constitutive relation [18].

The total energy may be split into parts in a variety of ways. A division between nondispersive and dispersive contributions u^{ND} and u^{DS} may be defined by

$$u^{ND} = \frac{1}{2}\mathbf{E} \cdot \mathbf{D} + \frac{1}{2}\mathbf{H} \cdot \mathbf{B}, \tag{31}$$

$$u^{DS} = \frac{\varepsilon_0}{\pi}\int_0^\infty d\omega \hat{\sigma}(\omega) [(\partial_t \mathbf{F})^2 + \omega^2 \mathbf{F}^2 - \mathbf{F} \cdot \mathbf{E}], \tag{32}$$

which gives rise to energy balance equations for the two parts

$$\partial_t u^{ND} + \nabla \cdot \mathbf{S} = -W', \tag{33}$$

$$\partial_t u^{DS} = W', \tag{34}$$

where the dissipation from the nondispersive to the dispersive energy density is given by

$$\begin{aligned} W' &= \partial_t \left[\frac{\varepsilon_0}{\pi}\int_0^\infty d\omega \hat{\sigma}(\omega) [(\partial_t \mathbf{F})^2 + \omega^2 \mathbf{F}^2 - \mathbf{F} \cdot \mathbf{E}] \right] \\ &= \frac{1}{2}\partial_t \mathbf{P} \cdot \mathbf{E} - \frac{1}{2}\mathbf{P} \cdot \partial_t \mathbf{E} = \frac{1}{2}\partial_t \mathbf{D} \cdot \mathbf{E} - \frac{1}{2}\mathbf{D} \cdot \partial_t \mathbf{E}, \end{aligned} \tag{35}$$

where the equation of motion for the material field $\mathbf{F}$ has been used, in a way similar to the derivation of Eq. (27). The nondispersive to dispersive dissipation W' is approximately zero in case dispersion is small. The total energy may then be approximated by the nondispersive contribution u^{ND}. This leads to the default expression for the electromagnetic energy in a dielectric appearing in many textbooks [12,29].

B. Momentum

The transport and dissipation of electromagnetic (linear) momentum is described by the momentum balance equation

$$\partial_t g_\alpha^{EM} + \partial_\beta T_{\alpha\beta}^{EM} = -f_\alpha, \tag{36}$$

where the momentum density g_α^{EM}, the momentum flux density (stress tensor) $T_{\alpha\beta}^{EM}$ and the density of the force on the material subsystem f_α are given by

$$g_\alpha^{EM} = \varepsilon_0 \epsilon_{\alpha\beta\gamma} E_\beta B_\gamma, \tag{37}$$

$$T_{\alpha\beta}^{EM} = -\varepsilon_0 E_\alpha E_\beta - \frac{1}{\mu_0} B_\alpha B_\beta + \left(\frac{\varepsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2 \right) \delta_{\alpha\beta}, \tag{38}$$

$$f_\alpha = -\partial_\beta P_\beta E_\alpha + \epsilon_{\alpha\beta\gamma} \partial_t P_\beta B_\gamma. \tag{39}$$

These expressions correspond to the Abraham momentum density, the Maxwell stress tensor, and the Lorentz force density. The momentum balance equation for the electromagnetic field can be derived from Maxwell's equations in a straightforward manner [12].

The Lorentz force density can be written as the sum of temporal and spatial derivatives. This implies the existence of a momentum balance equation without a source term, i.e., an equation that expresses the conservation of the total momentum of the combined field-matter system. This rewriting is done in a number of steps. First, using Faraday's law it follows that

$$f_\alpha = f'_\alpha + \partial_t(\epsilon_{\alpha\beta\gamma} P_\beta B_\gamma) + \partial_\beta \left(-E_\alpha P_\beta + \frac{1}{2}\mathbf{E} \cdot \mathbf{P} \delta_{\alpha\beta} \right), \tag{40}$$

where

$$f'_\alpha = \frac{1}{2}P_\beta \partial_\alpha E_\beta - \frac{1}{2}E_\beta \partial_\alpha P_\beta = \frac{1}{2}D_\beta \partial_\alpha E_\beta - \frac{1}{2}E_\beta \partial_\alpha D_\beta. \tag{41}$$

A second step is the use of the equation of motion for the material field $\mathbf{F}$, Eq. (14), in order to eliminate $\mathbf{E}$ in favor of $\mathbf{F}$ in the expression for f'_α

$$f'_\alpha = \frac{\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)(F_\beta\partial_\alpha E_\beta - E_\beta\partial_\alpha F_\beta)$$
$$= \frac{\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)(F_\beta\partial_\alpha\partial_t^2 F_\beta - \partial_t^2 F_\beta\partial_\alpha F_\beta)$$
$$= \partial_t\left[\frac{\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)(F_\beta\partial_\alpha\partial_t F_\beta - \partial_t F_\beta\partial_\alpha F_\beta)\right]. \quad (42)$$

A third step is rewriting this expression using the following identity:

$$0 = \partial_\alpha\left[\frac{\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)(\partial_t^2 F_\beta + \omega^2 F_\beta - E_\beta)F_\beta\right]$$
$$= \frac{\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)[\partial_\alpha(\omega^2 F_\beta F_\beta - E_\beta F_\beta)$$
$$+ F_\beta\partial_\alpha\partial_t^2 F_\beta + \partial_\alpha F_\beta\partial_t^2 F_\beta]$$
$$= \partial_t\left[\frac{\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)(F_\beta\partial_\alpha\partial_t F_\beta + \partial_t F_\beta\partial_\alpha F_\beta)\right]$$
$$- \partial_\alpha\left[\frac{\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)[(\partial_t\mathbf{F})^2 - \omega^2\mathbf{F}^2 + \mathbf{F}\cdot\mathbf{E}]\right]. \quad (43)$$

This identity follows from the equation of motion of the material field Eq. (14). This gives that

$$f'_\alpha = \partial_t\left(\frac{2\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)F_\beta\partial_\alpha\partial_t F_\beta\right)$$
$$- \partial_\alpha\left[\frac{\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)[(\partial_t\mathbf{F})^2 - \omega^2\mathbf{F}^2 + \mathbf{F}\cdot\mathbf{E}]\right]. \quad (44)$$

The Lorentz force density can now be expressed as

$$f_\alpha = \partial_t g_\alpha^{MT} + \partial_\beta T_{\alpha\beta}^{MT}, \quad (45)$$

where the material momentum density and momentum flux density are given by

$$g_\alpha^{MT} = \epsilon_{\alpha\beta\gamma}P_\beta B_\gamma + \frac{2\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)F_\beta\partial_\alpha\partial_t F_\beta, \quad (46)$$

$$T_{\alpha\beta}^{MT} = -E_\alpha P_\beta - \frac{\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)[(\partial_t\mathbf{F})^2 - \omega^2\mathbf{F}^2]\delta_{\alpha\beta}. \quad (47)$$

Conservation of momentum of the total system is expressed by

$$\partial_t g_\alpha + \partial_\beta T_{\alpha\beta} = 0, \quad (48)$$

where the momentum density and stress tensor of the total system are

$$g_\alpha = g_\alpha^{EM} + g_\alpha^{MT} = \epsilon_{\alpha\beta\gamma}D_\beta B_\gamma + \frac{2\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)F_\beta\partial_t\partial_\alpha F_\beta, \quad (49)$$

$$T_{\alpha\beta} = T_{\alpha\beta}^{EM} + T_{\alpha\beta}^{MT} = -E_\alpha D_\beta - H_\alpha B_\beta + \left(\frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{H}\cdot\mathbf{B}\right)\delta_{\alpha\beta}$$
$$- \frac{\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)[(\partial_t\mathbf{F})^2 - \omega^2\mathbf{F}^2 + \mathbf{F}\cdot\mathbf{E}]\delta_{\alpha\beta}. \quad (50)$$

The total system momentum proposed here corresponds to the pseudomomentum of Gordon [2], the wave momentum of Nelson [3], and the canonical momentum of Garrison and Chiao [4].

According to Nelson, the wave momentum is the sum of momentum and pseudomomentum. The momentum contribution from the material subsystem in the present theory corresponds to Nelson's pseudomomentum contribution to the wave momentum. A difference with Nelson is in the general form of the momentum density and stress tensor. These quantities are not unique in the sense that terms can be shifted from the density to the flux density and vice versa. In particular, any multiple of the identity Eq. (43) can be added or subtracted from the total momentum conservation law Eq. (48). An example of such a redefinition of the momentum density and stress tensor using the identity Eq. (43) is

$$g'_\alpha = \epsilon_{\alpha\beta\gamma}D_\beta B_\gamma - \frac{2\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)\partial_t F_\beta\partial_\alpha F_\beta, \quad (51)$$

$$T'_{\alpha\beta} = -E_\alpha D_\beta - H_\alpha B_\beta + \left(\frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{H}\cdot\mathbf{B}\right)\delta_{\alpha\beta}$$
$$+ \frac{\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)[(\partial_t\mathbf{F})^2 - \omega^2\mathbf{F}^2 + \mathbf{F}\cdot\mathbf{E}]\delta_{\alpha\beta}. \quad (52)$$

These forms correspond quite closely to the density and flux density of wave momentum of Nelson [3]. Apparently, an independent requirement is needed to justify the form of these quantities. The point of view taken here is motivated by an analysis of the relation between energy and momentum of wave packets, and will be discussed in the next section. It turns out that the present choice, Eqs. (49) and (50), results in transport of energy and momentum with the same velocity, as opposed to the alternative choice, Eqs. (51) and (52), which leads to transport of energy and momentum at different velocities [19]. It seems natural to have energy and momentum travel at the same speed, which implies that Eqs. (49) and (50) are the correct forms of the density and flux density of the total momentum.

Similar to the energy case the total momentum can be divided into nondispersive and dispersive parts, with densities

$$g_\alpha^{ND} = \epsilon_{\alpha\beta\gamma}D_\beta B_\gamma, \quad (53)$$

$$g_\alpha^{DS} = \frac{2\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)F_\beta\partial_\alpha\partial_t F_\beta, \quad (54)$$

and flux densities

$$T_{\alpha\beta}^{ND} = -E_\alpha D_\beta - H_\alpha B_\beta + \left(\frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{H}\cdot\mathbf{B}\right)\delta_{\alpha\beta}, \quad (55)$$

$$T_{\alpha\beta}^{DS} = -\frac{\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)[(\partial_t\mathbf{F})^2 - \omega^2\mathbf{F}^2 + \mathbf{F}\cdot\mathbf{E}]\delta_{\alpha\beta}. \quad (56)$$

The nondispersive contributions to the momentum density and momentum flux density are recognized as the Minkowski momentum density and the Minkowski stress tensor, respectively. The momentum balance equations for the two parts are

$$\partial_t g_\alpha^{ND} + \partial_\beta T_{\alpha\beta}^{ND} = -f'_\alpha, \quad (57)$$

$$\partial_t g_\alpha^{DS} + \partial_\beta T_{\alpha\beta}^{DS} = f'_\alpha, \quad (58)$$

where the Minkowski force density f'_α is given by (41). This force density is approximately zero when dispersion may be neglected. In that case the total momentum may be approximated by the nondispersive (Minkowski) momentum. In general, however, the dispersive terms need to be taken into account. The importance of including dispersive contributions has also been stressed by Nelson [3] and Garrison and Chiao [4].

C. Angular momentum

Our treatment of angular momentum will be brief, as it is quite similar to the case of linear momentum treated previously. The angular momentum quantities are simply found from the linear momentum quantities by taking the cross product with the position vector.

An issue frequently popping up in discussions about angular momentum conservation is the symmetry, or lack of it, of the stress tensor. It appears that dispersion can result in an asymmetric stress tensor, although the medium is isotropic. This can be seen as follows. The dielectric displacement **D** at a specific time t depends on the electric field **E** at all previous times. The electric field at these times $t' < t$ is not necessarily oriented in the same direction as the electric field at time t. It follows that **D** and **E** at time t are not necessarily parallel, implying that the stress tensor defined by Eq. (50) is generally asymmetric. This seemingly points to nonconservation of angular momentum. However, it turns out that the antisymmetric part of the stress tensor can be expressed as a time derivative of a quantity, which may be interpreted as contributing to the internal angular momentum

$$\epsilon_{\alpha\beta\gamma}T_{\beta\gamma} = \epsilon_{\alpha\beta\gamma}P_\beta E_\gamma = \frac{2\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)\epsilon_{\alpha\beta\gamma}F_\beta(\partial_t^2 F_\gamma + \omega^2 F_\gamma)$$
$$= \partial_t\left(\frac{2\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)\epsilon_{\alpha\beta\gamma}F_\beta\partial_t F_\gamma\right). \quad (59)$$

The internal angular momentum contribution depends on the cross product of the material field and the time derivative of the material field. It follows that this contribution is only nonzero if the orientation of the fields changes with time, which corroborates the qualitative argument given previously. An alternative, equally valid, way of dealing with this asymmetry is to absorb it into a redefinition of the linear momentum density.

Conservation of angular momentum is thus expressed by

$$\partial_t j_\alpha + \partial_\beta M_{\alpha\beta} = 0, \quad (60)$$

where the total angular momentum density and angular momentum flux density are given by

$$j_\alpha = \epsilon_{\alpha\beta\gamma}r_\beta g_\gamma + \frac{2\varepsilon_0}{\pi}\int_0^\infty d\omega\hat{\sigma}(\omega)\epsilon_{\alpha\beta\gamma}F_\beta\partial_t F_\gamma, \quad (61)$$

$$M_{\alpha\beta} = \epsilon_{\alpha\mu\nu}r_\mu T_{\nu\beta}. \quad (62)$$

Division of the total angular momentum into field and matter contributions, and into dispersive and nondispersive contributions are completely analogous to the linear momentum case. A division into spin and orbital parts can be developed along the lines of Refs. [20,21], but will not be pursued here.

IV. WAVE PACKETS

Certain interesting features of the auxiliary field model become apparent when studying wave packets. In this section, one-dimensional propagation along the z axis of a linearly polarized wave packet is considered. Then the electric field only has a nonzero x component given by

$$E_x(z,t) = \int_{-\infty}^\infty \frac{d\omega}{2\pi}\hat{E}(z,\omega)\exp(-i\omega t), \quad (63)$$

where

$$\hat{E}(z,\omega) = \hat{E}(\omega)\exp[ik(\omega)z]. \quad (64)$$

Here $\hat{E}(\omega) = \hat{E}(-\omega)^*$, because $E_x(z,t)$ is real, and where the (magnitude of) the wave vector is given by

$$k(\omega) = [n(\omega) + i\kappa(\omega)]\frac{\omega}{c}, \quad (65)$$

with $n(\omega)$ the refractive index and $\kappa(\omega)$ the absorption coefficient. These are related to the dielectric function by

$$\hat{\varepsilon}(\omega) = [n(\omega) + i\kappa(\omega)]^2, \quad (66)$$

so that the real and imaginary parts can be written as

$$\hat{\varepsilon}_r(\omega) = n(\omega)^2 - \kappa(\omega)^2, \quad (67)$$

$$\hat{\varepsilon}_i(\omega) = \frac{\hat{\sigma}(\omega)}{\omega} = 2n(\omega)\kappa(\omega). \quad (68)$$

The other field components of interest are

$$D_x(z,t) = \varepsilon_0\int_{-\infty}^\infty \frac{d\omega}{2\pi}\hat{\varepsilon}(\omega)\hat{E}(z,\omega)\exp(-i\omega t), \quad (69)$$

$$B_y(z,t) = \mu_0 H_y(z,t) = \int_{-\infty}^\infty \frac{d\omega}{2\pi}\frac{k(\omega)}{\omega}\hat{E}(z,\omega)\exp(-i\omega t), \quad (70)$$

$$F_x(\omega',z,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{\omega'^2 - (\omega + i\gamma)^2} \hat{E}(z,\omega)\exp(-i\omega t). \tag{71}$$

Using these expressions, the density and flow of energy and linear momentum, and the density of the rate of work and force can be calculated. The attention is restricted to energy and linear momentum, as angular momentum does not play a role for the wave packets under discussion.

In the following, the shorthand

$$\int D(\omega,\omega')f(\omega,\omega') \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\omega d\omega'}{(2\pi)^2} f(\omega,\omega')\hat{E}(z,\omega) \times \hat{E}(z,\omega')^* \exp[-i(\omega-\omega')t], \tag{72}$$

will be used, which is convenient as most relevant quantities are bilinear in field components. The integrand may be split into parts $f(\omega,\omega')=f_S(\omega,\omega')+f_A(\omega,\omega')$, where $f_S(\omega,\omega') = f_S(\omega',\omega)^*$ and $f_A(\omega,\omega')=-f_A(\omega',\omega)^*$. As $D(\omega,\omega') = D(\omega',\omega)^*$ it follows that only the part $f_S(\omega,\omega')$ contributes to the integral. This often helps to simplify equations.

A. Energy

For the field part of the energy density it is found that

$$u^{EM} = \frac{1}{2}\varepsilon_0 E_x{}^2 + \frac{1}{2\mu_0} B_y{}^2 = \frac{1}{2}\varepsilon_0 \int D(\omega,\omega')[1 + \sqrt{\hat{\varepsilon}(\omega)\hat{\varepsilon}(\omega')^*}]. \tag{73}$$

The material part is more involved

$$u^{MT} = \frac{\varepsilon_0}{\pi}\int_0^{\infty} d\omega_0 \hat{\sigma}(\omega_0)[(\partial_t F_x)^2 + \omega_0{}^2 F_x^2] = \frac{1}{2}\varepsilon_0 \int D(\omega,\omega')X(\omega,\omega'), \tag{74}$$

with

$$\begin{aligned} X(\omega,\omega') &= \frac{2}{\pi}\int_0^{\infty} d\omega_0 \hat{\sigma}(\omega_0) \frac{\omega_0{}^2 + \omega\omega'}{[\omega_0{}^2 - (\omega+i\gamma)^2][\omega_0{}^2 - (\omega'-i\gamma)^2]} \\ &= \frac{1}{(\omega+i\gamma)^2 - (\omega'-i\gamma)^2}\frac{2}{\pi}\int_0^{\infty} d\omega_0 \hat{\sigma}(\omega_0) \times \left[\frac{\omega_0{}^2 + \omega\omega'}{\omega_0{}^2 - (\omega+i\gamma)^2} - \frac{\omega_0{}^2 + \omega\omega'}{\omega_0{}^2 - (\omega'-i\gamma)^2}\right] \\ &= \frac{1}{(\omega+\omega')(\omega-\omega'+2i\gamma)}\frac{2}{\pi}\int_0^{\infty} d\omega_0 \hat{\sigma}(\omega_0) \times \left[\frac{(\omega+i\gamma)^2 + \omega\omega'}{\omega_0{}^2 - (\omega+i\gamma)^2} - \frac{(\omega'-i\gamma)^2 + \omega\omega'}{\omega_0{}^2 - (\omega'-i\gamma)^2}\right] \\ &= \frac{1}{\omega-\omega'+i\gamma}\frac{2}{\pi}\int_{-\infty}^{\infty} d\omega_0 \hat{\sigma}(\omega_0) \times \left[\frac{\omega}{\omega_0{}^2 - (\omega+i\gamma)^2} - \frac{\omega'}{\omega_0{}^2 - (\omega'-i\gamma)^2}\right] \\ &= \frac{[\hat{\varepsilon}(\omega)-1]\omega - [\hat{\varepsilon}(\omega')^*-1]\omega'}{\omega-\omega'+i\gamma} \\ &= \left(\frac{\hat{\varepsilon}_r(\omega)\omega - \hat{\varepsilon}_r(\omega')\omega'}{\omega-\omega'} - 1\right) + \frac{i[\hat{\sigma}(\omega)+\hat{\sigma}(\omega')]}{\omega-\omega'+i\gamma}, \end{aligned} \tag{75}$$

where it has been used that γ is infinitesimally small and where the expression (20) for the dielectric function is used. The total energy density may now be divided into two parts in yet a third way, namely into a propagating and nonpropagating part, the nonpropagating part being due to dissipation alone

$$u = u^{PR} + u^{NP}, \tag{76}$$

where the propagating and nonpropagating parts are

$$u^{PR} = \frac{1}{2}\varepsilon_0 \int D(\omega,\omega')\left(\frac{\hat{\varepsilon}_r(\omega)\omega - \hat{\varepsilon}_r(\omega')\omega'}{\omega-\omega'} + \sqrt{\hat{\varepsilon}(\omega)\hat{\varepsilon}(\omega')^*}\right), \tag{77}$$

$$u^{NP} = \frac{1}{2}\varepsilon_0 \int D(\omega,\omega')\frac{i[\hat{\sigma}(\omega)+\hat{\sigma}(\omega')]}{\omega-\omega'+i\gamma}. \tag{78}$$

The nonpropagating contribution may be further rewritten using the Fourier representation of the step function

$$\frac{i}{\omega-\omega'+i\gamma} = \int_{-\infty}^{\infty} dt'\,\theta(t')\exp[i(\omega-\omega')t'], \tag{79}$$

to the time integral of the rate of work

$$u^{NP} = \int_{-\infty}^{t} dt'\,W''(t'), \tag{80}$$

where the rate of work is given by

$$W'' = \frac{1}{2}\varepsilon_0 \int D(\omega,\omega')[\hat{\sigma}(\omega)+\hat{\sigma}(\omega')]. \tag{81}$$

This may be written as the product of the dissipative part of the current density and the electric field

$$W'' = j_x(z,t)E_x(z,t), \tag{82}$$

where the dissipative current density is given by

$$j_x(z,t) = \varepsilon_0 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\hat{\sigma}(\omega)\hat{E}(z,\omega)\exp(-i\omega t). \tag{83}$$

This dissipative current density is the time derivative of the dissipative part of the dielectric polarization, i.e., the part of the dielectric polarization that involves only the imaginary part of the susceptibility. The remaining, conservative part of the dielectric polarization contributes to the energy of the propagating wave. It follows that the nonpropagating part of the total energy may be identified as the local energy of the

continuous reservoir of oscillators into which the wave dissipates energy. The reservoir gains energy by dissipation and, because of causality, depends only on the electric fields at previous times. The energy flux is directed along the z axis and has a magnitude

$$S_z = \frac{1}{2}\varepsilon_0 c \int D(\omega,\omega')[\sqrt{\hat{\varepsilon}(\omega)} + \sqrt{\hat{\varepsilon}(\omega')^*}]. \tag{84}$$

It turns out that the nonpropagating energy density satisfies

$$\lim_{t\to-\infty} u^{NP} = 0, \tag{85}$$

$$\lim_{t\to+\infty} u^{NP} = \int_{-\infty}^{\infty} dt' W'' = \varepsilon_0 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\hat{\sigma}(\omega)|\hat{E}(z,\omega)|^2. \tag{86}$$

It follows that the loss in the propagating part of the energy density over the total duration of the pulse is always positive, as $\hat{\sigma}(\omega)$ is positive for all ω. This irreversibility is in agreement with expectations. The integral of the propagating energy density and energy flux density over the duration of the pulse follow as

$$\int_{-\infty}^{\infty} dt u^{PR} = \frac{1}{2}\varepsilon_0 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\left[\frac{d[\hat{\varepsilon}_r(\omega)\omega]}{d\omega} + |\hat{\varepsilon}(\omega)|\right]|\hat{E}(z,\omega)|^2, \tag{87}$$

$$\int_{-\infty}^{\infty} dt S_z = \varepsilon_0 c \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} n(\omega)|\hat{E}(\omega)|^2. \tag{88}$$

Consider now a pulse

$$E_x(z,t) = \frac{1}{2}E_0(z,t)\exp(-i\omega_0 t) + \frac{1}{2}E_0(z,t)^* \exp(i\omega_0 t), \tag{89}$$

where ω_0 is a carrier frequency and where $E_0(z,t)$ is a slowly varying envelope function for all z concerned. It then follows that the spectrum $\hat{E}(z,\omega)$ has narrow peaks at $\pm\omega_0$ so that the dielectric function can be taken constant across the integration range. In this approximation the propagating energy, rate of work, and energy flux density are given by

$$u^{PR} = \frac{1}{4}\varepsilon_0\left[\frac{d[\hat{\varepsilon}_r(\omega_0)\omega_0]}{d\omega_0} + |\hat{\varepsilon}(\omega_0)|\right]|E_0(z,t)|^2, \tag{90}$$

$$W'' = \frac{1}{2}\varepsilon_0\hat{\sigma}(\omega_0)|E_0(z,t)|^2, \tag{91}$$

$$S_z = \frac{1}{2}\varepsilon_0 n(\omega_0) c|E_0(z,t)|^2. \tag{92}$$

If the absorption is small the nonpropagating contribution may be neglected and the total energy (which is equal to the propagating energy in this limit) may be approximated as

$$u = \frac{1}{2}\varepsilon_0 n(\omega_0)\frac{d[n(\omega_0)\omega_0]}{d\omega_0}|E_0(z,t)|^2 = \frac{d[n(\omega_0)\omega_0]}{d\omega_0}\frac{S_z}{c}, \tag{93}$$

proving that the energy of the wave packet travels at the group velocity $c/n_g(\omega)$ with the group refractive index

$$n_g(\omega) = \frac{d[n(\omega)\omega]}{d\omega}. \tag{94}$$

It appears that $E_0(z,t)$ is a slowly varying function if $E_0(0,t)$ is a slowly varying function, provided that z is sufficiently small, irrespective of whether ω_0 is close to a resonance or not [22]. In this transparency regime the group velocity can exceed the speed of light in vacuum and can even be negative, as demonstrated experimentally in Ref. [23]. The arrival time of a pulse can be given a well-defined meaning, even in these exotic regimes [24]. In turn, if z is sufficiently large $E_0(z,t)$ cannot be a slowly varying function, even if $E_0(0,t)$ is, and if ω_0 is far away from a resonance. In this latter regime, the asymptotic regime, a different treatment is needed [25,26].

B. Momentum

The nondispersive (Minkowski) part of the momentum density is given by

$$g_z^{ND} = D_x B_y = \frac{\varepsilon_0}{2c}\int D(\omega,\omega')[\hat{\varepsilon}(\omega)\sqrt{\hat{\varepsilon}(\omega')^*} + \hat{\varepsilon}(\omega')^*\sqrt{\hat{\varepsilon}(\omega)}]. \tag{95}$$

The dispersive part of the momentum density is

$$\begin{aligned} g_z^{DS} &= \frac{2\varepsilon_0}{\pi}\int_0^{\infty} d\omega_0 \hat{\sigma}(\omega_0) F_x \partial_t \partial_z F_x \\ &= \frac{\varepsilon_0}{2c}\int D(\omega,\omega')Y(\omega,\omega')[\sqrt{\hat{\varepsilon}(\omega)}\omega^2 + \sqrt{\hat{\varepsilon}(\omega')^*}\omega'^2], \end{aligned} \tag{96}$$

with

$$\begin{aligned} Y(\omega,\omega') &= \frac{2}{\pi}\int_0^{\infty} d\omega_0 \hat{\sigma}(\omega_0)\frac{1}{[\omega_0{}^2 - (\omega + i\gamma)^2][\omega_0{}^2 - (\omega' - i\gamma)^2]} \\ &= \frac{\hat{\varepsilon}(\omega) - \hat{\varepsilon}(\omega')^*}{(\omega+\omega')(\omega-\omega'+i\gamma)} \\ &= \frac{1}{(\omega+\omega')(\omega-\omega'+i\gamma)} \\ &\quad \times\left[\hat{\varepsilon}_r(\omega) - \hat{\varepsilon}_r(\omega') + i\left(\frac{\hat{\sigma}(\omega)}{\omega} + \frac{\hat{\sigma}(\omega')}{\omega'}\right)\right]. \end{aligned} \tag{97}$$

The total momentum can be rewritten using

$$2\frac{\sqrt{\hat{\varepsilon}(\omega)}\omega^2+\sqrt{\hat{\varepsilon}(\omega')}^*\omega'^2}{(\omega+\omega')(\omega-\omega'+i\gamma)} = \frac{\sqrt{\hat{\varepsilon}(\omega)}\omega+\sqrt{\hat{\varepsilon}(\omega')}^*\omega'}{\omega-\omega'+i\gamma}+\frac{\sqrt{\hat{\varepsilon}(\omega)}\omega-\sqrt{\hat{\varepsilon}(\omega')}^*\omega'}{\omega+\omega'}, \tag{98}$$

so that a division of total momentum into propagating and nonpropagating parts

$$g_z = g_z^{PR} + g_z^{NP}, \tag{99}$$

can be made, such that

$$g_z^{PR} = \frac{\varepsilon_0}{4c}\int D(\omega,\omega')\left[2\sqrt{\hat{\varepsilon}(\omega)\hat{\varepsilon}(\omega')}^*[\sqrt{\hat{\varepsilon}(\omega)}+\sqrt{\hat{\varepsilon}(\omega')}^*] + \frac{\hat{\varepsilon}_r(\omega)-\hat{\varepsilon}_r(\omega')}{\omega-\omega'}[\sqrt{\hat{\varepsilon}(\omega)}\omega+\sqrt{\hat{\varepsilon}(\omega')}^*\omega'] + \frac{\hat{\varepsilon}(\omega)-\hat{\varepsilon}(\omega')^*}{\omega+\omega'}[\sqrt{\hat{\varepsilon}(\omega)}\omega-\sqrt{\hat{\varepsilon}(\omega')}^*\omega']\right], \tag{100}$$

$$g_z^{NP} = \frac{\varepsilon_0}{4c}\int D(\omega,\omega')\frac{i[\hat{\sigma}(\omega)/\omega+\hat{\sigma}(\omega')/\omega']}{\omega-\omega'+i\gamma} \times[\sqrt{\hat{\varepsilon}(\omega)}\omega+\sqrt{\hat{\varepsilon}(\omega')}^*\omega'] = \frac{\varepsilon_0}{4c}\int D(\omega,\omega')\frac{i[\hat{\sigma}(\omega)+\hat{\sigma}(\omega')]}{\omega-\omega'+i\gamma}[\sqrt{\hat{\varepsilon}(\omega)}+\sqrt{\hat{\varepsilon}(\omega')}^*]. \tag{101}$$

The nonpropagating momentum density may be written as the integral of the force density

$$g_z^{NP} = \int_{-\infty}^{t} dt' f_z''(t'), \tag{102}$$

with

$$f_z'' = \frac{\varepsilon_0}{4c}\int D(\omega,\omega')[\hat{\sigma}(\omega)+\hat{\sigma}(\omega')][\sqrt{\hat{\varepsilon}(\omega)}+\sqrt{\hat{\varepsilon}(\omega')}^*]. \tag{103}$$

The flow of z momentum in the z direction is given by the sum of the nondispersive (Minkowski) stress tensor component

$$T_{zz}^{ND} = \frac{1}{2}E_xD_x+\frac{1}{2}H_yB_y = \frac{1}{4}\varepsilon_0\int D(\omega,\omega')[\hat{\varepsilon}(\omega)+\hat{\varepsilon}(\omega')^*+2\sqrt{\hat{\varepsilon}(\omega)\hat{\varepsilon}(\omega')}^*], \tag{104}$$

and the dispersive stress tensor component

$$T_{zz}^{DS} = -\frac{\varepsilon_0}{\pi}\int_0^\infty d\omega_0\hat{\sigma}(\omega_0)[(\partial_t F_x)^2-\omega_0{}^2F_x{}^2+F_xE_x] = \frac{\varepsilon_0}{2}\int D(\omega,\omega')Q(\omega,\omega'), \tag{105}$$

with

$$Q(\omega,\omega') = \frac{2}{\pi}\int_0^\infty d\omega_0\hat{\sigma}(\omega_0)\left[2\frac{\omega\omega'+\omega_0{}^2}{[\omega_0{}^2-(\omega+i\gamma)^2][\omega_0{}^2-(\omega'-i\gamma)^2]} - \frac{1}{\omega_0{}^2-(\omega+i\gamma)^2}-\frac{1}{\omega_0{}^2-(\omega+i\gamma)^2}\right] = -\frac{2}{\pi}\int_0^\infty d\omega_0\hat{\sigma}(\omega_0)\frac{(\omega-\omega')^2}{[\omega_0{}^2-(\omega+i\gamma)^2][\omega_0{}^2-(\omega'-i\gamma)^2]} = -(\omega-\omega')\frac{\hat{\varepsilon}(\omega)-\hat{\varepsilon}(\omega')^*}{\omega+\omega'}.$$

The zz component of the total stress tensor then follows as

$$T_{zz} = \frac{1}{4}\varepsilon_0\int D(\omega,\omega')\left(\hat{\varepsilon}(\omega)+\hat{\varepsilon}(\omega')^*+2\sqrt{\hat{\varepsilon}(\omega)\hat{\varepsilon}(\omega')}^* - 2(\omega-\omega')\frac{\hat{\varepsilon}(\omega)-\hat{\varepsilon}(\omega')^*}{\omega+\omega'}\right). \tag{106}$$

The nonpropagating momentum density satisfies

$$\lim_{t\to-\infty} g_z^{NP} = 0, \tag{107}$$

$$\lim_{t\to+\infty} g_z^{NP} = \int_{-\infty}^{\infty} dt' f_z'' = \frac{\varepsilon_0}{c}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{\sigma}(\omega)n(\omega)|\hat{E}(z,\omega)|^2, \tag{108}$$

from which it may be concluded that the dissipation of propagating momentum integrated over the entire pulse is always positive. Each Fourier component of the integrated momentum dissipation is a factor $n(\omega)/c$ times the Fourier component of the integrated energy dissipation. The time integral of the propagating momentum density and flux density are

$$\int_{-\infty}^{\infty} dt g_z^{PR} = \frac{\varepsilon_0}{c}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left[|\hat{\varepsilon}(\omega)|-\kappa(\omega)^2+\frac{1}{2}\omega\frac{d\hat{\varepsilon}_r(\omega)}{d\omega}\right] \times n(\omega)|\hat{E}(z,\omega)|^2 = \frac{\varepsilon_0}{2c}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left[\frac{d(\hat{\varepsilon}_r(\omega)\omega)}{d\omega}+|\hat{\varepsilon}(\omega)|\right]n(\omega)|\hat{E}(z,\omega)|^2, \tag{109}$$

$$\int_{-\infty}^{\infty} dt T_{zz} = \frac{1}{2}\varepsilon_0 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}[\hat{\varepsilon}_r(\omega) + |\hat{\varepsilon}(\omega)|]|\hat{E}(z,\omega)|^2$$
$$= \varepsilon_0 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} n(\omega)^2 |\hat{E}(z,\omega)|^2. \quad (110)$$

Similar to the dissipation of energy it follows that each Fourier component of the integrated density and flux density of the propagating momentum is a factor $n(\omega)/c$ times the Fourier component of the integrated density and flux density of the propagating energy. It is this relation between energy and momentum that has motivated the choice of the density and flux density of momentum given by Eqs. (49) and (50) over the forms given in Eqs. (51) and (52). It is mentioned that the dispersive momentum flux density does not contribute to the time integral of the total momentum flux density, the only nonzero contribution comes from the nondispersive (Minkowski) stress tensor.

For the narrow-band pulse it is found that

$$g_z^{PR} = \frac{1}{2}\frac{\varepsilon_0}{c}\left[\frac{d[\hat{\varepsilon}_r(\omega_0)\omega_0]}{d\omega_0} + |\hat{\varepsilon}(\omega_0)|\right] n(\omega_0)|E_0(z,t)|^2, \quad (111)$$

$$f_z'' = \frac{1}{2}\frac{\varepsilon_0}{c}\hat{\sigma}(\omega_0) n(\omega_0)|E_0(z,t)|^2, \quad (112)$$

$$T_{zz} = \frac{1}{2}\varepsilon_0 n(\omega_0)^2 |E_0(z,t)|^2. \quad (113)$$

If the absorption is small the nonpropagating momentum density may be neglected, and the total momentum may be approximated as

$$g_z = \frac{1}{2}\varepsilon_0 n(\omega_0)^2 \frac{d[n(\omega_0)\omega_0]}{d\omega_0}|E_0(z,t)|^2 = \frac{d[n(\omega_0)\omega_0]}{d\omega_0}\frac{T_{zz}}{c}, \quad (114)$$

proving that the momentum of the wave packet travels at a speed equal to the group velocity, just as the energy. The ratios of the density, flux density, and dissipation of energy and the density, flux density, and dissipation of momentum are all equal to

$$\frac{u^{PR}}{g_z^{PR}} = \frac{u^{NP}}{g_z^{NP}} = \frac{S_z}{T_{zz}} = \frac{W''}{f_z''} = \frac{c}{n(\omega_0)}, \quad (115)$$

i.e., the ratio between total energy and total momentum of the wave packet is equal to the phase velocity.

V. DISCUSSION

The momentum conservation law that has been derived in this paper applies to homogeneous dielectrics only, as deformations of the medium are excluded from the start. A more general theory should address the deformability of the material medium. Then the kinetic energy, kinetic momentum, hydrostatic forces (for fluids), and elastic forces (for solids), and effects such as electrostriction (change in density as a function of the electric field) need to be taken into account [27–29]. This would also give rise to separate conservation laws for momentum and pseudomomentum as then there are two independent continuous translation symmetries, one reflecting uniformity of space and one reflecting homogeneity of matter in the undeformed reference state.

Instead of such a first principles approach we may also introduce inhomogeneity in an *ad hoc* manner by making the conduction function space dependent, i.e., by replacing $\hat{\sigma}(\omega)$ by $\hat{\sigma}(\mathbf{r},\omega)$ everywhere. This does not alter the equations of motion of the model, nor the expression of the conservation of energy. The total momentum is no longer conserved because of the broken translational symmetry (the Lagrangian density depends explicitly on the spatial coordinates). It turns out that now

$$\partial_t g_\alpha + \partial_\beta T_{\alpha\beta} = -f_\alpha^{inh}, \quad (116)$$

where the dissipation of momentum due to the inhomogeneity is given by

$$f_\alpha^{inh} = -\frac{\varepsilon_0}{\pi}\int_0^\infty d\omega \partial_\alpha \hat{\sigma}(\mathbf{r},\omega)[(\partial_t \mathbf{F})^2 - \omega^2 \mathbf{F}^2 + 2\mathbf{F}\cdot\mathbf{E}]. \quad (117)$$

The implication is that inhomogeneities are accompanied by forces on the system. As a consequence, the total field-plus-matter system considered so far must be an open system, as external forces are needed to maintain the static inhomogeneity of the system when an electromagnetic field is applied. These external forces can be identified as the mechanical forces that have been excluded from the description in the beginning. The open character of the system has also been noticed by Garrison and Chiao as important for the applicability of the total momentum [4].

An explicit expression for the force density may be found in the Fourier domain (similar to the expressions derived for the wave packets studied in Sec. IV)

$$f_\alpha^{inh} = -\frac{1}{2}\varepsilon_0 \int D(\mathbf{r},\omega,\omega')\partial_\alpha\left[\frac{\hat{\varepsilon}(\mathbf{r},\omega)\omega' + \hat{\varepsilon}(\mathbf{r},\omega')^*\omega}{\omega+\omega'}\right], \quad (118)$$

where the shorthand $D(\mathbf{r},\omega,\omega')$ is defined by

$$\int D(\mathbf{r},\omega,\omega') f(\omega,\omega') \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\omega d\omega'}{(2\pi)^2} f(\omega,\omega')\hat{E}_\alpha(\mathbf{r},\omega) \times \hat{E}_\alpha(\mathbf{r},\omega')^* \exp[-i(\omega-\omega')t]. \quad (119)$$

This gives the time integral

$$\int_{-\infty}^{\infty} dt f_\alpha^{inh} = -\frac{1}{2}\varepsilon_0 \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}|\mathbf{E}(\mathbf{r},\omega)|^2 \partial_\alpha \hat{\varepsilon}_r(\mathbf{r},\omega), \quad (120)$$

so that the dissipation of momentum integrated over the duration of the interaction between the medium and the electromagnetic field is proportional to the spectral average of the product of the square of the electric field and the gradient of the real part of the dielectric function. This agrees with the

Helmholtz force expression when the latter is restricted to static incompressible media [27,28].

Of particular importance is the case of an interface between two otherwise homogeneous media. According to (116) the stress tensor must be discontinuous across the interface. This discontinuity is restricted to the flux across the interface of the momentum component normal to the interface. The flux across the interface of momentum components parallel to the interface is continuous due to the continuity requirements of the different fields (parallel components of **E** and **H** continuous, normal component of **D** and **B** continuous). The discontinous flux of normal momentum must be balanced by a mechanical flux of normal momentum, such as a pressure difference between the two media. The total force exerted by the first medium on the adjacent second medium is found by evaluating the stress tensor at the interface in the first medium. A similar view is found in Landau and Lifshitz [29], where it is shown that in the electrostatic limit the Minkowski stress tensor must be used to calculate the thermodynamic equilibrium forces on a dielectric body. The same conclusion is obtained by Gordon for optical frequencies and negligible dispersion [2]. Both results are generalized to arbitrary dispersive and absorbing media if the flux density of the total momentum (50) is used to calculate the force on a dielectric body.

According to a different point of view, the Lorentz force is the basic quantity, and the force on a dielectric body is found by integrating this force over the volume of the body [5–8]. This approach is equivalent to using the Abraham momentum and the Maxwell stress tensor. A variation of this approach is due to Mansuripur [8] who argues that a mechanical momentum density, equal to $\mathbf{P}\times\mathbf{B}/2$ in media with small dispersion and dissipation, accompanies a pulse of light in a dielectric, and that this contribution should count as electromagnetic momentum as well. It follows that the total momentum is then the average of the Abraham and Minkowski forms. The evaluation of radiation forces from the Lorentz force does not depend on this interpretation of what the total electromagnetic momentum is. The two views on how radiation forces should be calculated are incompatible in some cases, notably the case of a dielectric slab immersed in a different dielectric, and can thus be tested experimentally. This variation on the Jones-Leslie experiment [30] will be discussed in a separate paper.

An important result of this paper is that the ratio of the total energy and total momentum is given by the phase velocity. This is consistent with the assignment of an energy $E=\hbar\omega$ and momentum $\mathbf{p}=\hbar\mathbf{k}$ to a single photon ($\mathbf{k}$ is the wave vector in the medium), which gives $E/p=\omega/k=c/n$. This suggests that in the quantized version of the auxiliary field model the total momentum proposed here should correspond to a momentum $\hbar\mathbf{k}$ per quantum. Garrison and Chiao argue that this is the case based on the requirement that the total electromagnetic momentum operator should be the generator of translations [4]. The ratio between energy and momentum being the phase velocity agrees with the phase-matching condition in spontaneous down conversion [3,4], with experiments on the photon drag effect in semiconductors (see discussion in [6]), and recently with experiments on the recoil of an atom in a Bose-Einstein condensate when it absorbs a photon [31]. This perspective on photon momentum may be tested theoretically by using the well-developed quantum theory of light in dielectric media [10,14–17,32,33] to find out if indeed the proposed total momentum corresponds to $\hbar\mathbf{k}$ per photon. It is mentioned that this also offers a look on the Casimir effect in general linear dielectrics. In contrast to the use of the Minkowski or Maxwell stress tensor [34], it may well be that the flux density of the total momentum is the relevant quantity for calculating the Casimir force.

Finally, it is mentioned that the auxiliary field model can be extended in a number of ways. A generalization to deformable media has been mentioned already. A derivation of the model from microscopical principles, including statistical mechanical principles to incorporate dissipation, would justify the use of the auxiliary field model and elucidate the circumstances under which the model can be applied to describe experimental results. Other applications of the auxiliary field model are in the description of other types of media, in particular anisotropic and bianisotropic media, spatially dispersive media, and nonlinear media.

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