E-15 TIME- VERSUS FREQUENCY-DOMAIN MODELLING OF SEISMIC WAVE PROPAGATION

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Abstract

Realistic three-dimensional elastic modelling is feasible though still costly on present-day parallel computers, when using time-domain finite-difference methods. Frequency-domain methods are competitive for two-dimensional problems, although the gain in speed is not sufficiently dramatic to justify the increase in code complexity. In three dimensions, iterative solvers appear to be necessary. We report some experiments with BICGSTAB2 and a preconditioner based on separation of variables. These show unacceptable performance degradation for higher frequencies. Still, frequency domain methods may be attractive for migration and inversion, if only a reduced number of frequencies are used.

Introduction

Since the 70s of the previous century, time-domain finite-difference methods for the modelling of wave propagation have matured to a level that they can be routinely used on an industrial scale – at least for two-dimensional problems. In the following section, we review some of the experiences with these types of methods within Shell and point out potential areas of improvement. Frequency domain methods are less mature, the main stumble block being the numerical solution of the indefinite matrices that arise from the discretisation of Helmholtz-type equations. Nevertheless, frequency-domain methods may be competitive for two-dimensional problems. This is discussed in the 3rd section. In three dimensions, this is no longer true unless a fast iterative or perhaps even direct method can be found. We have included some test results for an attempt to build a preconditioner based on separation of variables.

Time-domain modelling

Explicit time-domain finite-difference schemes are an attractive choice for time-domain seismic modelling. They are relatively easy to parallelise and use. Staggered grids (Madariaga, 1976; Virieux, 1986; Igel *et al.*, 1995) are often used for elastic problems.

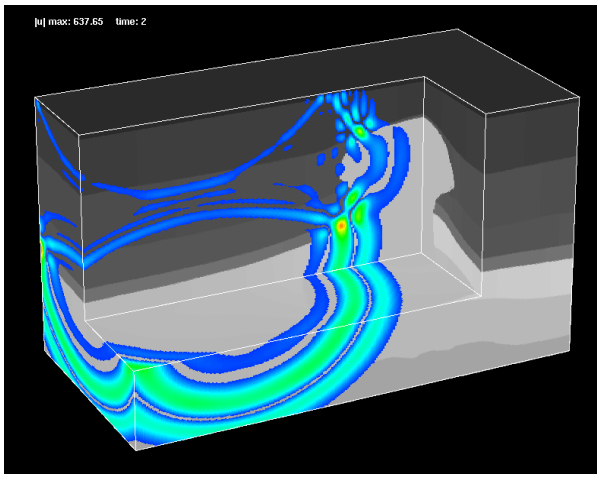

Figure 1. Example of a 3D wave equation simulation.

Within Shell, a parallel seismic modelling code for equations ranging from the simple constant-density acoustic equation up to anisotropic elastics has been in use since 1994 (Mulder, 1997). The code is mainly applied for forward modelling of two-dimensional problems. Full-scale 3D elastic modelling is feasible on current hardware, but still costly. Recent timings on Shell's 1024 nodes Linux cluster show that 6 seconds of data for a medium-size realistic model (1001^3 points, 4 m grid spacing) would take

about 1.5 week on $1+6^3 = 217$ nodes, each node having a Pentium III at 1 GHz. The default discretisation is 2nd order in time and 8th order in space. Figure 1 shows an example.

However, there are a number of limitations when using finite differences. Sharp contrasts in material properties cause a degradation of the accuracy for two reasons. Firstly, the pressure and displacements are continuous but not differentiable, leading to an error of 2nd order in the grid spacing. Secondly, for gridded models, the position of the interface between two materials is undefined within a grid spacing, resulting in a 1st order error. Because several points per wavelength are required anyway to control numerical errors even in constant models, the contrast can be smoothed over a few grid points to reduce these interface errors. Nevertheless, it does not seem to pay to go to very high or even spectral order. Other problems are related to the boundaries, in particular absorbing boundary conditions, rough topography, and borehole modelling.

A natural alternative is the use of finite elements to represent interfaces, assuming that element boundaries are aligned with the interfaces. To avoid the numerical solution of a large sparse linear system, mass lumping and explicit time stepping can be used. This can be done without loss of accuracy if special elements are constructed. For one-dimensional problem, the Gauss-Lobatto points are the obvious choice for the element nodes (Mulder, 1999). For two-dimensional scalar problems, a number of triangular elements can be found in (Cohen *et al*., 1993). The extension to tetrahedral elements can be found in (Mulder, 1996; Chin-Joe-Kong *et al*., 1999). A comparison to a finite-difference method shows that the mass-lumped finite-element method outperforms the finite-difference method in efficiency, at least for two-dimensional problems. In 3D, the elements are so complicated that this is questionable. Alternatively, one could use the Cartesian product of the Gauss-Lobatto points to go to more than one space dimensions (Cohen and Fauqueux, 2000). In general, this requires a grid of distorted rectangles or blocks. The associated grid generation problem is far from easy. Some of elements may become so small that implicit time stepping is necessary anyway. Semi-implicit methods (explicit if the stability limit is satisfied, implicit otherwise) may be the way to go for these schemes.

Frequency-domain modelling

Complexity

At first sight, frequency- instead of time-domain modelling appears to be inefficient, because the frequency-domain approach requires the solution of a large sparse-matrix problem. However, if many shots need to be treated, results of the matrix LU-decomposition can be reused and the method may still be competitive (Marfurt and Shin, 1989). An efficient LU-decomposition can be obtained with a nested dissection ordering (George and Liu, 1981), which reduces the cost of a band-matrix LU decomposition by a factor of $O(n)$. Table 1 lists complexity estimates for two- and three-dimensional problems, discretised with n points per spatial coordinate. The number of shots is n_s and the number of frequencies n_ω. The stability limit of an explicit time-stepping method leads to an $O(n)$ number of time steps for a given time span. This explains the complexity estimate for the time-domain method. In the frequency domain, the LU decomposition is costly. However, for the 2D case, the method still may compete with the time-domain method if we assume that n_s and n_ω are both $O(n)$ and the $O(\log n)$ factor is small. In 3D, the complexity explodes and an iterative method appears to be necessary.

Table 1. Complexity of time- and frequency domain methods.

complexity	2D	3D
number of grid points	n^2	n^3
time domain method	$n_s\, O(n^3)$	$n_s\, O(n^4)$
band LU decomposition	$n_\omega\, O(n^4)$	$n_\omega\, O(n^7)$
LU decomposition with nested dissection	$n_\omega\, O(n^3)$	$n_\omega\, O(n^6)$
application of LU (nested dissection)	$n_s\, n_\omega\, O(n^2 \log n)$	$n_s\, n_\omega\, O(n^4 \log n)$

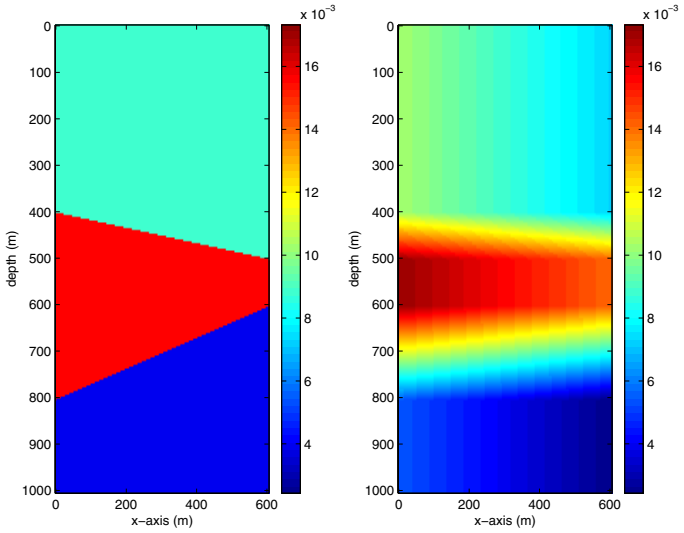

Figure 2. Original slowness model (left) and model used for preconditioner (right). The slowness scale is in $m^{-1}s$.

Table 2. Convergence for wedge model.

Freq (Hz)	Nr. of iterations
1	3
10	14
20	47
30	173
40	1508
50	> 5000

Iterative methods for Helmholtz-type equations for the interior problem have been hardly studied by numerical analysts and for good reasons. Here we will present results obtained for a method based on separation of variables.

Separation of variables

Separation of variables (Heikkola *et al.*, 2000) can be used to construct an exact solver for simple models. This solver can then be used as a preconditioner for BICGSTAB2 (Van der Vorst, 1992), a conjugate-gradient method for non-symmetric problems. Consider the simple constant-density acoustic operator $L = -k^2 - \Delta$ with laplacian Δ and wave-number $k = \omega/c$. Let $\kappa = k^2 = \kappa(x,y) + \kappa(z)$. First. let the problem be discretised on a $n_1 \times n_2 \times n_3$ grid. Then, the eigenvalues and eigenvectors of the one-dimensional operator $L_3 = -\kappa(z) - D_{zz}$, where D_{zz} is the discretisation of the second derivative in the z direction, are computed. For each of the n_3 eigenvalues, the projection of the original equation on the corresponding eigenvector is determined. In this way, n_3 independent two-dimensional problems of size $n_1 \times n_2$ in x and y are obtained. These are readily solved by nested dissection. The results for all the eigenvectors are finally combined into one solution. It should be noted that, because of absorbing boundary conditions, the method is not an exact solver even if the wave-number field is separable.

We have tested this idea for a two-dimensional problem. A first example is the wedge model shown in Fig. 2. The size of the

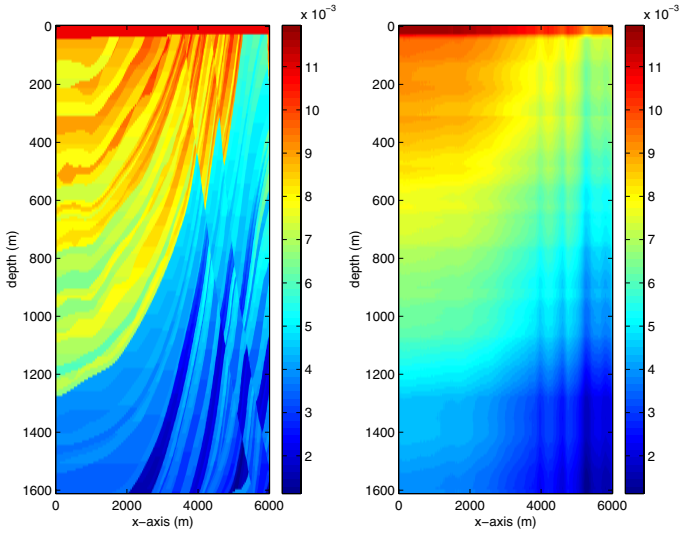

Figure 3. Part of the Marmousi model (left) and its separable approximation (right).

Table 3. Convergence for part of Marmousi.

Freq. (Hz)	Nr. of iterations
1	3
10	114
20	> 5000

model is 151 points in x and 251 points in z. The sampling is 4 m in each direction. The slowness is shown on the left of Fig. 2, and a separable approximation on the right. The latter is used for the preconditioner. The iterations are stopped either when their number exceeds 5000 or when the residual is smaller than 10^{-12}. Table 2 lists the number of iterations. Fast convergence is obtained for the lower frequencies, but the performance degrades for higher frequencies. A second example is based on a part of the well-known Marmousi model and is shown in Fig. 3. The grid sampling is 8 m and 751×201 points are used. Convergence results are listed in Table 3 and again show a degradation with increasing frequency.

Conclusions

Time-domain methods for the wave equation modelling have reached a level of maturity where they can routinely be used on an industrial scale for two-dimensional problems. Realistic 3D simulations are still costly but feasible on present-day hardware. Frequency-domain methods are competitive in two dimensions, thanks to the nested-dissection ordering, but not in three. Our experiences with a preconditioner based on separation of variables were disappointing.

Because frequency-domain methods do not offer a dramatic speed-up over time-domain methods, their usefulness for modelling appears to be limited. Their increase of code complexity over time-domain methods and the problem of finding an efficient solver for the three-dimensional case do not seem to justify their use. However, if these codes are to be used for processing purposes, in particular for migration and inversion, the balance changes. In that case, an additional order of magnitude can be gained by using only a limited number of frequencies (Pratt, 1990, 1999; Plessix *et al.*, 2001).

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