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An Approximate Solution to the Electromagnetic Problem for Two Half Spaces

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SUMMARY

We present analytical approximations for the computation of the diffusive electromagnetic field generated by a current source at the interface between two homogeneous half spaces. We obtain different expressions for the near and the far field. The method is applicable to arbitrarily large resistivity contrasts and is very fast. It can be useful for the validation of computer modelling codes and for the computation of the primary background field when a primary-secondary formulation is used.

Introduction

The accurate computation of the electric field, which requires the solution of the Maxwell equations in conducting media, is not an easy task in the majority of the geophysical settings. In the particular case of land measurements, the almost static response of the highly resistive air, the notorious air wave, is especially inconvenient.

Low frequencies, typically below 10 Hz, must be used in order to reach deep targets. For these low frequencies in conducting media, waves traveling with the speed of light may be ignored because their amplitudes are very small and their wavelengths larger than the size of the earth. Therefore, conduction currents are dominating over displacement currents.

Here we consider the problem of two homogeneous half spaces. We can use its solution for the validation of computer modelling codes and for the computation of the primary background field when a primary-secondary formulation is used. In the latter case, we need the electric field in a large part of the computational domain, wherever the actual conductivity differs from the homogeneous background, so we require a fast method for the evaluation of the primary field.

Raiche and Coggon (1975) provide exact expressions for two half spaces when one of them has zero conductivity and permittivity, with a fourier convention opposite to ours. Their formulas are wrong if source and receiver are positioned on the same vertical line, but this can be easily fixed. The more general case of a finite conductivity contrast between the two half spaces requires the numerical evaluation of Hankel transforms. We refer to a recent paper by Kong (2007) and references therein. We avoid the use of numerical Hankel transforms by approximations that allow for an evaluation of these transforms in closed form. Different approximations are required for the near and the far field.

Method

Problem statement

The Maxwell equations in a conductive medium read:

$$\sigma\hat{E} + \nabla\times(i\omega\mu)^{-1}\nabla\times\hat{E} = -J^e,$$

Here $\hat{E} = \hat{E}(x, y, z, \omega)$ is the electric field in the frequency domain, ω is 2π times the frequency, σ the electric conductivity, ε the electric permittivity, and μ is the magnetic permeability. For the low frequencies commonly used, we have $\omega\varepsilon/\sigma \to 0$.

We consider two homogeneous half spaces with conductivities σ_0 for $z < 0$ and σ_1 for $z > 0$. An electric point current source oriented in the x-direction is located at the origin. The expressions for the electric field in the Fourier domain are Sommerfeld-type integrals, also known as Hankel transforms. These are semi-infinite range integrals with Bessel function kernels, including a decaying function. There are methods to compute Sommerfeld-type integrals dealing with electromagnetic wave propagation that are more or less applicable to the electromagnetic diffusion problem; see Michalski (1998, 2005), among others. We can avoid their numerical integration or the use of fast Hankel transform (Anderson, 1979) by deriving approximation that allow for closed-form evaluation of the integrals.

Near-field approximation

In general, we can say that *short* distances are of the order of magnitude of the smallest skin depth, $\sqrt{2/(\omega\mu\sigma)}$, or less. A good approximation for the electric field for relatively short distances can be determined by expanding the expression of the vertical propagation coefficient Γ_0 when $\sigma_0 \to \sigma_1$. This approximation is valid for large values of the wave number α, which would lead to a progressively improving fitting for shorter distances.

The vertical propagation factor is given by:

$$\Gamma_n = \sqrt{i\omega(\alpha^2 + \mu_0\sigma_n)}, \quad \alpha = \sqrt{{\alpha_x}^2 + {\alpha_y}^2},$$

for $n = 0$ or 1 and with horizontal wave numbers α_x and α_y that have been rescaled by $\sqrt{i\omega}$. This results in

$$\Gamma_n = \Gamma_m + \frac{i\omega\mu_0\left(\sigma_n - \sigma_m\right)}{2\Gamma_m} - \frac{\left[i\omega\mu_0\left(\sigma_n - \sigma_m\right)\right]^2}{8\Gamma_m^3} + \frac{3\left[i\omega\mu_0\left(\sigma_n - \sigma_m\right)\right]^3}{48\Gamma_m^5} \ldots$$

where $m = 1-n$ and $n = 0$ or 1. Although this series may seem to diverge due to the increasing powers of the difference $(\sigma_n - \sigma_m)$ – for large values of α – the increasing power of the factor Γ_m in the denominator actually makes the series convergent. As a consequence, the original semi-infinite integral can be replaced by a series of simpler integrals that can be evaluated in closed form. This is a suitable approximation only for small values of horizontal and vertical offsets, $\rho = \sqrt{x^2 + y^2}$ and z respectively. Since ρ is in the argument of the Bessel function and z is in the exponential of the Sommerfeld-type integral, for large values of α and small ρ there will never be a strongly oscillatory behaviour, and for large α and small z there will never be a strongly damped behaviour.

As an example, Figure 1 presents a comparison of the result of the electric field (vertical component) in the upper half space $\hat{E}_{z;0}$ for the fast Hankel transform and the near-field approximation. For this particular case, the upper half space is lossless (i.e., $\sigma_0 \to 0$), the electric conductivity in the lower half space $\sigma_1 = 0.1$ S/m, and the frequency is 0.25 Hz. Note that the approximation for the near field is effectively valid for short vertical distances z and horizontal distances ρ, say within a radius of approximately twice the skin depth (i.e., 2×3 km = 6 km).

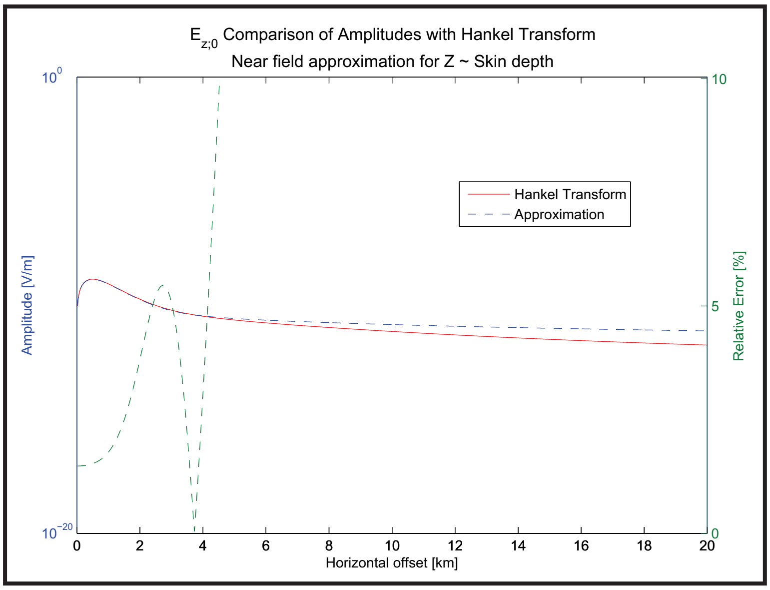

Figure 1: $\hat{E}_{z;0}(x, y = 0, z, \omega)$ resulting from the fast Hankel transform and from the near-field approximation for $z \approx$ skin depth ≈ 3 km.

In this particular example, the expression for the x- and z-components of the electric field in the upper half space can be approximated by

$$\hat{E}_{x;0} = \frac{\hat{J}_1^e}{2\pi\sigma_1}\left[(\partial_x^2 + \partial_z^2)(r^{-1}) - \partial_z\mathcal{I}\right], \quad \hat{E}_{z;0} = \frac{\hat{J}_1^e}{2\pi\sigma_1}\partial_x\mathcal{I}, \tag{1}$$

where

$$\mathcal{I} = \int_0^\infty \sqrt{i\omega}\Gamma_1 e^{-\sqrt{i\omega}\alpha|z|} J_0(\sqrt{i\omega}\,\rho\alpha)\,d\alpha \approx \frac{|z|}{r^3} - \frac{\kappa_1^2}{2}\log(r + |z|) + \ldots$$

We have introduced $\kappa_n = \sqrt{i\omega\mu_0\sigma_n}$. J_0 is the Bessel function of the first kind and order 0 and $r = \sqrt{\rho^2 + z^2}$. With air, we do not need approximations for the field components inside the conductive half space:

$$E_{z;1} = \frac{\hat{J}_1^e}{2\pi\sigma_1}\partial_x\partial_z\Phi_1, \quad E_{y;1} = -\frac{\hat{J}_1^e}{2\pi\sigma_1}\partial_x\partial_y\partial_z\Psi_1, \quad E_{x;1} = \frac{\hat{J}_1^e}{2\pi\sigma_1}\left(\partial_y^2\partial_z\Psi_1 - \partial_z^2\Phi_1\right).$$

Here $\Phi_n = e^{-\kappa_n r}/r$ and $\Psi_1 = I_0\left(\frac{1}{2}\kappa_1(r-z)\right)K_0\left(\frac{1}{2}\kappa_1(r+z)\right)$. Also, $E_{y;0} = \frac{\hat{J}_1^e}{2\pi\sigma_1}\frac{3xy}{r^5}$ in the air.

With a finite contrast, the lowest-order terms for an observation point in the upper half space provide

$$\hat{E}_{x;0} \approx \frac{\hat{J}_1^e}{2\pi}\left[\frac{\partial_x^2\Phi_0}{\sigma_0+\sigma_1} - \tfrac{1}{2}i\omega\mu_0\Phi_0\right], \quad \hat{E}_{y;0} \approx \frac{\hat{J}_1^e\partial_x\partial_y\Phi_0}{2\pi(\sigma_0+\sigma_1)}, \quad \hat{E}_{z;0} \approx \frac{\hat{J}_1^e\partial_x\partial_z\Phi_0}{2\pi(\sigma_0+\sigma_1)}.$$

For an observation point in the lower half space, we have approximations for $\hat{E}_{x;1}$, $\hat{E}_{y;1}$, and $\hat{E}_{z;1}$ that obey the same formulas but with σ_0 interchanged with σ_1 and κ_0 with κ_1.

Far-field approximation

In general, we can say that *large* distances are those beyond the skin depth associated with the half space with the lowest electrical conductivity. A good approximation for the electric field for relatively large distances can be determined by expanding the expression of the vertical propagation factor when $\alpha \to 0$. Because α is in the argument of the Bessel function and the exponential of the Sommerfeld-type integral, there will never be a strongly oscillatory and/or damped behaviour.

Since the approximation is only accurate for small α, large values of ρ are needed to have good solutions. However, for small α, the vertical wave-number is similar to that of a planar field "propagating" in the vertical direction, which corresponds to an accurate representation for large z. For small z, large values of α are required to have substantial damping to make the contribution to the total sum in the integral go to zero. Hence, the far-field approximation is valid for larger horizontal offsets ρ, but also for large vertical offsets z.

As an example, Figure 2 presents a comparison of the result of the electric field (vertical component) in the upper half space $\hat{E}_{z;0}$ for the fast Hankel transform and the far-field approximation. The same model has been used as for the previous example. Both are profiles for a vertical distance z around the skin depth (3 km) and 5 times the skin depth. Note that the approximation for the far field is effectively valid for large vertical distances z and all horizontal distances ρ, say beyond a radius of approximately 4 times the skin depth, i.e., 12 km.

For this particular example, the integral can be approximated by a series

$$\mathcal{I} = \kappa_1\left[1+\kappa_1^{-2}\partial_z^2\right]^{1/2}\frac{1}{r} = \kappa_1\left[(r^{-1}) + \tfrac{1}{2}\kappa_1^{-2}\partial_z^2(r^{-1}) - \tfrac{1}{8}\kappa_1^{-4}\partial_z^4(r^{-1}) + \ldots\right],$$

which then can be inserted into equation (1).

Approximation for short vertical distances z

For large contrasts of σ, the combination of near and far-field approximations may be insufficient to cover the whole 3-D space. Especially when a lossless medium is involved, part of the intermediate field (i.e., between 2 and 3 times the skin depth) cannot be computed accurately.

We can apply an approximation for small z to obtain the intermediate field near the interface. For small z and arbitrary σ_0 and σ_1, the Sommerfeld-type integrals can be evaluated analytically by taking advantage of the fact that $\Gamma_0 \approx \Gamma_1 - \sqrt{\kappa_1^2-\kappa_0^2}$ in the exponential. When $\sigma_0 \to 0$, the expression for the integral becomes

$$\mathcal{I} \approx e^{\kappa_1|z|}\partial_z^2\left\{I_0\left(\tfrac{1}{2}\kappa_1\left(r-|z|\right)\right)K_0\left(\tfrac{1}{2}\kappa_1\left(r+|z|\right)\right)\right\},$$

which can be inserted into equation (1).

I_0 and K_0 are modified Bessel functions of the first and second kind, both of order 0.

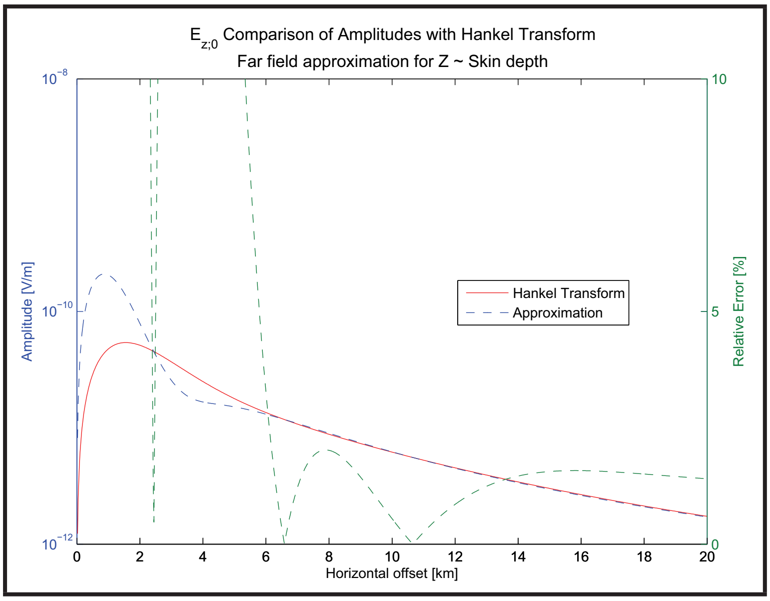

Figure 2: $\hat{E}_{z;0}(x, y = 0, z, \omega$ resulting from the fast Hankel transform and from the far-field approximation for $z \approx$ skin depth $\approx$ 3 km.

Interpolation

We can quickly compute the full space solution from a few slices by interpolation, using the relations

$$\hat{E}_x(x, y, z, \omega) = \hat{E}_x(x = y, z, \omega) + \cos(2\phi)\hat{E}_y(x = y, z, \omega),$$
$$\hat{E}_y(x, y, z, \omega) = \sin(2\phi)\hat{E}_y(x = y, z, \omega), \quad \hat{E}_z(x, y, z, \omega) = \cos(\phi)\hat{E}_z(x, y = 0, z, \omega).$$

Here $x = \rho\cos\phi$ and $y = \rho\sin\phi$.

Conclusions

We have presented a method for computing the electric field of a point current source at the flat interface between two homogeneous media with an arbitrary conductivity constrast. We only listed approximations for the the x- and z-components of the electric field in the special case when one of the half spaces has a zero conductivity, as happens for air. Different approximations were found for the near field, for the far field, and for short vertical offsets, respectively. The y-component in air can be found in closed form, as can the three electric field components in the conductive half space.

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