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The Dynamic Acoustic Inverse Problem in Laterally Homogeneous Media

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SUMMARY

Inverse problems are ubiquitous in many fields of science and can only be solved directly in very specific simple cases. There are many kinematic and dynamic inverse solution methods that require a good starting model to be able to proceed to a correct reconstruction. We investigated the application of the Gelfand-Levitan method to the problem of finding acoustic parameters from seismic data. The method allows for a direct reconstruction of the model from the data if the medium is laterally homogeneous. A synthetic numerical example shows that we can reconstruct a smooth version of a given 1D model up to a predictable depth for typical marine data that include multiples.

Introduction

Inverse problems occur in many branches of science and mathematics where the values of some model parameters must be obtained from the observed data. Examples of inverse problems can be found in geophysics, medical imaging (such as computed axial tomography and EEG), remote sensing, ocean acoustic tomography, non-destructive testing, and astronomy. Inverse problems are typically ill-posed, in contrast to the well-posed problems obtained when modelling physical situations for which the model parameters or material properties are known. Of the three conditions for a well-posed problem suggested by Jacques Hadamard – existence, uniqueness, and stability of the solution(s) – the condition of stability is most often violated. While inverse problems are usually formulated in infinite-dimensional spaces, limitations to a finite number of measurements and the practical consideration of recovering only a finite number of unknown parameters may lead to the problems being recast in a discrete form. In this case, the inverse problem will typically be ill-conditioned. Regularization may be used to introduce mild assumptions on the solution and to prevent overfitting. Because for most inverse problems, it is very complicated to obtain a direct result by analytical means, there are many practical techniques that attempt to find the model parameters by matching modelled to observed data. For such methods, the initial guess of the model parameters plays a big role. In seismics, inverse problems can be divided into kinematic and dynamic problems. Kinematic problems deal only with the propagation times of signals through a medium, whereas dynamic problems involve the full recorded wavefield.

With seismic data, the inverse problem can be stated as follows: given the recorded data at the Earth's surface or in a well, reconstruct medium parameters such as the P-wave velocity V_p, S-wave velocity V_s, density ρ, etc. Kinematic methods like Kirchhoff migration can reconstruct the seismic horizons, the boundaries between layers with different parameters. Dynamic methods like full waveform tomography allow the reconstruction of the model parameters between layers as well as the horizons. But both methods are similar in the sense that they need a proper starting model, the reference medium that is the initial guess for the algorithm. Here, we investigate a mathematical method that directly provides an initial model for further inverse reconstruction.

Theory

In order to find a correct starting model, one can try to estimate that model by analytical means, i.e., do it mathematically for a simple averaged model for which a unique solution can be obtained. Here, we will consider an acoustic medium. This means that only P-waves are present. Also, for simplicity, we restrict ourselves to the 2D case. Extension to the 3D case is trivial.
The acoustic 2D equation reads

$$\frac{1}{\rho v^2}U_{tt}(x,z,t)-\frac{\partial}{\partial z}\left(\frac{1}{\rho}U_z(x,z,t)\right)-\frac{\partial}{\partial x}\left(\frac{1}{\rho}U_x(x,z,t)\right)=0\ , \qquad (1)$$

where v is the acoustic velocity, ρ is the density, and U_{tt} is the second time-derivative of the pressure, $U(x; z; t)$. The spatial derivatives in the horizontal and vertical directions are U_x and U_z, respectively. As usual in the mathematical formulation of a problem, we should specify a zero initial condition: $U(x; z; t<0) = 0$, which means that there was not any disturbance before the source started. The boundary condition here serves as a source: $U(x; z=0; t) = f(t)\cdot\delta(x)$, where $f(t)$ is the source signature and $\delta(x)$ is the Dirac delta function representing a point source. In order to solve the inverse problem, we will use the recorded data:

$$U_z(x, z=0, t) = R(x, t)\ . \qquad (2)$$

Thus, the inverse problem consists in obtaining model parameters $v(x; z)$ and $\rho(x; z)$ from initial data $R(x; t)$. We solve the inverse problem with the Gelfand-Levitan method, which is restricted to horizontally homogeneous media, i.e., Earth models where velocity and density depend only on the depth z, although the method can also be extended to cases with a slight lateral inhomogeneity [4].

The Gelfand-Levitan method

The Gelfand-Levitan method can be used for the reconstruction of acoustic parameters. This method was originally developed in 1951 for the Schrödinger equation by Gelfand and Levitan [1], and by Kay and Moses [2]. Blagovestchenskii [3] applied the method to the 1D wave equation. Also, the method was extended to the 2D and 3D acoustic wave equation. For other theoretical results, see [5]–[6].

The Gelfand-Levitan method (GLM) has the remarkable property that existence and uniqueness of the solution of the inverse problem can be proven in the case of a horizontally homogeneous medium with the following types of source time functions: $\delta(t)$, $\delta'(t)$, $\delta''(t)$, …, $\delta^{(n)}(t)$, …

Let us assume that the velocity $v(x; z)$ depends only on depth, i.e. $v = v(z)$, and that the density has a constant value, taken as 1 for simplicity. Let $R(t; k_x)$ be the Fourier transform in x of the initial data $R(t; x)$ at a single value of the wavenumber k_x:

$$R(t,k_x) = \int R(t,x)\cos(x\,k_x)dx. \tag{3}$$

Let $R_+(t, k_x)$ be the odd continuation of the function $R(t, k_x) - \delta(t)$ to negative time. Then, for a delta function source, GLM leads to the following integral equation with real parameter y:

$$\Theta(t,y) - \frac{1}{2}\int_{-y}^{y} d\tau\, \Theta(t,y)\int_{\tau}^{y} d\eta\, R_+(t-\eta,k_x) = \frac{1}{2}\int_{-y}^{y} d\tau\, R_+(t-\tau,k_x). \tag{4}$$

One can obtain the velocity $v(z)$ at an arbitrary value k_x by using the solution $\Theta(t,y)$ and the following equations:

$$\frac{\left(\sqrt{\sigma(y)}\right)''}{\sqrt{\sigma(y)}} + k_x^2\frac{1}{\sigma^2(y)} = -2\left(\lim_{t\to y}\Theta(t,y)\right)'_y; \qquad z = \int_0^y v(y)dy; \qquad v(y) = \frac{1}{\sigma(y)};$$

To solve the ordinary differential equation, one must know $v(0)$ and $v_z(0)$, the velocity and its vertical derivative at the surface, which are usually known, especially for a marine acquisition. Eq. 4 is an integral Fredholm equation of the second kind and can be solved by the means of linear algebra when discretized.

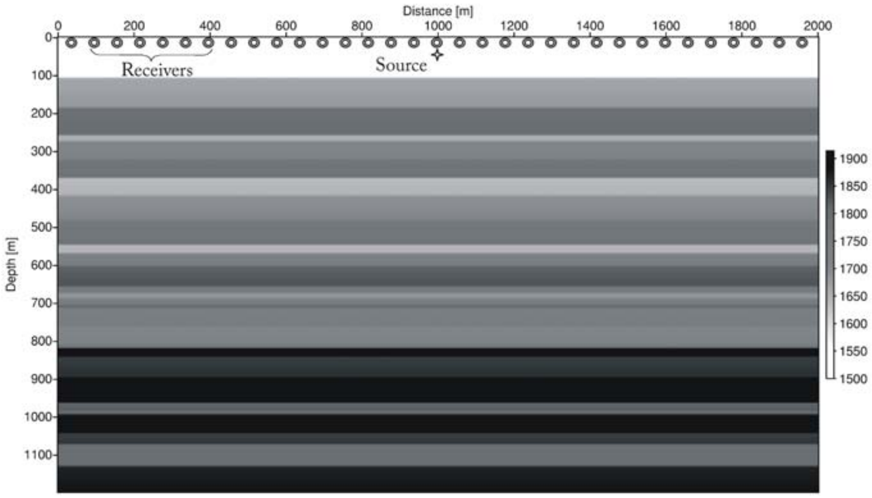
Because a delta function is a mathematical abstraction, it can only approximately be represented in a numerical scheme. Instead of a delta function, one can use a Ricker wavelet, the second time-derivative of a Gaussian function, to avoid the singularity. Also, a Gaussian of narrow width is the simplest way to represent the delta function source numerically. This means we can actually perform a double numerical integration of the data in time to obtain the delta function approximation.

Numerical example

A horizontally homogeneous acoustic model with a water layer on top is shown in Fig. 1. A set of seismic receivers with 12-m spacing is located at a depth h = 4 m. A point seismic source is placed at 14-m depth and at the centre of the aperture. Fig. 2 shows how the velocity changes with depth. The GLM can be implemented in the following way. Seismograms that are recorded by receivers after initiation of the source serve as input data. First we perform a Fourier transform of the data in the space domain (over x). In the discrete case, eq. 3 can be easily rewritten as

$$R(t,k_x) = \Delta x\sum_{n=1}^{N} R(t,x_n)\cos(x\,k_x),$$

where x_n are the positions of the receivers, numbered from *1* to *N*. It is important to mention that initially we need a zero boundary condition at the surface of the model, which is the same as having zero pressure at the free surface. According to eq. 2, the recorded data we actually need in this case is the vertical derivative of the pressure. We can easily obtain this derivative from the recorded pressure P at some small depth h by subtracting the free-surface pressure $P(0) = 0$ and dividing by the depth h.



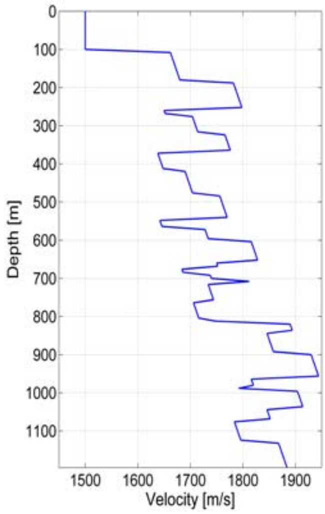
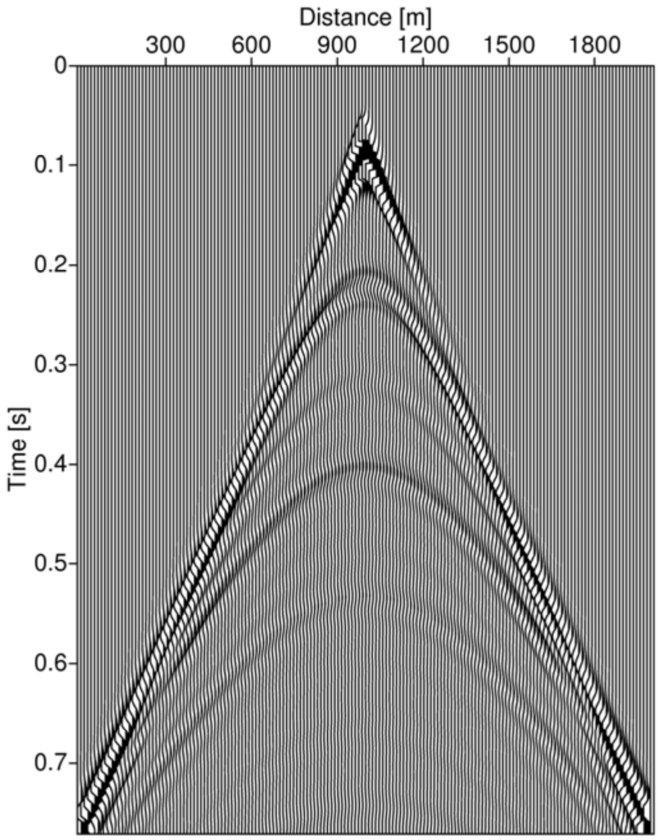


Figure 1. The P-velocity model used for the numerical experiment. The grey scale lists velocities in m/s.

Figure 2 Plot of the P-velocity versus depth.

We computed the pressure seismogram shown in Fig. 3 with a finite-difference code, using a Ricker wavelet with a peak frequency of 30 Hz. Multiple reflections from the sea bottom and free surface are clearly visible. These multiples are harmful for kinematic methods but they are fully used in our dynamic approach since it involves the full wavefield. Fig. 4 displays the dataset after applying the spatial Fourier transform.



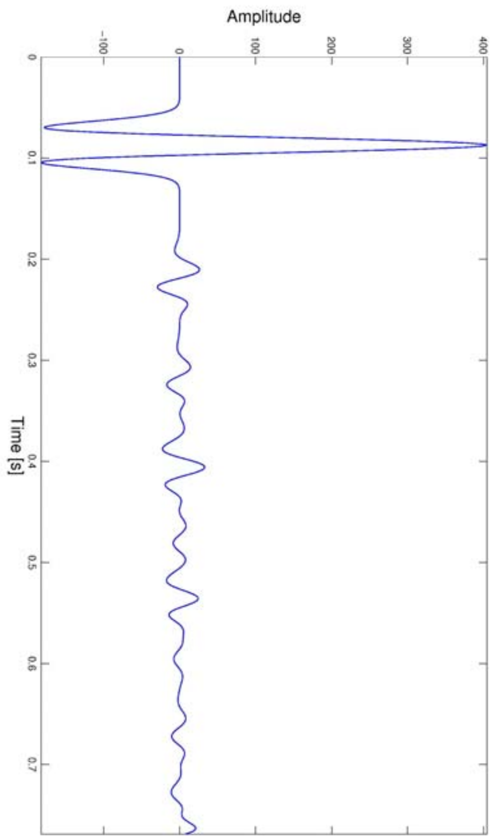


Figure 3 Acoustic pressure wavefield computed with a finite-difference code.

Figure 4 Dataset after applying a spatial Fourier transform.

Fig. 5 shows the result of the velocity reconstruction. It represents the true profile reasonably well if the spatial extent of the variations is bigger than the dominant wavelength. Previously [3], it was shown that the method provides better results for smooth velocity models. The physical depth limit of the reconstruction is given by the theory [3,5,6]. For the current experiment, the theoretical depth limit is approximately 550 m, which is confirmed by the experiment.

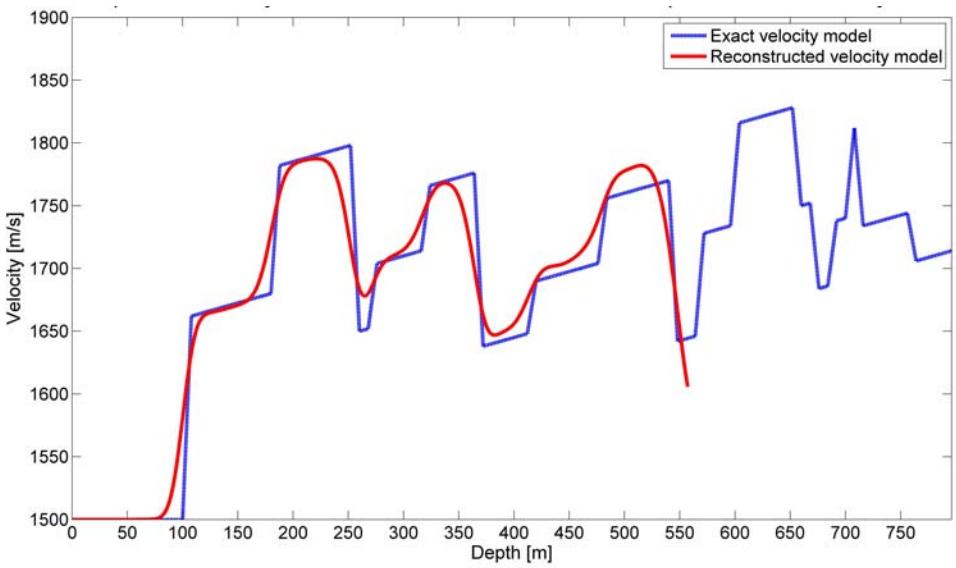


Figure 5 Comparison of the reconstructed and true velocity model.

Conclusion

We applied the Gelfand-Levitan method for the reconstruction of a solution of the dynamic acoustic inverse problem in a laterally homogeneous media. The method is proved to be efficient and accurate for smooth velocity distributions. The numerical experiment shows that the theoretical depth limit coincides with the actual one. One of the main disadvantages of the method is low stability in a presence of noise. The method can be useful for the construction of a starting model for, for instance, full waveform inversion. This is important for inversion in geophysics and other fields of science.

Acknowledgements

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