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Spectral Sensitivity Analysis of FWI in a Constant-gradient Background Velocity Model

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SUMMARY

Full waveform inversion suffers from local minima, due to a lack of low frequencies in the data. A reflector below the zone of interest may help in recovering the long-wavelength components of a velocity perturbation, as demonstrated in a paper by Mora. Because smooth models are more popular as initial guesses for FWI, we consider the Born approximation for a perturbation in a reference model with a constant velocity gradient. Analytic expressions are found that describe the spatial wavenumber spectrum of the recorded seismic signal as a function of the spatial spectrum of the inhomogeneity. We study this spectrum in more detail in terms of sensitivities. Since the velocity model is inhomogeneous near the perturbation, we need to specify its depth. We compare these sensitivities and find that low frequencies are extremely useful for the first stages of inversion – a well-known fact. However, also the high-frequency data contain some information about the low spatial wavenumbers in the perturbation, which offers opportunities for inversion in the absence of low frequencies in the data. We observe that the longer wavenumbers are better resolved in the deeper parts of the model if large enough offsets are available.

Introduction

Full waveform inversion usually needs low frequencies in the data to recover structures with a slow spatial variation. A reflector below the zone of interest may, however, help in recovering the long-wavelength components of a velocity perturbation, as demonstrated by Mora (1989). With the Born approximation for the perturbation in a reference model consisting of two homogeneous isotropic acoustic halfspaces, analytic expressions are found that describe the spatial spectrum of the recorded seismic signal as a function of the spatial spectrum of the inhomogeneity. Kazei et al. (2012) studied this spectrum in more detail by separately considering direct, reflected and head waves. Here, we generalize this sensitivity analysis to a velocity model with a constant vertical gradient, where the only waves from a point source are the diving ones. The sensitivity will then depend on depth and frequency. With a modification of Mora's approach, we investigate the spectral resolution at a given depth and frequency and compare the results for the gradient model with our earlier sensitivity results for a model consisting of two constant-velocity acoustic halfspaces.

Quasi-plane waves

We consider the 2-D constant-density acoustic wave equation. The pressure $\hat{p}(\mathbf{r},t)$ as function of position $\mathbf{r}$ and time t should obey the standard wave equation

$$\frac{1}{c^2}\frac{\partial^2 \hat{p}}{\partial t^2} - \Delta \hat{p} = \hat{f}(\mathbf{r}-\mathbf{r}_0,t) \xrightarrow{\text{Fourier } \omega, x} \left[\frac{\omega^2}{c^2(z)} - k_x^2 + \frac{\partial^2}{\partial z^2}\right] p(k_x,z,\omega) = f(k_x,z,\omega), \qquad (1)$$

with velocity $c(\mathbf{r})$ and source function $\hat{f}(\mathbf{r}-\mathbf{r}_0,t)$. The assumption of a horizontally uniform velocity model, $c(z)$, and Fourier transforms in time and in the horizontal coordinate, x, lead to a 1-D Helmholtz equation with the transformed pressure, $p(k_x,z,\omega)$, depending on the horizontal wavenumber k_x, depth z and angular frequency ω. A solution of this wave equation in a horizontally uniform media may be decomposed into 'quasi-plane' waves (Brekhovskikh and Godin, 1998):

$$\hat{p}(\mathbf{r},t) = \int_{-\infty}^{\infty} dk_x' e^{-ik_x' x} \int_{-\infty}^{\infty} d\omega' e^{i\omega' t} p(k_x',z,\omega'). \qquad (2)$$

In any local domain where the velocity is constant and the source function, f, equals zero, a solution of equation 1 is a linear combination of two exponentials, $\exp\left(\pm iz\sqrt{(\omega/c)^2 - k_x^2}\right)$, corresponding to upgoing $(+)$ and downgoing $(-)$ wavefields. Note that our choice of Fourier convention is the conjugate of the usual one. A single quasi-plane wave immediately reduces to a standard plane wave if the medium is homogeneous:

$$\hat{p}(\mathbf{r},t) = e^{-ik_x x + i\omega t} p(k_x,z,\omega) \xrightarrow{\text{homogeneous media}} \hat{p}(\mathbf{r},t) = e^{-ik_x x \pm iz\sqrt{(\omega/c)^2 - k_x^2} + i\omega t}. \qquad (3)$$

Locally, we have a solution to equation 1 of the form $A(z)\exp(izB)$, where A is an amplitude that varies slowly with z and B is a constant that is positive for an upgoing wave. Therefore, we can define the propagation vector at depth z as $(k_x, B(z))$, analogously to that for a plane wave. Of course, this technique to find the propagation direction should be restricted to a region where this approximation is valid.

Constant velocity gradient

The exact frequency-domain solution for a point source in infinite medium with constant velocity gradient was established by Pekeris (1946) for the 3-D and by Kuvshinov and Mulder (2006) for the 2-D case. We use the notation of the last for the Green function of equation 1,

$$G(k_x,\hat{z}_>,\omega) = \exp(-ik_x x_s)\sqrt{\hat{z}_<\hat{z}_>} I_\nu(k_x\hat{z}_<) K_\nu(k_x\hat{z}_>), \qquad (4)$$

for a source at the point (x_s,z_s) and receiver at the depth z_r. Here, I_ν and K_ν are modified Bessel functions of order $\nu = i\sqrt{(\omega/\alpha)^2 - 1/4}$. The velocity is $c(z) = c_0 + \alpha z$, $\hat{z} = z + c_0/\alpha$ for all z, $\hat{z}_< = \min(\hat{z}_s,\hat{z}_r)$

and $\hat{z}_> = \max(\hat{z}_s, \hat{z}_r)$. Later on, we will consider scattering by a perturbation at depth $z_p > z_a$, with $a = s$ or r and then $\hat{z}_< = \hat{z}_a$ and $\hat{z}_> \simeq \hat{z}_p$. The asymptotic expansions for high frequencies or large imaginary ν have attracted considerable attention in the second half of 20th century (Olver, 1954; Dunster, 1990, e.g.). Olver (1954) derived formulas for I_ν and K_ν at high frequencies. With the definition $Y = \sqrt{Z^2 + \nu^2}$, these formulas are

$$I_\nu(Z) \sim \frac{1}{\sqrt{2\pi}} \left(\frac{Z}{Y+\nu}\right)^\nu \frac{e^Y}{\sqrt{Y}}, \quad K_\nu(Z) \sim \sqrt{\frac{\pi}{2}} \left(\frac{Z}{Y+\nu}\right)^{-\nu} \frac{e^{-Y}}{\sqrt{Y}}. \tag{5}$$

Gupta (1965) used these asymptotic expansions for the calculation of the reflection coefficient of a linear transition layer and proved them to coincide with the standard 1-D ray method to first order. Unfortunately, Gupta had to use Olver's formulas at the boundaries of their domain of validity, so his conclusions are doubtful. Expansions in ascending powers of Z for the modified Bessel functions can be found in (Abramowitz and Stegun, 1964, e.g.). This series, valid for all purely imaginary ν, have an infinite radius of convergence in the Z-plane and no singularities on the positive Z semi-axis:

$$I_\nu(Z) = (Z/2)^\nu \sum_{s=0}^{\infty} \frac{(Z/2)^{2s}}{s!\Gamma(\nu+s+1)}, \quad K_\nu(Z) = \frac{\pi[I_{-\nu}(Z) - I_\nu(Z)]}{2\sin(\nu\pi)}. \tag{6}$$

From formula 4, we see that if k_x is small, corresponding to a wave propagating almost in the vertical direction, then Z becomes small too and the first term in the series for I_ν is very close to the sum of the whole series. At small depths and angles of incidence we have

$$I_\nu(Z) \simeq (Z/2)^\nu = \exp\Big(\nu \ln(Z/2)\Big) \sim \exp\Big(\nu \ln(Z_0/2)\Big) \exp\Big(\nu \Delta Z/Z_0\Big). \tag{7}$$

Here, we have assumed that we are near a small but non-zero Z_0. We let $Z = Z_0 + \Delta Z$ and can capture the plane-wave part of the exact solution by using a Taylor expansion of the power in equation 7. The imaginary part of ν defines the direction of wave propagation of I_ν solution. If $\mathrm{Im}(\nu) > 0$, the wave is upgoing, otherwise it is downgoing. If the real part of ν is positive, then I_ν may be disregarded in equation 6 and formula 5 shows good agreement with the total sum for K_ν at relatively small values of Z. Nevertheless, for wave propagation at not too low frequencies, $\omega/\alpha > \frac{1}{2}$, so ν is purely imaginary ν and the $I_{\pm\nu}$ functions are of the same absolute value. In this case, we have upgoing as well as downgoing waves in the K_ν solution. Finally, we should mention that Olver's asymptotics proved to be good for I_ν functions up to the turning point and the K_ν functions have to be decomposed into $I_{\pm\nu}$ functions to obtain a proper asymptotical expansion at purely real frequencies in the same region.

Consequences for inversion

We modify Mora's (1989) technique of sensitivities to a background model with a constant velocity gradient. The Born approximation is used for the scattered field. After Fourier transforms in time and in the horizontal coordinates of source and receiver, the wavefield perturbation is given by

$$\delta u_{sg}(k_s, k_r) = \int k_0^2 \delta W(\mathbf{r}) G(k_s, \mathbf{r}) G(k_r, \mathbf{r}) d\mathbf{r}, \quad k_0 = \frac{\omega}{c_0}, \tag{8}$$

where $\delta W(\mathbf{r})$ is a relatively small perturbation of the background squared slowness. With both Green's functions in analytical form and a target area around a depth z_p deeper than the sources and receivers at z_a ($a = s$ or r), we define the following coefficients at z_p: $C(k_a) = \sqrt{\hat{z}_a \hat{z}_p} I_\nu(k_a \hat{z}_a)$ and $C_\pm(k_a) = \mp\frac{\pi}{2\sin(\nu\pi)} I_{\pm\nu}(k_a \hat{z}_p)$. Then, introducing local coordinates (x, z) with their origin at an expected centre of the perturbation, (x_p, z_p), we obtain

$$\delta u_{sg} \sim C(k_s)C(k_r) \int dx\, e^{i(k_s+k_r)x} \int dz\, \delta W(x,z) \Big[C_-(k_s)e^{-iB(k_s)z} + C_+(k_s)e^{+iB(k_s)z}\Big] \Big[C_-(k_r)e^{-iB(k_r)z} + C_+(k_r)e^{+iB(k_r)z}\Big]. \tag{9}$$

This reveals a simple linear dependence between the spatial Fourier transforms of a perturbation in the model parameters and a perturbation of the single-frequency data on the horizontal coordinates of the source and receiver, just as in the case of a perturbation in a homogeneous layer (Mora, 1989). An appropriate constant $B(k_a, z_p)$ can be found either numerically by evaluating logarithmic derivatives of $I_{\pm\nu}$, leading to $B_{exact} = -i\frac{\partial}{\partial z}\ln I_\nu\left(k_a(z+\hat{z}_p)\right)$ at $z = 0$, or by using Olver's approximation, providing $B_{ray} = \sqrt{\omega^2/c^2(z_p) - k_a^2}$ for high frequencies. The comparison of these coefficients shows that they don't differ much up to the turning point, even for low frequencies. Therefore, we can use the ray approximation for B in the next section.

Sensitivities

The relation between the spatial wavenumber spectrum of the wavefield perturbation and of the inhomogeneity can be expressed as the sensitivity

$$S(K_x, K_z; \omega, z_p) = \left| \frac{\delta u_{sg}\left(k_s(K_x, K_z), k_r(K_x, K_z)\right)}{\delta \widetilde{W}(K_x, K_z)} \right|. \tag{10}$$

Here K_x, K_z are coordinates in the spatial perturbation spectrum $\delta\widetilde{W}(K_x, K_z)$ (Wu and Toksöz, 1987, e.g.). Figures 1–3 show that at lower frequencies, the sensitivities are more uniform. This makes

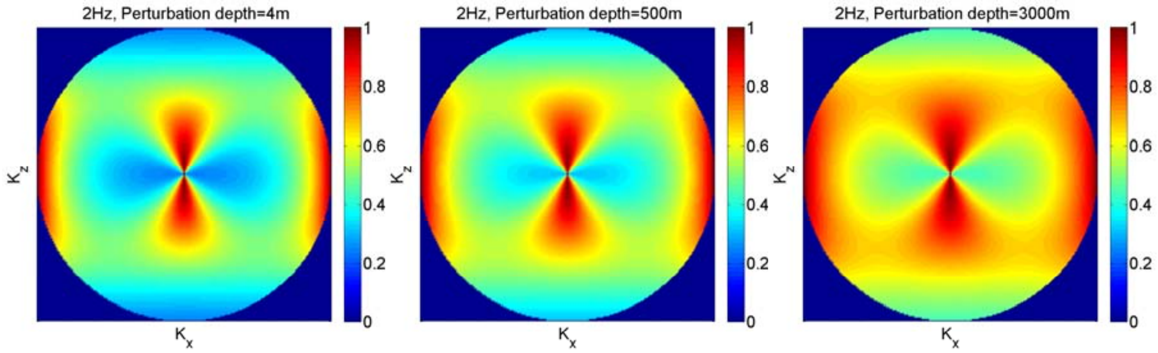

Figure 1 *Sensitivities for small (4 m, left), intermediate (500 m, center), and large (3000 m, right) depths of the perturbation. The scales of the diagrams are proportional to* $\omega/c(z)$ *(*$c_0 = 1500\,m/s$*,* $\alpha = 0.5$*,* $\omega = 2\,Hz$*).*

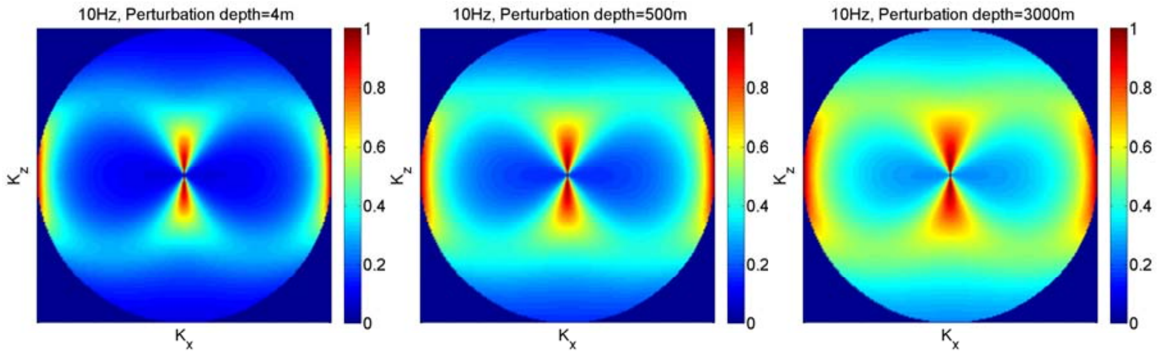

Figure 2 *Same as Figure 1, but for* $\omega = 10\,Hz$*.*

inversion at low frequencies better conditioned in the spectral domain. At high frequencies, the spectral sensitivities are almost zero in the larger part of the domain, but there is some non-zero sensitivity at low wavenumbers of the perturbation near the centre of the inhomogeneity. The high sensitivities near the vertical edges as well as the vertical peak in the centre are produced by waves travelling almost horizontally, as one can conclude by considering vectors pointing to the source and receiver along rays

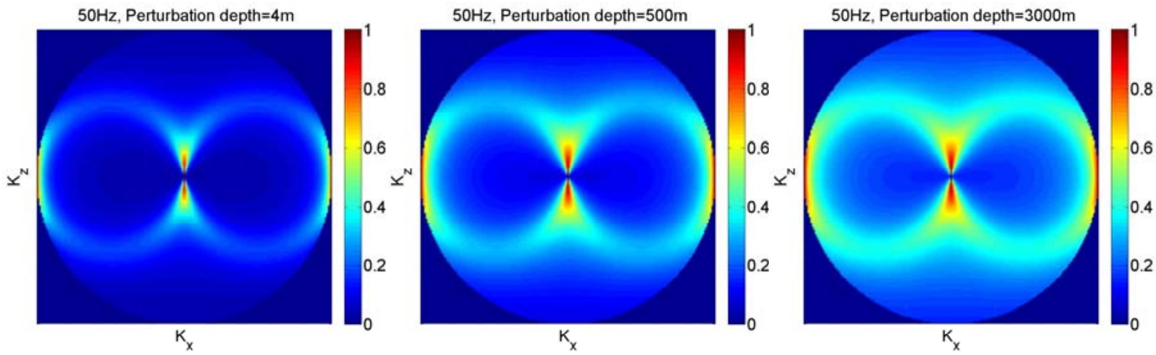

Figure 3 *Same as Figure 1, but for* $\omega = 50\,Hz$.

as Devaney (1984),Wu and Toksöz (1987) and Mora (1989) did for homogeneous background models. These peaks are related to rather low vertical wavenumbers in the perturbation spectrum and will be present if the offset is sufficiently long to allow for diving waves that travel nearly horizontal at that depth. The bright circular regions in the sensitivities at high frequencies bare similarity to those of the head waves at very low contrasts (Kazei et al., 2012).

Conclusions

Low frequencies are very useful for the first stages of FWI. However, large offsets in acquisition can help to mitigate the lack of very low frequencies in the data. We have shown that waves travelling almost horizontally in a model with a constant vertical velocity gradient can help to reconstruct the low spatial wavenumbers of an acoustic velocity perturbation. Even if these waves have only high frequencies, their vertical slowness vectors can be small enough to resolve the low vertical wavenumbers in the perturbation spectrum.

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