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A Comparison of Triangular Mass-lumped Finite Elements for 2D Wave Propagation

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SUMMARY

Mass-lumped continuous finite elements provide accurate solutions of the second-order acoustic or elastic wave equation in complex geological settings, in particular in the presence of topography and large impedance contrasts. Elements of higher polynomial degree have better accuracy than those of lower degree but are also more costly. Here, the performance of elements of degree 1 to 6 is compared for a simple acoustic test problem. The element of degree 6 is new. The numerical test confirm the increase of accuracy with the polynomial degree of the basis functions. In terms of computational cost, the element of degree 4 performs best in the specific example. For higher degrees, the cost of having extra nodes and a more restrictive stability constraint on the time step can no longer be compensated by having a smaller number of larger triangular elements. Only at very high accuracy, the new element of degree 6 wins.

Introduction

Finite-difference methods for modelling wave propagation are less accurate in the presence of large impedance contrasts or topography. Finite-elements maintain their formal accuracy if element boundaries follow the interfaces between contrasts. Higher-order methods are usually more efficient than lower-order methods. The standard finite-element discretization of the second-order wave equation results in a large sparse mass matrix. One way to avoid the computational cost of inverting this matrix is mass-lumping. On rectangular-shaped elements, this leads to the well-known spectral methods based on Legendre-Gauss-Lobatto points. Triangles and tetrahedra offer more flexibility in meshes, but mass lumping is less trivial in that case. If one wants to maintain a symmetric distribution of nodes inside the element, a recipe for elements of arbitrarily high degrees is not available.

For the linear element with shape functions of polynomial degree 1, mass lumping works well, but higher-order elements lose their accuracy. To preserve accuracy, polynomials of still higher degree in the interior that vanish on the edges can be added (Fried and Malkus, 1975; Tordjman, 1995; Cohen et al., 1995a). The construction of these elements is far from trivial and requires the solution of a nonlinear system with polynomials that has exponential complexity, causing symbolic methods based on Gröbner bases to fail for higher degrees. Also, numerical optimization methods have difficulties. Up till now, there exist mass-lumped triangular elements of degree 1 and 2 (Fried and Malkus, 1975), 3 (Tordjman, 1995; Cohen et al., 1995a), 4 (Mulder, 1996), and 5 (Chin-Joe-Kong et al., 1999) on the edges, as well as tetrahedral elements of degree 2 (Mulder, 1996) and 3 (Chin-Joe-Kong et al., 1999). Here, a new element of degree 6 is presented. The triangular elements of degree 6 and 7, constructed by Giraldo and Taylor (2006), have their accuracy too low by one order for the second-order wave equation.

Elements of higher degree have more nodes per element and also a smaller stability limit for explicit time stepping. Because of their better accuracy, elements of larger size can be used, but it is not obvious if the reduction in the number of elements can compensate for the increased cost per element and the smaller time step. This is investigated by a numerical test.

Method

The 2D acoustic wave equation,

$$\frac{1}{\rho c^2}\frac{\partial^2 p}{\partial t^2} - \nabla \cdot \frac{1}{\rho}\nabla p = w(t) f,$$

with pressure $p(t,\mathbf{x})$, source term $f(\mathbf{x})$ and wavelet $w(t)$ in a medium with density $\rho(\mathbf{x})$ and velocity $c(\mathbf{x})$ is discretized by triangular finite elements. The solution, p, is represented by polynomials of degree M in each triangle. To enable mass lumping without loss of accuracy, we have to include polynomials of degree $M_f \geq M$ in the interior that vanish on the edges. Mass lumping is then equivalent to numerical quadrature and should be exact for polynomials up to degree $Q = M + M_f - 2$. This requirement leads to a system of equations that is linear in the integration weights w and polynomial in the coefficients that describe the nodes. Only symmetric arrangements of nodes will be considered that agree with the symmetry of the triangle. The reference element has vertices $(0,0)$, $(1,0)$ and $(0,1)$. Given one node with coordinates (α,β), the other nodes in the same equivalence class follow by taking the first two elements of all permutations of the set $\{\alpha,\beta,1-\alpha-\beta\}$. Table 1 lists the 6 different equivalence classes for the triangle.

Table 1 *Equivalence classes for the triangle. For each class, one of the nodes is specified on the reference element. The number of nodes per class is n_c.*

class	node	n_c	type
1	$(0,0)$	3	vertices
2	$(1/2,0)$	3×1	edge midpoints
3	$(\alpha,0)$	3×2	interior edge points
4	$(1/3,1/3)$	1	centre
5	(α,α)	3	interior
6	(α,β)	6	interior

Consistency conditions (Keast, 1986; Chin-Joe-Kong et al., 1999) limit the choice of integration nodes to cases where the number of unknowns, involving the parametrization of the nodes and the integration weights, equals the number of equations. This leads to so-called rule patterns. Here, this requirement is relaxed by allowing the number of unknowns to exceed the number of equations. The solution of the nonlinear system can in theory be found by constructing a Gröbner basis for the system. Given the exponential complexity of the problem, this soon becomes impossible on present-day computers with increasing degree. An alternative is numerical root finding, which has its own problems as there are many local minima and the conditioning of the underlying linearized problem, if Newton's method is used, can be poor. Nevertheless, the last approach has provided new elements of degree 6.

Results

New elements

Table 2 lists the integration weights and node positions for elements of degree 1 to 6. Those of degree 6 with 46 nodes are new. Version 1, unfortunately, is degenerate and not uni-solvent. Version 2 is special in the sense that the system of polynomial equations has less equations than unknowns. Several solutions were found. The values in the table correspond to an element that has a less restrictive time-stepping stability limit than other elements determined up till now.

Figure 1 shows the distributions of nodes for the elements of degree 1 to 6 listed in the table.

Table 2 *Triangular elements of degree M, enriched with polynomials of degree M_f in the interior. Numerical quadrature is exact up to and including degree $q = M + M_f - 2$. Multiple versions are denoted by* v*. For each of the equivalence classes, the integration weights and parameters that define the node positions, modulo symmetry, are listed.*

M	M_f	q	v	class	weight	position parameters	
1	1	1		1	$1/6$		
2	3	3		1	$1/40$		
				2	$1/15$		
				4	$9/40$		
3	4	5		1	$(8-\sqrt{7})/720$		
				3	$(7+4\sqrt{7})/720$	$1/2-\sqrt{1/(3\sqrt{7})-1/12}$	
				5	$7(14-\sqrt{7})/720$	$(7-\sqrt{7})/21$	
4	5	7		1	$1/315$		
				2	$4/315$		
				3	$3/280$	$\frac{1}{2}(1-1/\sqrt{3})$	
				5	$163/2520-47\sqrt{7}/8820$	$\left(5-\sqrt{7}\right)/18$	
					$163/2520+47\sqrt{7}/8820$	$\left(5+\sqrt{7}\right)/18$	
5	7	10	1	1	0.000709423970679245979296007		
				3	0.00348057864048921065844268	0.132264581632713985353888	
					0.00619056500367662911411813	0.363298074153686045705506	
				5	0.0116261354596175711394984	0.0575276844114101056608175	
					0.0459012376307628573770191	0.256859107261959076063891	
					0.0345304303772827935283885	0.457836838079161101938503	
				6	0.0272785759699962595486715	0.0781925836255170219988860,	0.221001218759890007978128
6	9	13	1*	1	0.0000779588662078435375178633		
				2	0.00339681189525519904330250		
				3	0.000665051266109643505640721	0.0279179934866402859608917	
					0.00362789876510517486953904	0.241092107358188959592334	
				4	0.0350469796677262076302804		
				5	0.0230539870369004268211275	0.208636402173497335322249	
				6	0.00661471763751599238359865	0.0257804392908054300630399,	0.0867676995664192358459690
					0.0170275218459173680417388	0.0472834372836282533842341,	0.365554923752573525334004
					0.0169179833927653607471877	0.0741366666993740353225476,	0.183636359341933759692246
					0.0193746182487836911462744	0.155330109405837631767247,	0.489368383573739089292973
6	9	13	2	1	0.000454987233833027795019629		
				2	0.00355887313687377058494543		
				3	0.00197187750035259785092208	0.0971182187637092591147781	
					0.00326405309041122550397833	0.265081457109796002621167	
				4	0.0312076201071798865578207		
				5	0.00751308598168073829542106	0.0472995857353385920258670	
					0.0180228944641299796145248	0.416718033745028399150151	
					0.0141271046812814012527394	0.170113123718258152434644	
				6	0.0183416350514485381364864	0.0527759379996054919028731,	0.366011227036634594000994
					0.0152607589589274236884861	0.0580443861948438701313498,	0.174910053392499448673606
					0.0174552659654307749558317	0.171896598740400978122421,	0.290177530266989875101690

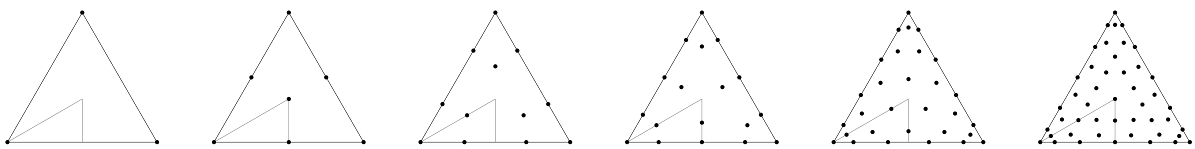

Figure 1 *Nodes for the elements of degree M from 1 to 6, mapped to the equilateral triangle. The number of nodes is 3, 7, 12, 18, 30, and 46, for increasing degree.*

Stability estimates

A standard second-order time discretization of the wave equation is stable when the time step Δt obeys the CFL condition (Courant et al., 1928), $\Delta t \leq 2/\sqrt{\sigma(\mathbf{L})}$, where $\sigma(\mathbf{L})$ is the spectral radius of $\mathbf{L} = \mathbf{M}_{\mathrm{L}}^{-1}\mathbf{K}$, the inverse lumped mass matrix applied to the stiffness matrix. Finding the spectral radius for the general case is difficult. The largest eigenvalue is bounded by the largest row sum of absolute values of $\mathbf{L}$. Sharper estimates can be found by considering simpler examples, for instance, by Fourier analysis (Zhebel et al., 2012, e.g.). Here, we choose $\Delta t \leq \texttt{CFL}\min(d_i/c)$, where $\texttt{CFL}$ is a constant and $\min(d_i/c)$ denotes the minimum over all elements in the computational domain of the ratio of the diameter, d_i, of the inscribed circle and the the local sound speed, c, for each triangle. The constant, $\texttt{CFL}$, is estimated by considering the spectral radius σ_0 of $\mathbf{L}_0$ for a unit sound speed on a single reference element with natural (Neumann) boundary conditions. Then, $\texttt{CFL} = 2/(d_{i,0}\sqrt{\sigma_0})$, where $d_{i,0} = 2-\sqrt{2}$ is the diameter of the inscribed circle of the reference element. The results are listed in Table 3.

***Table 3** CFL-numbers for the reference element.*

M	M_f	q	v	CFL
1	1	1		1.14
2	3	3		0.367
3	4	5		0.210
4	5	7		0.128
5	7	10	1	0.0512
6	9	13	2	0.0387

Higher-order time stepping can be obtained by the Cauchy-Kovalewski or Lax-Wendroff procedure (Lax and Wendroff, 1960) that substitutes higher time-derivatives with spatial derivatives using the partial differential equation. This approach is also known as the modified equation approach (Shubin and Bell, 1987) or Dablain's scheme (Dablain, 1986). The time-stepping order M_t can be taken as the smallest even integer larger than or equal to $M+1$, with M the degree of the spatial discretization on the edges. The corresponding $\texttt{CFL}$ should then be multiplied by 1, 1.732, 1.376, 2.317 for $M_t = 2$, 4, 6, and 8, respectively. The spatial discretization has a formal accuracy of order $M+1$, assuming that interpolation with the basis functions is required to read off the wavefield at an arbitrary receiver location. At the nodes, superconvergence may be observed (Cohen et al., 1995b).

Performance

Elements of higher degree have more nodes and are therefore more costly than elements of lower degree. Also, the stability limit is more restrictive for elements of higher degrees. For computational efficiency, this should be compensated by the use of coarser meshes with less elements. A simple test problem is a standing wave of the form $\cos(kx)\cos(kc_0t)$ with wavenumber $k = 2\pi/200\ \mathrm{m}^{-1}$ and natural (Neumann) boundary conditions on a domain of size [0,1] km by [0,1] km. The sound speed c_0 is set to 1 km/s and the code ran from time 0 to 0.5 s at a time step for half the CFL-number, to be on the safe side.
Figure 2 plots the error, measured in the standard quadratic norm over all nodes, as a function of the number of degrees of freedom, which equals the number of nodes in this case. Clearly, higher degrees have better accuracy. In terms of elapsed computer time, however, the element of degree 4 performs best. Only at very high accuracy, the elements of degree 5 and especially 6 become competitive. With second-order time stepping for all degrees, not shown here, a similar conclusion follows.
In this test, the stiffness matrices were assembled on the fly (Mulder, 1996). For the lowest degree 1, this is inefficient, but the direct approach was not considered here.
Figure 3 shows an example of wave propagation in an environment with topography. The source is located near the second of the three peaks.

Conclusions

Mass-lumped triangular elements with a symmetric arrangement of nodes are now available up to degree 6. Higher-degree elements are more accurate but also more costly than lower-degree elements. For a simple standing wave problem, elements of degree 4 are the most efficient in terms of computational time for a given accuracy.

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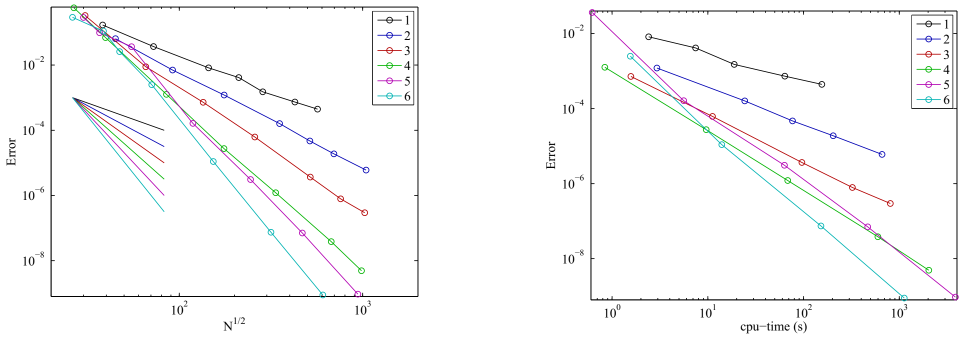

(a) (b)

Figure 2 *Performance for standing wave problem. (a) The maximum error as a function of $\sqrt{N}$, with N the total number of degrees of freedom, for elements of degrees 1 to 6 shows that the higher the degree, the better the accuracy for given N. (b) The same error as a function of elapsed time on a single core shows that elements of degree 4 tend to be the most efficient. Only at very high accuracy, the new element of degree 6 wins.*

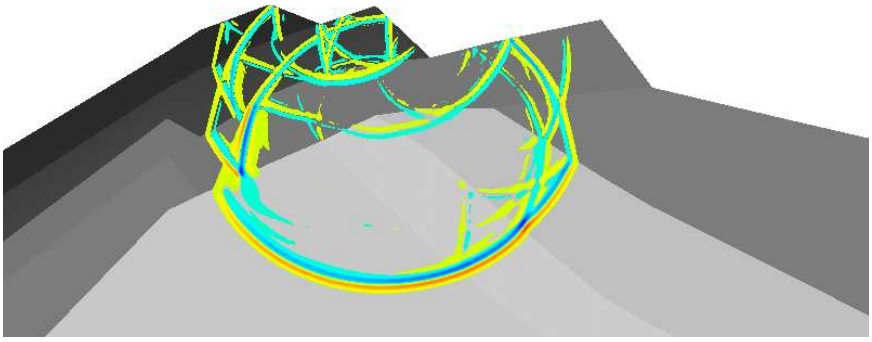

Figure 3 *Wavefield for a 2D example with topography.*

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