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Multiparameter Elastic Imaging Improved by Preconditioning with an Incomplete Inverse Hessian Approximation

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SUMMARY

The result of multiparameter isotropic elastic linearized inversion carries information about the reflectivity of each of the three elastic parameters, but these reflectivities are cross-coupled, i.e., a perturbation in one of the elastic parameters appear as a cross talk in the reflectivity image of the other. We propose an efficient way to decouple the reflectivity images of the three elastic parameters reliably within a multiparameter frequency-domain linearized approach. This is achieved by computing selected elements of the Hessian matrix and subsequent construction of its approximate sparse inverse rather than the inverse of its sparse approximation. The resulting preconditioning matrix can be treated either as an improved imaging condition for multiparameter elastic reverse-time migration or as an efficient preconditioner for the conjugate gradient method. The numerical study shows the potential of the proposed preconditioning method and provides an insight into the best attainable resolution and quality of elastic reflectivity images in the context of linearized inversion.

Introduction

Classic seismic migration provides a map of the reflectivity, a single image from which the interpreter can distinguish the shape and contrast of reflectors and other important information. However, a single migration image is not enough since an isotropic elastic medium can be characterized by three parameters, e.g., density, compressional-wave velocity and shear-wave velocity. To estimate the reflectivity in each elastic parameter, one can utilize elastic multiparameter inversion techniques, either linearized as in least-squares migration/inversion (Beydoun and Mendes, 1989, e.g.) or non-linear as with full waveform inversion (FWI) (Fichtner, 2010, e.g.). These techniques involve iterative minimization of the misfit functional and require an initial or background medium model that allows the computation of the initial misfit and its gradient with respect to the model parameters. The linearized inverse problem assumes a smooth and kinematically accurate background model that always remains the same, whereas the sharp local perturbations are iteratively updated. Unlike FWI, the linearized approach is useful for estimation of the reflectivity of each individual elastic parameter with better resolution, assuming that sufficiently accurate smooth model parameters were obtained by another type of inversion such as travel-time tomography. Østmo et al. (2002) described the advantages of the linearized approach over the full non-linear problem when applying iterative migration to the acoustic constant-density wave equation. The linearized approach can be regarded as a straightforward implementation of "multiparameter elastic migration", i.e., the technique that can provide three high-resolution true-amplitude reflectivity images, one for each parameter.

Although the result of linearized inversion carries information about the reflectivity of each elastic parameter, these reflectivities are cross-coupled, i.e., a perturbation in one of the elastic parameters will appear as cross talk in the reflectivity image of the other. Here, we suggest an efficient way to reliably decouple the reflectivity images of the three elastic parameters within a frequency-domain linearized approach. This is achieved by computing selected elements of the Hessian matrix and subsequent construction of its approximate inverse rather than the inverse of its approximation. The resulting preconditioning matrix can be treated either as an improved imaging condition for multiparameter elastic reverse-time migration or as a preconditioner for the conjugate gradient method (CGM), to decrease the number of iterations.

A brief review of the theory is followed by a numerical study for the case of three point scatterers in a homogeneous isotropic elastic background model. The numerical results show the potential of the proposed preconditioning method and provide an insight into the best attainable resolution and quality of elastic reflectivity images in the context of linearized inversion.

Theory

The frequency-domain equations of motion in an isotropic elastic medium can be formally written as $\mathrm{L}\,\mathbf{u} = -\omega^2\rho\mathbf{u} - \nabla\cdot\left[\lambda(\nabla\cdot\mathbf{u})\mathbf{I} + \mu\left(\nabla\mathbf{u} - [\nabla\mathbf{u}]^{\mathrm{T}}\right)\right] = \mathbf{f}$. The operator L acts on the displacement vector $\mathbf{u}$, the source term is expressed by $\mathbf{f}$, ρ is the mass density, λ and μ are the Lamé parameters, ω is the angular frequency and $\mathbf{I}$ is the identity tensor. For imaging, a formal elastic parameter $\mathbf{m}$ can be treated as a sum of a known background value $\mathbf{m_0}$ and a perturbation $\mathbf{m_s}$ obeying $\mathbf{m} = \mathbf{m_0} + \mathbf{m_s}$. Linearization with the Born approximation produces two equations, $\mathrm{L}_0\mathbf{u_0} = \mathbf{f}$ and $\mathrm{L}_0\mathbf{u_s} = -\mathrm{L}_s\mathbf{u_0}$, where $\mathbf{u_0}$ is the incident wavefield due to a source in the background model. The wavefield scattered by the perturbations is $\mathbf{u_s} = \mathbf{u} - \mathbf{u_0}$. The term L_0 is the elastic wave operator for the background parameters ρ_0, λ_0, μ_0, and $\mathrm{L_s}$ is the operator with the perturbations ρ_s, λ_s and μ_s. Instead of the perturbation parameter m_s, it is more convenient to define the reflectivity as $\widetilde{m} = m_s/m_0$. The inverse problem consists in finding an optimal reflectivity model $\widetilde{\mathbf{m}}_{opt}$ among all reflectivity models $\widetilde{\mathbf{m}}$ that best satisfies the observed wavefield $\mathbf{u}_{obs}$ by minimization of the least-squares misfit functional $J(\widetilde{\mathbf{m}}) = 1/2\sum_{\omega}\sum_{\mathbf{x}_s,\mathbf{x}_r}\|\mathbf{u}(\widetilde{\mathbf{m}}) - \mathbf{u}_{obs}\|^2$. The optimal reflectivity model corresponds to the minimum of the functional $J(\widetilde{\mathbf{m}})$ and hence to the zero its gradient with respect to the model parameters. The linear approximation of the gradient at $\widetilde{\mathbf{m}}_{opt}$ around a nearby initial reflectivity model leads to the linear system $H\widetilde{\mathbf{m}}_{opt} = -\boldsymbol{\nabla}J$, where $\boldsymbol{\nabla}J$ is the gradient of J with respect to the model parameters and H is the Hessian, i.e. the matrix of second derivatives of J with respect to those parameters. The formulae for the gradient and the Hessian are well known (Fichtner, 2010, e.g.). Since the explicit computation and inversion of the complete Hessian matrix is extremely resource-intensive for large-scale applications, an iterative method like the CGM, which does not require explicit matrix computation, is usually applied. The number of iterations necessary for convergence depends on the conditioning of the problem. If the number of iterations is large, a preconditioning can speed up convergence: $PH\widetilde{\mathbf{m}}_{opt} = -P\nabla J$, where P is a symmetric positive-definite matrix (Axelsson, 1996). This preconditioner should be an effective approximation of the Hessian's inverse that is able to bring the total computational cost of all the iterations below that of the full inverse Hessian. This involves the cost of its construction, its application at each iteration and the total number of iteration steps. If we assume that the preconditioner is close enough to the inverse of actual Hessian matrix, then the model can be approximated simply as $\widetilde{\mathbf{m}}_{opt} \approx -P\nabla J$. In this case the preconditioner represents an improved imaging condition for multiparameter elastic reverse-time migration.

The preconditioner

The main challenge with constructing a preconditioner is how to design an approximate inverse Hessian matrix that will be effective and computationally easy to construct. Plessix and Mulder (2004) as well as Hu et al. (2011) noted that the dominant Hessian values are concentrated close to the diagonal, corresponding to the neighbourhood of a given subsurface point. The strongest correlation corresponds to the self-correlation, i.e., the diagonal elements of the Hessian. The simplest preconditioner will be an inverse of the diagonal of the Hessian and is known as Jacobi (Axelsson, 1996). However, in multiparameter elastic inversion, each medium point is described with three parameters, hence it is more natural to take the inverse of the block-diagonal approximation of the Hessian, where each block is a 3×3 matrix with elements corresponding to the second derivatives of the misfit functional computed in single spatial position. Although these preconditioners are commonly used in practice, they neglect the interaction of scattered wavefields from neighbouring points of the medium (Beydoun and Mendes, 1989) and can be inefficient (Plessix and Mulder, 2004; Anikiev et al, 2013). Therefore, a more elaborate preconditioning strategy is necessary. The simplest inclusion of 4 neighbouring spatial grid points (in 2D) around each grid point will require the computation of only 4 additional off-diagonal elements, since the Hessian is symmetric. In theory, the more neighbouring points we include, the better the approximation is expected to become but also the less sparse. Hu et al. (2011) reported that including more non-zero elements in the approximate Hessian does not necessarily result in a more accurate approximation of the inverse of the Hessian matrix. This may be related to the fact that such a sparse approximation of the Hessian does not necessarily mean its inverse will be a good approximation of the inverse Hessian. Eventually, it is more important to construct an accurate approximation of the inverse of the Hessian.

Recently, Kaporin (2012) introduced a new preconditioning method, called Approximate Inverse Triangular factorization or Incomplete Inverse Cholesky (IIC) factorization. The method is based on the principle of K-optimality, requiring the derived preconditioner to have a minimum K-condition number, which is a better alternative to the standard spectral condition number (Kaporin, 2012). This is one of the advantages of this method compared to Incomplete LU, Incomplete Cholesky, Approximate Triangular factorizations and other methods (Saad and van der Vorst, 2000, e.g.) and, possibly, the method of Hu et al. (2011). The idea of IIC preconditioning can be translated to the problem of constructing an approximation to the inverse Hessian. First, the Hessian matrix is supposed to be sparse and we assume that its sparsity pattern (the locations of the non-zero matrix elements) is known. Second, these non-zero elements must be computed and stored. Third, these values are used for construction of an incomplete version of the inverse Hessian. To conclude, for the construction of an IIC preconditioner for the Hessian matrix, one needs to know the locations and values of its non-zero elements. Obviously, the cost of constructing such a preconditioner will depend on the assumed sparsity pattern of the Hessian. Therefore, combining the idea of Hessian's sparsity pattern with the IIC preconditioning method we obtain an efficient preconditioning technique. The quality of preconditioning should be proportional to the amount of information involved in the Hessian (related to the number of neighbouring grid nodes), or, in other words, inversely proportional to the sparsity of Hessian approximation. The Hessian element measures the interaction of the scattered wavefields between two medium points when modelled and imaged. If these points are close to each other, the scattered energy cannot be resolved and interaction of wavefields is significant (Fichtner, 2010). Therefore, the "neighbourhood area" should include the first Fresnel zone to make sure that all significant wavefield interactions are involved. In order to test the proposed preconditioning strategy, we designed a simple numerical example.

Numerical Example

We considered a model with three point scatterers in a homogeneous isotropic elastic background to study the decoupling of the elastic parameters and the effect of the new preconditioner. The background medium had a density, ρ, of 2 g/cm^3, a P-wave velocity, α, of 2 km/s, and an S-wave velocity, β, of 1.2 km/s. Three point scatterers were located at the same depth of 750 m and had horizontal coordinates, −500 m, 0 m and 500 m, respectively. Each scatterer represented the perturbation in one of the elastic parameters. The leftmost scatterer corresponded to a perturbation of the density, ρ, the central scatterer was related to a perturbation in the P-wave impedance, $Z_\alpha=\rho\alpha$, and the third rightmost scatterer represented a perturbation in the S-wave impedance, $Z_\beta=\rho\beta$. This choice of the three parameters was inspired by Tarantola (1986). For simplicity, each of the three perturbations was set to a reflectivity value of 1. A line of 81 shots (from −2000 to 2000 m along the *x* axis with a 50-m spacing) was placed on the surface in the same plane y = 0 m as the scatterers. Therefore, only P- and SV-waves were involved. 161 receivers were also deployed on the surface along the *x*-axis, between −2000 and 2000 m, at a 25-m interval. A vertical-force source with a 10-Hz Ricker wavelet was used as a source function. The scattered wavefield for a given frequency was constructed from the 3-D Green functions in a homogeneous background (Wu and Aki, 1985, e.g.). We used 111 discrete frequencies from 0 to 30 Hz to compute the wavefields. The reflectivity images were computed within a 2-D rectangular zone of 2000 m by 1000 m with grid spacing of 25 m around the actual positions of the scatterers.

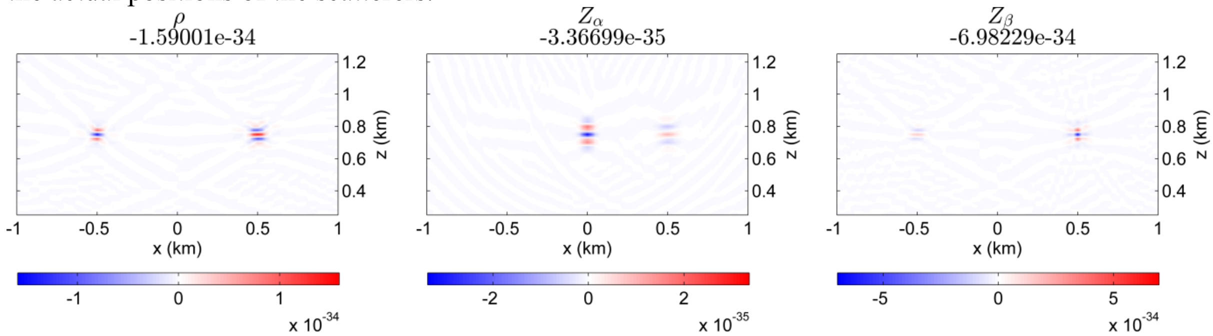

Figure 1 *Gradient components with respect to the three elastic parameters: mass density (left), P-wave impedance (centre) and S-wave impedance (right). The number above each image shows the maximum absolute value of the corresponding gradient component.*

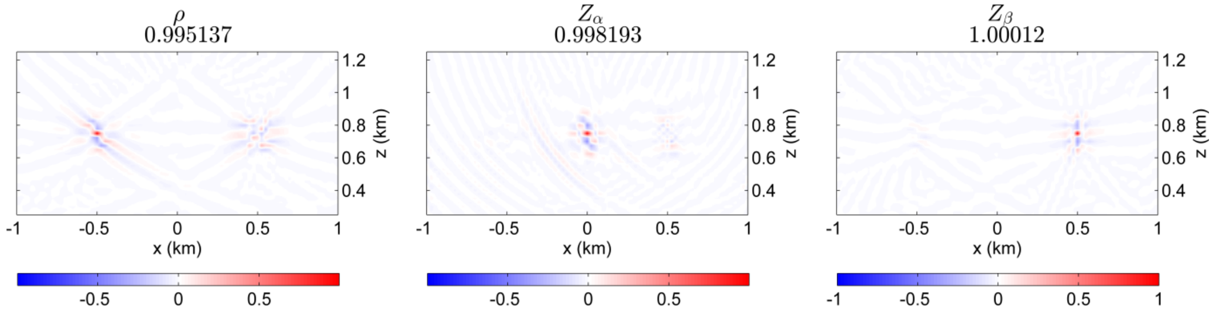

Figure 2 *Reflectivities for the three elastic parameters obtained with the help of preconditioning: mass density (left), P-wave impedance (centre) and S-wave impedance (right). The number above each image shows the maximum absolute value of the corresponding reflectivity distribution.*

Figure 1 illustrates the cross talk between the elastic parameters. It displays the components of the gradient of the misfit functional with respect to density (left), P-wave impedance (centre) and S-wave impedance (right). These images can be treated as reflectivity maps for each of the three elastic parameters. Obviously, the density reflectivity shows two perturbations, the left one corresponds to the true density scatterer and the right one is the result of cross talk between density, ρ, and S-wave impedance, Z_β. The right-hand image shows similar cross talk between ρ and Z_β, whereas the central image reveals the cross talk between two impedances. Apart from the apparent problem of coupling and cross talk, the gradient amplitudes are far from the true reflectivity: the number above each image in Figure 1 shows the maximum absolute value of the gradient component. Ideally these numbers should equal 1, corresponding to the true reflectivity in each case, but they are much smaller.

The radius of the first Fresnel zone at the depth of scatterers is approximately 275 m for P-waves and 210 m for S-waves. However, our tests showed that the Hessian sparsity pattern approximation with the radius of wavefield interaction zone equal to 50 m already gives appropriate results. Figure 2 shows the reflectivity images after preconditioning, i.e., after multiplication by the IIC preconditioner based on the sparsity pattern that corresponds to a "neighbourhood area" of 50-m radius (12 neighbouring grid nodes) and Hessian sparsity of approximately 0.4%. The quality of the reflectivity images in Figure 2 is better when compared to Figure 1: the resolution is higher and the cross talk between parameters is reduced. The maximum absolute values are close to 1, which perfectly matches the true perturbations at the scatterer positions. The iterative approach with CGM, preconditioned by proposed method, leads to a further improvement of the reflectivity images (results not shown).

Conclusions

We have proposed to use Incomplete Inverse Cholesky factorization to construct an approximation of the inverse of the Hessian that can improve the three migration images for linearized frequency-domain isotropic elastic inversion. In a simple example, the method improves the resolution of imaging and reduces the cross talk between the perturbation parameters.

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