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A Multigrid-based Iterative Solver for the Frequency-domain Elastic Wave Equation

G. Rizzuti* (Delft Inversion) & W.A. Mulder (Shell GSI BV & Delft University of Technology)

SUMMARY

Efficient numerical wave modelling is essential for imaging methods such as reverse-time migration (RTM) and full waveform inversion (FWI). In 2D, frequency-domain modelling with LU factorization as a direct solver can outperform time-domain methods by one order. For 3-D problems, the computational complexity of the LU factorization as well as its memory requirements are a disadvantage and the time domain becomes more attractive. Recently, it has been shown that with compute cores in abundance, a parallel frequency-domain iterative solver can be a competitive alternative to the time-domain approach for 3-D acoustic RTM. The solver relies on a preconditioned Krylov subspace method, where the preconditioner involves the multigrid solution of a heavily damped wave equation. Here, we generalize this idea to the isotropic elastic wave equation in the frequency domain.

Introduction

Seismic imaging methods as reverse time migration (RTM, Baysal et al., 1983) or full wave-field inversion (FWI, Tarantola, 1984) are steadily gaining popularity within the seismic community. These techniques formulate imaging as an inverse problem and generally rely on gradient-based iterative minimization to find the best match between observed and the modelled data. They require repeated numerical wave propagation simulations for a given subsurface model. Since the size of the computational domain can be very large, especially for 3-D problems, the numerical modelling needs to be efficient.

In principle, the imaging problem can be equivalently formulated in the time or in the frequency domain, as pointed out, e.g., by Mulder and Plessix (2002) and Virieux and Operto (2009). The advantage of the frequency domain is the fact that the numerical work can be done for each frequency (and for each shot) independently and, therefore, in parallel. Also, only a limited subset of frequencies, well below what is prescribed by the Nyquist criterion, is needed for a successful inversion (Pratt, 1990; Plessix and Mulder, 2004). However, this requires the solution of a large, though sparse, linear system whose algebraic properties do not favour an iterative approach. On the other hand, direct methods as the classic LU factorization (George and Liu, 1981) remain impractical in terms of memory storage. For these reasons, time-domain modelling is usually preferred for large 3-D problems.

Nevertheless, frequency-domain modelling remains attractive and is the subject of active research, both with a direct and with an iterative approach. Wang et al. (2011, 2012) presented parallel multifrontal direct solvers for both the acoustic and the elastic wave equation. Among iterative methods, the multigrid solution of a heavily damped version of the wave equation as preconditioner for the Helmholtz equation has reasonable convergence properties (Erlangga et al., 2006; Plessix, 2007). The multigrid preconditioner has linear complexity with problem size, but the required number of outer iterations increases with frequency on a mesh of given size.

This paper extends the concept of multigrid preconditioning introduced by Erlangga et al. (2006) to the elastic isotropic wave equation. It contains a first theoretical assessment of the applicability of this idea and includes an 2-D example.

Method and Theory

In the frequency domain, the 2-D isotropic elastic linear system reads:

$$L(\omega) = -\rho\omega^2 I - D, \qquad D = \begin{pmatrix} \partial_x(\lambda+2\mu)\partial_x + \partial_z\mu\partial_z & \partial_x\lambda\partial_z + \partial_z\mu\partial_x \\ \partial_x\mu\partial_z + \partial_z\lambda\partial_x & \partial_x\mu\partial_x + \partial_z(\lambda+2\mu)\partial_z \end{pmatrix}. \tag{1}$$

Here, ω is the angular frequency, ρ the density. The Lamé parameters are λ and μ. We want to solve the equation $L\mathbf{v} = \mathbf{f}$, where $\mathbf{v} = (v_x, v_z)$ is the particle velocity and $\mathbf{f}$ the source function. We will consider two classic finite-difference discretizations, by Kelly et al. (1976) and by Virieux (1986).

In Erlangga et al. (2006), a good preconditioner for (1) is found to be $L(\tilde{\omega})^{-1}$, where $\tilde{\omega} = \omega\sqrt{1-\mathrm{i}\beta}$. The operator $L(\tilde{\omega})$ corresponds to a heavily damped wave equation, with the damping factor β; generally $\beta \geq 0.25$. Unlike the undamped version, the system $L(\tilde{\omega})\mathbf{v} = \mathbf{f}$ can be efficiently and approximately solved by the multigrid method with a number of operations independent of the grid size for a fixed frequency $\tilde{\omega}$.

Multigrid is a numerical method originally designed for finite-difference approximations of elliptic linear problems of the form $L^h\mathbf{v}^h = \mathbf{f}^h$, where the grid spacing is now explicitly denoted by h. A typical multigrid cycle starts from an initial guess $\mathbf{v}_0^h$ of the solution $\mathbf{v}^h$ and comprises the following steps: (i) we compute $\mathbf{v}_1^h$ such that the magnitude of the oscillatory or short-wavelength components of the error $\mathbf{e}_1^h = \mathbf{v}^h - \mathbf{v}_1^h$ is close to zero by a so-called smoothing operator; (ii) to solve the long-wavelength components, the error equation $L^h\mathbf{e}_1^h = \mathbf{r}_1^h$ is approximated on or restricted to the coarse grid by a similar but smaller problem $L^{2h}\tilde{\mathbf{e}}_1^{2h} = \tilde{\mathbf{r}}_1^{2h}$. Its solution $\tilde{\mathbf{e}}_1^{2h}$ is then projected back on or prolongated to the fine grid to obtain the coarse-grid correction $\tilde{\mathbf{e}}_1^h$ that updates the solution to $\mathbf{v}_2^h = \mathbf{v}_1^h + \tilde{\mathbf{e}}_1^h$. Because long wavelengths become shorter on coarser grids, a proper smoother can remove all the error components by considering many instead of two coarse-grid levels (see Briggs et al., 2000, for an elementary introduction on multigrid).

The multigrid method fails for the Helmholtz equation, because the operator $L(\omega)$ becomes indefinite (the eigenvalues change sign at higher frequencies). An elementary smoothing operator will amplify the long-wavelength components of the error and the coarse-grid correction will update the solution in the wrong 'direction' (Elman et al., 2001). However, multigrid can be applied to $L(\tilde{\omega})$.

A standard tool for assessing the effectiveness of multigrid is the so-called local mode analysis (Trottenberg et al., 2001, e.g.). It provides estimates of relevant quantities as smoothing and ideal two-grid convergence factors by exploiting the fact that the monochromatic grid functions $\varphi_\theta^h = \exp(i\mathbf{x}\cdot\theta/h)$ are formal eigenvectors for differential operators such as (1), under the simplifying assumptions of homogeneity and unboundedness or periodicity of the physical medium. On the basis of local mode analysis, we have chosen and validated the multigrid ingredients for the damped elastic wave equation, but here we will only show the results of the smoothing analysis and omit the two- and three-grid convergence estimates. Note that the elastic wave equation requires the analysis of a system of equations, unlike the acoustic case, and we therefore have to consider the different modes of propagation, the P- and S-waves. Apart from that, the design and analysis of multigrid for the damped elastic wave equation closely follows the acoustic case. The most significant difference lies within the smoothing procedure. Starting from an initial guess $\mathbf{v}_0$ of the solution, the classic point-wise Jacobi smoothing considered in Erlangga et al. (2006) is defined by the iteration $\mathbf{v}_1 = \mathbf{v}_0 + \alpha D^{-1}\mathbf{r}_0$, $\mathbf{r}_0$ being the residue, D the diagonal of L and α a real-valued smoothing parameter that needs to be tuned. In terms of error reduction, $\mathbf{e}_1 = S\mathbf{e}_0$, with smoothing operator $S = I - \alpha D^{-1}L$. Smoothing analysis involves the spectral decomposition $S\varphi_\theta = \lambda_\theta\varphi_\theta$. For effective smoothing, the quantity $\rho(S) = \max\{\lambda_\theta : \pi/2 < \theta_. < \pi\}$ should be small. This is, indeed, the case for the scalar Helmholtz equation. However, since in the elastic case different modes of propagation are present, point-wise Jacobi will produce smoothing 'anisotropy' — not to be confused with physical anisotropy — meaning that only one of the two modes will be sufficiently smoothed. This results in a good smoothing of the error only in the horizontal direction for the component v_x, and only in the vertical direction for the component v_z. Because of this, we propose line-Jacobi smoothing given by

$$S = I - \alpha\tilde{D}^{-1}L, \qquad \tilde{D} = \begin{pmatrix} \partial_x(\lambda+2\mu)\partial_x + \mathrm{diag}(\partial_z\mu\partial_z) & \\ & \mathrm{diag}(\partial_x\mu\partial_x) + \partial_z(\lambda+2\mu)\partial_z \end{pmatrix}. \tag{2}$$

The operator $\tilde{D}$ can be easily inverted and parallelized, since it corresponds to many independent 1-D problems. Figure 1 depicts its smoothing properties.

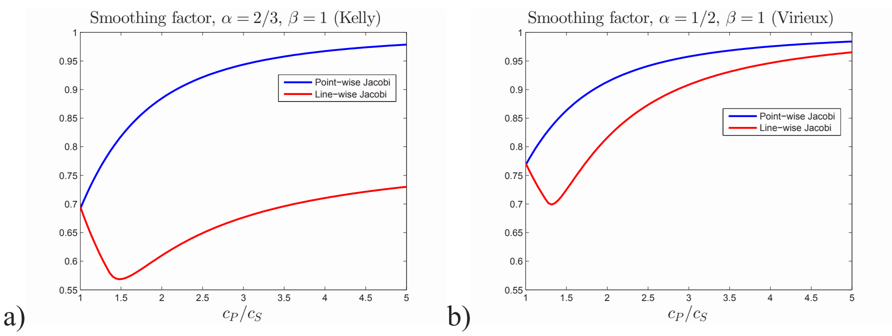


Figure 1 *Comparison of point-wise and line-wise Jacobi smoothing factors as a function of the velocity ratio c_P/c_S: (a) Kelly et al. (1976) and (b) Virieux (1986) (the considered grid corresponds to 10 samples per S-wave wavelength).*

Examples

We have applied the numerical scheme described in the above to a highly heterogeneous problem, a subset of the Marmousi2 model (Martin et al., 2006), with one modification: the acoustic layer with replaced by an elastic one to avoid low S-wave velocities. We used Kelly's fourth-order finite-difference

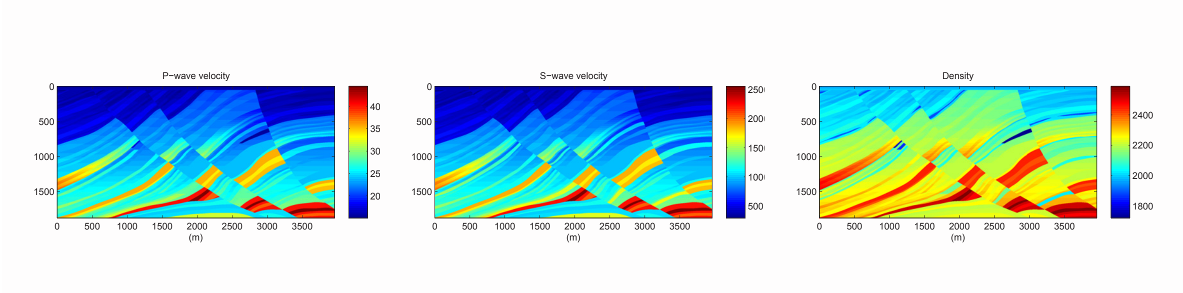
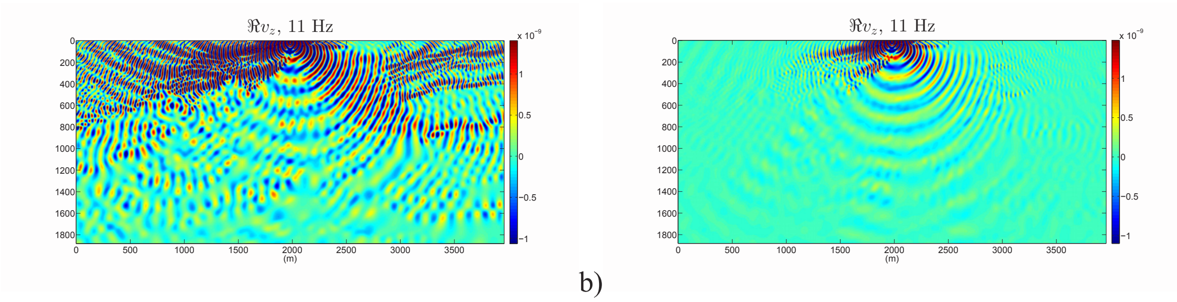


Figure 2 *Subset of the elastic Marmousi2 model with (a) P-velocity, (b) S-velocity and (c) density.*

scheme with simple sponge layers for absorbing boundary conditions. The outer iteration scheme is Bi-CGSTAB. The preconditioner corresponds to a damping factor $\beta = 2$.

We chose standard ingredients for multigrid: full-weighting and bilinear interpolation as grid-transfer operators, Galerkin coarse-grid operators and a F$(1,1)$-cycle (Briggs et al., 2000). For each of the considered frequencies, we imposed 10 samples per minimum S-wave wavelength. Figure 4 displays the convergence history for the frequencies 3, 6 and 11 Hz, The solution for the last one is shown in Figure 3. We also considered the performance in the presence of model attenuation, determined by the P and S quality factors $1/\tilde{c}^2_{P,S} = 1/c^2_{P,S}(1 + \mathrm{i}/Q_{P,S})$, where $c_{P,S}$ is the P- or S-wave velocity, respectively.



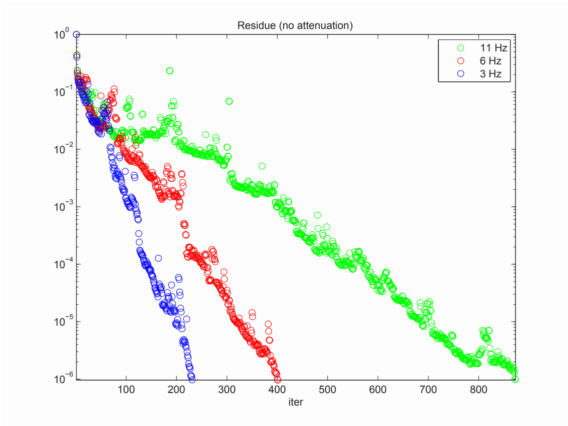
a) b)

Figure 3 *Real part of the vertical component of the solution at a frequency of 11 Hz with (a) no attenuation and (b) $Q_P = 40$, $Q_S = 40$.*

Conclusions

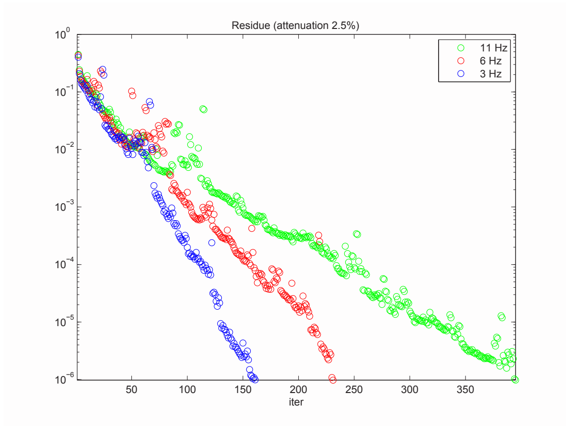
We extended the work of Erlangga et al. (2006) on multigrid preconditioning for the acoustic Helmholtz equation to the 2-D isotropic elastic case and demonstrated its effectiveness. Of course, the approach has the same limitation as the scalar case in that the number of required outer iterations increases linearly with frequency as evident from Figure 4. Here, the number of points per wavelength is kept fixed, so the grid size increases along. A combination of multigrid and deflation as preconditioner (Erlangga and Nabben, 2008) might help to make the number of required iterations independent of grid size — a property of the classic multigrid method applied to the Laplace equation.

For applications in 2-D, a direct solver in the frequency domain will be more efficient for RTM and FWI than an iterative method or a time-domain approach. In 3-D, approximate direct solvers Wang et al. (2011, 2012) are an option, but the time-domain is often preferred. However, Knibbe et al. (2014) demonstrated that with an abundance of compute cores, a multigrid-based iterative approach may compete with or even outperform a time-domain method. This motivates further work on the elastic generalization of the method.



Frequency	Grid size	Iterations
3	257×481	230
6	449×865	400
11	833×1665	872

a)



Frequency	Grid size	Iterations
3	257×481	160
6	449×865	230
11	833×1665	394

b)

Figure 4 *Convergency history for the subset of the Marmousi2 model with (a) no attenuation or (b)* $Q_P = Q_S = 40$.

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