MULTIGRID RELAXATION FOR THE EULER EQUATIONS

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1. Introduction

Implicit time-discretization, combined with upwind space-differencing, yields a fast and robust method for finding stationary solutions of the Euler equations. Particularly successful is the switched evolution/relaxation (SER) scheme, which provides a smooth switching between explicit time-integration and Newton's method for finding zero values of a given function. For one-dimensional problems quadratic convergence can be obtained, as shown in an earlier paper [1]. In two dimensions the exact inversion of the linear system arising in the implicit formulation is too costly. Various approximate solvers are described in [2].

In this paper an efficient approximate solver based on the multigrid method for the solution of large linear systems is described. An outline of the basic multigrid concepts can be found in [3]. The method is applied to compute the two-dimensional transonic flow through a channel with a circular bump at one wall. Numerical results are presented for single-grid and multigrid, with first- and second-order spatial accuracy.

2. Method

Let the system of hyperbolic equations in two dimensions be given by:

$$\frac{\partial \mathbf{w}}{\partial t} = -\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} \equiv \mathbf{r}(\mathbf{w}) \tag{1}$$

Here $\mathbf{f}(\mathbf{w})$ and $\mathbf{g}(\mathbf{w})$ are the fluxes, $\mathbf{w}$ is the vector of conserved state quantities and $\mathbf{r}(\mathbf{w})$ is the residual, the function that must be made to vanish. The implicit scheme of our choice is the linearized "backward Euler" scheme:

$$\mathbf{L}^n \Delta_t \mathbf{w}^n = [\frac{\Delta x \Delta y}{\Delta t^n} - \Delta x \Delta y \left(\frac{d\mathbf{r}}{d\mathbf{w}}\right)^n] \Delta_t \mathbf{w}^n = \Delta x \Delta y \; \mathbf{r}^n(\mathbf{w}^n) \tag{2}$$

The superscript n denotes values at a time t^n, while $\Delta t^n = t^{n+1} - t^n$ and $\Delta_t \mathbf{w}^n = \mathbf{w}^{n+1} - \mathbf{w}^n$. The discrete values $\mathbf{w}_{ij}$ of the state quantity are obtained by volume-averaging. The local residual $\mathbf{r}_{ij}(\mathbf{w})$ is computed, for the present purpose, by a first-order upwind-difference scheme on a 5-point stencil: $\mathbf{r}_{ij}(\mathbf{w}) = \mathbf{r}(\mathbf{w}_{i-1,j}, \mathbf{w}_{i,j-1}, \mathbf{w}_{ij}, \mathbf{w}_{i+1,j}, \mathbf{w}_{i,j+1})$. The timestep Δt^n is determined by:

$$\Delta t^n = \varepsilon / RES^n ,$$
$$RES^n = \max_{ijk} (|r^n_{ijk}| \; / \; |w^n_{ijk}| + h^n_{ijk}) . \qquad (3)$$

Here h is a bias to prevent division by zero; in case of the isenthalpic Euler equations used here, $h_{ij1}=0$ and $h_{ij2}=h_{ij3}=\rho c$. If Δt^n is small, the implicit scheme behaves very much like an explicit time-accurate scheme. Once the solution is getting closer to the steady state, Δt^n becomes larger and the scheme automatically switches to Newton's method. The constant ε controls the relative variation of w and is usually taken to be 1.

The inversion of the linear system (2) can be carried out efficiently by a multigrid scheme. Its basic ingredients are: (i) relaxation, (ii) restriction and (iii) prolongation. Symmetric Gauss-Seidel relaxation is used here for its excellent short-wave damping. Restriction is carried out by adding the values of the neighbouring zones and placing them on the coarser grid. The matrix L is restricted by addition of the corresponding blocks. For non-uniform grids the multiplication by the local cell-volume $\Delta_i x \; \Delta_j y$ ensures the proper weighting. Since restriction only involves additions, its cost is but a small fraction of that of a relaxation sweep. Finally, prolongation is carried out by distributing the coarse grid solution uniformly over the fine grid.

The multigrid strategy used in this paper is a simple V-cycle. Before every restriction and after every prolongation symmetric Gauss-Seidel relaxation (consisting of 2 sweeps) is carried out. On the coarsest grid an exact inversion is applied.

The usual quantity "work" is computed here by adding the number of relaxation sweeps, weighting each grid-total with respect to the finest grid. Thus, the amount of work for a single-grid iteration is 2, and for a V-cycle about $5\frac{1}{3}$.

3. Results for a test problem

The method is tested on the two-dimensional problem of transonic flow through a straight channel. The flow runs along the x-direction and is obstructed by a circular arc on the lower wall. The channel has an x-coordinate running from -1.5 to 2.5 and a y-coordinate running from 0.0 to 2.0. The circular arc between x = -0.5 and 0.5 has a maximum thickness equal to 4.2% of the chord. Thin-airfoil theory is used to transfer the boundary conditions at the arc to the flow. For simplicity a uniform square grid is adopted. In this setting the isenthalpic Euler equations in conservation form are solved for an ideal gas with $\gamma = 1.4$. The free-stream values are chosen to be: $\rho_\infty = 1$, $u_\infty = 0.85$, $v_\infty = 0$, $c_\infty = 1$. For the unchoked case two boundary conditions at the inlet and one at the outlet should be specified. At the inlet the direction of the flow and the total pressure are given, at the outlet the static pressure is specified; these parameters are computed from the free-stream values. The fluxes on the boundaries are computed by using differences of characteristic variables, with the appropriate upwind-switching to determine between extrapolation or direct computation. Boundaries at the lower and upper wall are simulated by an extra zone with reflected state quantities.

For the upwind differencing of the internal flow the split fluxes as proposed in [4] are used, as they can be easily linearized. for second-order accuracy an incomplete linearization is adopted to give $L^n(\mathbf{w})$ the same structure as for the first-order scheme (see [1]). This will obviously lead to some loss in convergence speed, but greatly simplifies the computation and inversion of $L^n(\mathbf{w})$.

The linearization L^n is frozen now and then to save cpu-time, just as in [2]. Furthermore, the inverses of the main-diagonal blocks and the restricted blocks are stored in memory, so that a multigrid cycle can be carried out must faster during freezing.

Convergence histories for a 16x8, 32x16 and 64x32 grid are shown in Fig. 1, both for the first-order and second-order accurate solutions. In all cases the multigrid scheme is faster. The second-order runs are somewhat slower than the first-order ones, due to the incomplete linearization. Fig. 2 shows the pressure coefficient on bottom and top wall as computed from the first-order and second-order accurate solution, respectively.

4. Concluding Remarks

It has been demonstrated that the multigrid technique can be successfully applied to compute a stationary transonic solution of the Euler equations. For the two-dimensional test problem the gain in efficiency with respect to a single-grid scheme, both in terms of work and cpu-time, is of the order $N^{0.4}$, where N is the total number of zones. Consequently, the number of iterations required to obtain a converged solution, whether first- or second-order accurate, increases only slowly with N. This result certainly justifies the additional effort of coding the multigrid scheme.

Finally, the power of the method presented here is demonstrated by Fig. 3, showing the stationary solution and convergence history for the flow in a rotating galaxy.

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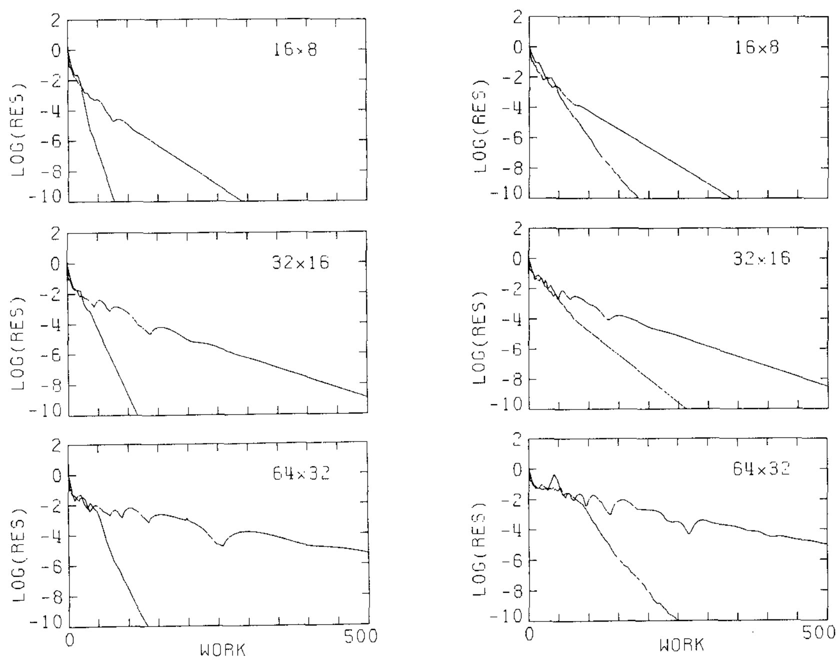

Fig. 1. Convergence histories for the first-order (left) and second-order (right) accurate solutions on 3 different grids. The residual RES is normalized by the initial value at t=0. In all cases the multigrid scheme is faster than the single-grid scheme. For the multigrid scheme, the total amount of work increases only slowly with the number of points.

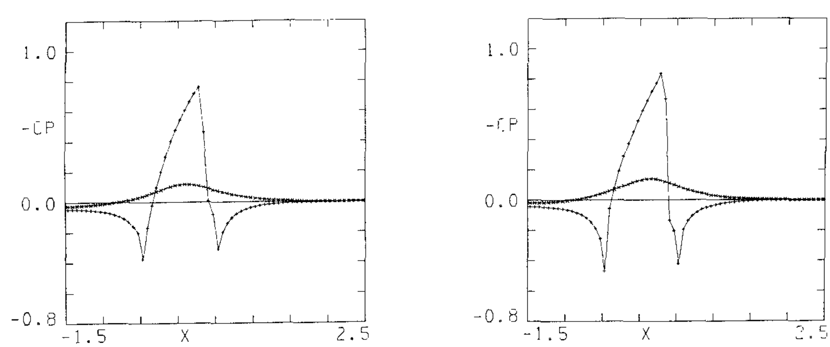

Fig. 2. Pressure coefficient on bottom (+) and top wall (x) for a 4.2% circular arc and a free-stream Mach number 0.85, as computed from the first-order (left) and second-order (right) solution on a 64x32 grid. Thin-airfoil theory is applied to transfer the boundary conditions to the flow.

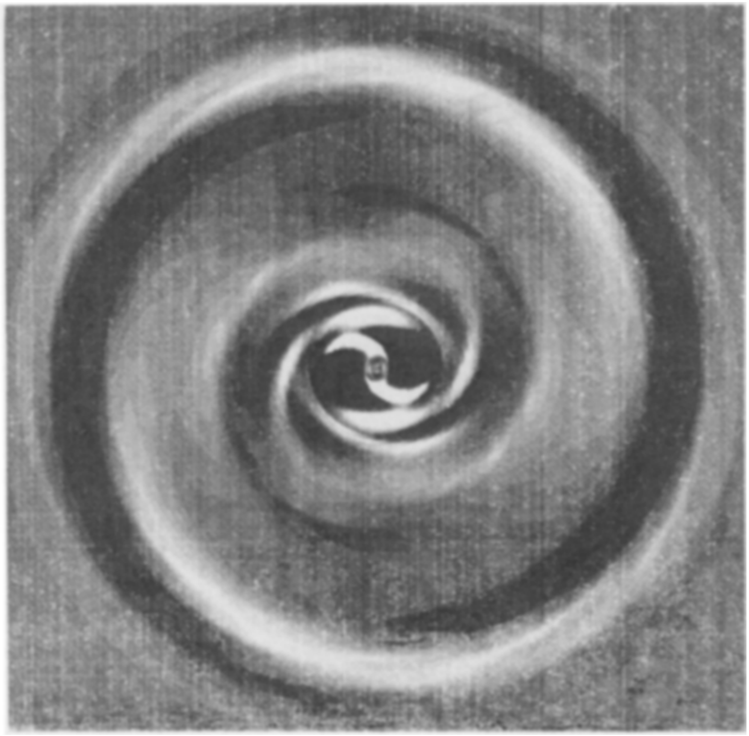

Fig. 3. Stationary spiral pattern in the co-rotating frame of a weakly barred galaxy. The gravitational potential consists of an axisymmetric part and a rotating $\cos(2\phi)$ perturbation, which ends at a co-rotation radius 8.36 . The polar grid (64x64 zones) covers one half-plane, with a radius R running from 0.3 to 30 and ΔR varying from 0.03 to 1.7. The computation is carried in single precision. Shown is the density divided by the average density per ring; the size of the figure is 40x40. Convergence histories for two different grids are shown below. The peaks in the initial phase are due to successive grid-refinement starting from a 4x4 grid.