Robust iterative methods for the Helmholtz equation applied in geophysical surveys*

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1 Introduction

Wave equation migration is becoming popular in seismic applications. This migration is currently based on a one-way scheme to allow applications in 3D, in which the full wave equation simulation is simply too expensive. It is already known, however, that one-way wave equations do not correctly image steep events and do not predict the amplitudes of the reflections [4].

In 2D, it is possible to perform the full wave equation in the frequency domain, since a direct solver with nested dissection ordering can efficiently solve the linear system obtained from discretizations of the two-way wave equation. In 3D, however, the linear system and its band size become too large which make the direct method inefficient. As an alternative, iterative methods can be used to solve the linear system.

Iterative methods, however, often result in a rather slow convergence or even fail to convergence if they are used to solve the linear systems arising from the wave equation. In such situations, a preconditioner can help accelerating the convergence. The indefinite property of the discretized wave equation, however, gives the fact that the classic preconditioner based on incomplete LU factorizations is not sufficient to improve the convergence. It is also often observed that slow convergence (or breakdown) may still occur [3].

In this paper we propose a preconditioner based on a complex shift of the Laplace operator. We call the preconditioner the "complex Shifted-Laplace (CSL) preconditioner". To solve the preconditioner in each iteration a multigrid

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algorithm [5] has been used. We show that an iterative method in combination with multigrid applied to CSL may become a robust solver for the wave equation.

2 The Helmholtz equation and the preconditioner

We are concerned with solutions of the wave equation in the frequency domain governed by the following PDE:

$$\mathcal{L}\phi \equiv \left(\partial_{xx} + \partial_{yy} + k^2(x,y)\right)\phi = g \text{ in } \Omega \in \mathbb{R}^2, \tag{1}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial \phi}{\partial n} - ik\phi\right) = 0 \text{ on } \Gamma = \partial\Omega, \tag{2}$$

where k is the wavenumber and can vary in Ω, while g is the source term. With a particural discretization, equations (1)-(2) reduce to the linear system

$$Ap = b, \;\; A \in \mathbb{C}^{N\times N}, \;\; p, b \in \mathbb{C}^N. \tag{3}$$

The matrix A is sparse, large, symmetric indefinite linear system.

For the preconditioner the following operator is used [2]:

$$\mathcal{M} \equiv \partial_{xx} + \partial_{yy} - ik^2(x,y). \tag{4}$$

Discretization of (4) always leads to a complex, symmetric positive definite matrix M. This type of matrix can be solved by any kind of methods, e.g. incomplete LU factorization and inner iteration using another short reccurence iterative method (e.q. COCG [6]). In our applications, multigrid has been used to solve the matrix M. Since structured grids are used, the geometric multigrid is sufficient. For coarse grid approximation, the Galerkin coarse grid approximation is used with the full weighting and bi-liniear interpolation as restriction and prolongation operator. Smoothing process is done by the red-black Gauss-Seidel iteration.

With respect to A preconditioned by M we have shown the following properties [2]:

- the spectrum $M^{-1}A$ is bounded above by one and has the minimal eigenvalue of order $\mathcal{O}(1/k)$
- the spectrum can be rotated by a rotation matrix such that all eigenvalue lies in the half complex plane. Furthermore, if Dirichlet conditions are used to replace the condition (2), the eigenvalues exactly lie on a circle $(z - \frac{1}{2}\sqrt{2})^2 = \frac{1}{2}\sqrt{2}$

In the view of analysis, the latter property is somewhat nice. If one, e.g. uses Manteuffel's result on Chebychev polynomial iterations, the Chebychev iterations can be guaranteed to converge. Using general algorithms, e.g. BiCGSTAB or GMRES, the rotation needs not be implemented as the methods are already able to handle the linear system having the above properties.

3 Results

We have used the method to solve the 2D Marmousi problem [1], a problem which mimics complicated earth's subsurface. The numerical performance is shown in Table 1. Results using ILU(1) are also included for comparison. For both cases we have used BiCGSTAB for the main iteration.

Table 1: *Numerical results of the Marmousi problem. Number of iterations and (in brackets) CPU time in sec. are shown for various frequency f*

f (Hz)	1	10	20	30
grid	751×201	751×201	1501×401	2001×534
No-Prec	17446(1375)	6623(538)	14687(4572)	–
ILU(A,0)	3058(365)	1817(219)	3854(1904)	–
MG(V(1,0))	16(9)	177(75)	311(537)	485(1445)
MG(V(1,1))	13(9)	169(94)	321(728)	498(1891)

As seen from the table, the preconditioner accelerates the BiCGSTAB convergence dramatically as compared with the unpreconditioned case. With respect to the CPU-time, a significant reduction results. The method also outperforms BiCGSTAB preconditioned with the standard ILU(0). Evidently, one pre-smoothing and no post-smoothing in one V-cycle are sufficient to achieve satisfactorily convergence.

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