Sub-grid finite-difference modeling of wave propagation and diffusion in cracked media

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Summary

We present a finite-difference modeling technique that can be applied to wave propagation and diffusion problems in media containing a large number of small-scale cracks. The embedding medium can be heterogeneous. The boundaries of the cracks are not represented in the finite-difference mesh but the cracks are incorporated as distributed point sources. The strength and distribution of these point sources is derived from the finite-difference operator. This technique allows for the use of grid cells that are considerably larger than the crack sizes. The method is accurate and computationally very fast.

Introduction

Heterogeneities that are much smaller than the seismic wavelength cannot be distinguished individually using seismic waves, but nevertheless can have a significant effect on the amplitude and phase of the transmitted wave field. In the long-wavelength limit, a homogeneous embedding containing small-scale heterogeneities effectively behaves as a homogeneous medium, in which small-scale heterogeneities manifest themselves through apparent anisotropy and/or attenuation and dispersion. Most methods concerning wave propagation in media with embedded inclusions are based on this concept of an effective medium. A similar problem arises for the evolution of diffusive fields in cracked media.

For the case of a large number of cracks embedded in a homogeneous medium, accurate integral-equation techniques have been developed for the wave equation by Muijres et al. (1998). Both Muijres' method and the effective medium methods are not applicable to the case of cracks in the direct vicinity of a boundary or embedded in a heterogeneous medium. Nevertheless, these situations might arise when studying, for instance, wave propagation through a cracked reservoir in a layered earth or when investigating the propagation of boundary waves in tunnel walls containing cracks. Finite-difference techniques are well suited for solving wave propagation problems in heterogeneous media. The presence of cracks in this type of methods can be accounted for by incorporating explicit boundary conditions at the crack location. This implies that each crack boundary has to be incorporated in the finite-difference mesh, requiring a prohibitive amount of grid points in the case of a large number of small-scale cracks.

In the present paper, a finite-difference technique is presented for the computation of wave propagation of scalar, two-dimensional waves in a heterogeneous medium containing a large number of small-scale cracks. Instead of imposing explicit boundary conditions at the crack boundaries, our method accounts for the presence of the cracks by introducing secondary point sources, the strength of which is computed using perturbation theory. In order to represent the point sources properly on a coarse finite-difference grid, an asymptotic method is used based on the integral representation of the scattered field of a small crack. We have developed the method for the two-dimensional scalar wave equation and diffusion equation and are currently considering the elastic case.

Description of the method

We consider a two-dimensional field quantity in an inhomogeneous medium containing a large number of small-scale cracks characterized by an explicit boundary condition. The field quantity, ϕ, is first split into the incident field, ϕ^{inc}, and scattered field, ϕ^{sc}, in the following way: $\phi(\mathbf{x},t) = \phi^{inc}(\mathbf{x},t) + \phi^{sc}(\mathbf{x},t)$. The incident field, which would be present in the absence of the cracks, satisfies the wave- or diffusion equation without any explicit boundary conditions on the cracks:

$$L(\mathbf{x},t)\ \phi^{inc}(\mathbf{x},t) = s(\mathbf{x},t)\ , \tag{1}$$

where the operator L corresponds to the wave operator $[c^{-2}\partial_{tt} - \Delta]$ in the case of the wave equation and to the diffusion operator $[\alpha^{-2}\partial_t - \Delta]$ in the case of the diffusion equation (with c the wave velocity and α the diffusion constant, respectively). Furthermore, s is the source that generates the incident field. It can be shown that the scattered field, accounting for the presence of the cracks, satisfies the following equation:

$$L(\mathbf{x},t)\ \phi^{sc}(\mathbf{x},t) = -\sum_m q_m(t) \cdot \mathrm{pointsource}(\mathbf{x}_m)\ , \tag{2}$$

where the m is the index labeling the m-th crack, pointsource($\mathbf{x}_m$) is either a monopole or dipole point source at the location of the crack and q_m is the secondary source strength, depending upon the incident field and the type of explicit boundary condition considered. With the aid of perturbation theory, this secondary source strength can be related to the incident field at the crack center $\mathbf{x}_m$. In the derivation of Eq.(2), we have assumed that the crack is small with respect to the length scale in which the incident field varies and we have neglected the interaction between cracks. For the computation of the incident field given by Eq.(1), we have used the finite-difference scheme by Dablain (1986). After computation of the incident field, the scattered field is computed by solving Eq.(2) using a similar scheme. The presence of the cracks is now accounted for by secondary point sources representing the cracks. The strength of each secondary source is related to the incident field. Computation of the scattered field in this way implies that a large number of point sources have to be considered that do not have to coincide with the grid points. Because point sources cannot be accurately represented in a finite difference scheme – in particular if the position of the crack does not coincide with a grid point – subgrid modeling is used. This means that the effect of each crack has to be distributed over its surrounding grid points. These grid points together should produce the same scattered field as the crack. Briefly, this is done as follows: First, the scattered field close to each crack is approximated by its asymptotic value taking into account the leading order terms for distances to the crack that are small compared to the wavelength. Then, the finite-difference operator is applied to this scattered field resulting in a numerically constructed source distribution of each crack. This source distribution is subsequently used as source term in stead of the pointsources on the right-hand side of Eq.(2). It can be shown, that, for the case of the wave equation, Eq.(2) can then be written as

$$L(\mathbf{x},t)\ \phi^{sc}(\mathbf{x},t) = \sum_m\ A_0(\mathbf{x},\mathbf{x}_m)q_m(t) + A_2(\mathbf{x},\mathbf{x}_m)\frac{1}{c^2}\partial_{tt}q_m(t) + A_4(\mathbf{x},\mathbf{x}_m)\frac{1}{c^4}\partial_{tttt}q_m(t). \tag{3}$$

The coefficients A_0, A_2 and A_4 depend on the actual finite-difference operator used. For the diffusion equation, a similar result is obtained with $\partial_{tt}q_m(t)$ replaced by $\partial_t q_m(t)$ and $\partial_{tttt}q_m(t)$ replaced by $\partial_{tt}q_m(t)$. The numerically constructed source distribution of all cracks over the grid points is now used as right-hand side of Eq.(2).

Numerical results

We have implemented this method for the two-dimensional scalar wave equation and the diffusion equation. As a first example, we show the validation of the method for the case of the diffusion equation. In Figure 1, we display the scattered diffusive field due to a small crack with the Neumann explicit boundary condition (implying that the normal derivative of the total field vanishes at the crack). Two curves are shown, being the analytical and the finite-difference solutions. Agreement is very good, despite the fact that the crack is much smaller than the grid size. To illustrate the versatility of our method, we have also simulated a cross-well experiment for the wave equation. We consider a low-velocity layer, containing cracks, between two high-velocity layers. The geometry is shown in Figure 2. A seismic source, situated

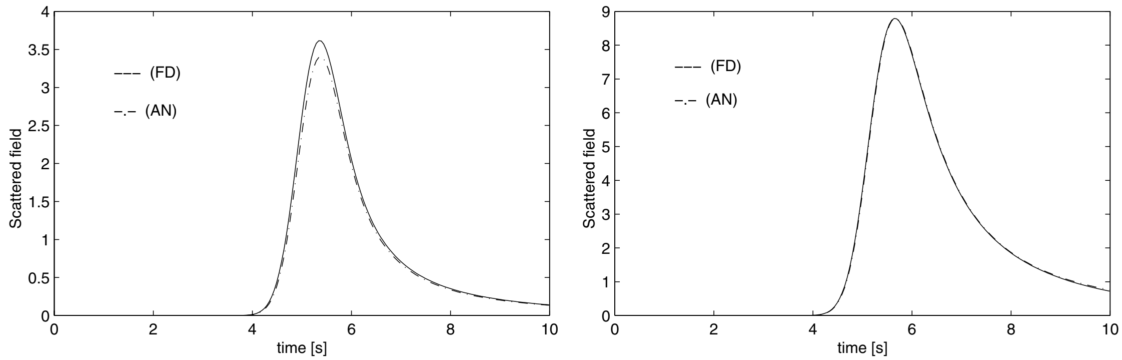

Figure 1: *Comparison of the scattered diffusive field computed with the finite-difference (FD) method of this paper and the analytical solution (AN). The two figures display the scattered field at different positions in the vicinity of the crack.*

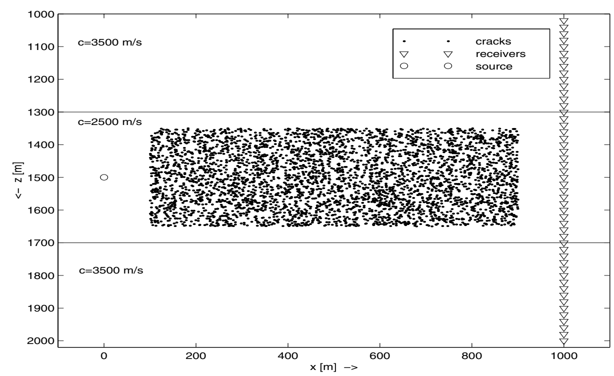

Figure 2: *Geometry of the cross-well simulation. A low-velocity layer ($c = 2500\,m/s$), containing 4000 randomly positioned cracks, is embedded in two high-velocity layers ($c = 3500\,m/s$). The seismic source is located in the low-velocity layer (at the left-hand side), whereas the receivers are positioned in a vertical borehole at the right.*

in the low-velocity layer, generates the wave field. The receivers are positioned in a vertical borehole. When the source is located in the low-velocity layer in between the two high-velocity layers, most energy emitted by the source will remain trapped in this layer as a "guided" wave. The density ρ is constant everywhere; the layer contains 4000 cracks. We consider four cases: the case without cracks, and the cases with horizontally, vertically, and randomly oriented cracks, respectively. Figure 3 shows the results for a receiver within the layer, located at $(1000, 1500)$, for all four cases. In Figure 3(a), we can distinguish the different arrivals of the reflections. Figure 3(b) is a detail of Figure 3(a). It clearly shows, that the signal for the case of horizontally oriented cracks coincides with the incident field until about $t = 0.44\,\mathrm{s}$. Horizontally oriented cracks have hardly any effect on the wave field as long as it propagates parallel to the cracks. After reflection, the direction of the wave field has changed giving rise to more scattering by the horizontally oriented cracks. The cracks have the largest effect when oriented vertically; randomly oriented cracks have intermediate effects. Compared to the incident field, we observe two effects: the presence of the cracks affects the amplitudes and also results in a time delay.

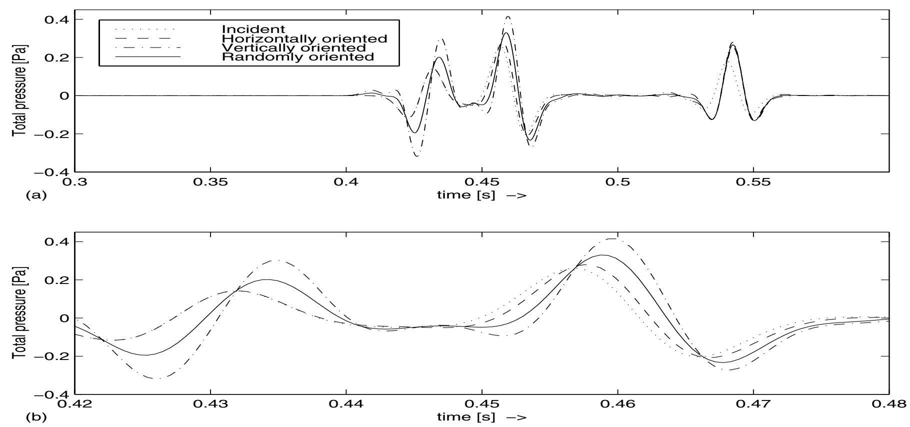

Figure 3: *Detailed comparison for a receiver at (1000,1500) for the case of Fig. 2. Results are shown for the following cases: (i) no cracks (incident field only), (ii) all cracks horizontally aligned, (iii) all cracks vertically aligned, (iv) cracks randomly oriented. (a) Results between 0.3 s and 0.6 s, (b) Detail between 0.42 s and 0.48 s.*

Conclusions

We have presented a finite-difference method for computing the scattering of two-dimensional fields by many small-scale cracks, characterized by an explicit boundary condition. As the cracks can be embedded in a heterogeneous medium, our method is an extension to methods based on integral equations which require a homogeneous background.

We have solved the problem of representing a small crack on a finite difference grid by deriving a suitable numerical source term. This has resulted in a substantial reduction of computing time, without appreciable loss of accuracy, as became clear from the comparisons with the method of Muijres et al. (1998).

So far, we have applied this method to the wave equation and the diffusion equation. Currently, we are applying the method to the two-dimensional elastodynamic P-SV wave equation. In that case, ϕ is a two-component vector containing the in-plane displacement components and L corresponds to the P-SV elastic wave operator.

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