A systematic approach to the construction of higher-order finite elements with mass lumping for the wave equation

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Summary

The higher-order finite-element scheme with mass lumping for triangles and tetrahedra is an efficient method for solving the wave equation. A number of lower-order elements have already been found. Here we continue the search for elements of higher order.

Elements are constructed in a systematic manner. The nodes are chosen in a symmetric way. Integration rules must be exact up to a certain degree to maintain an overall accuracy that is the same as without mass lumping. First, for given integration degrees, consistent rule structures are derived for which integration formulas are likely to exist. Then, as each rule structure corresponds to a potential element of certain order, the position of element nodes and the integration weights can be found by solving the related system of nonlinear equations.

With this systematic approach, all known elements have been reproduced and a new sixth-order triangular and fourth-order tetrahedral element have been found.

Introduction

The finite-difference method (FDM) is a popular numerical technique for the simulation of wave propagation through air, water, and solids. The method is relatively easy to implement and allows for straight-forward parallelisation. It is, however, difficult to model sharp contrasts in material properties on regular cartesian grids with higher-order finite-differences. Also, the FDM becomes less accurate near abrupt changes in the propagation velocity.

Finite elements for triangles and tetrahedra are better suited to model rough topography and sharp interfaces between different materials, because these can be fitted by the element boundaries. The finite-element method (FEM) in its original form requires the solution of a large sparse linear system of equations, which make the method costly. This cost can be avoided by mass lumping, a technique that replaces the large linear system by a diagonal matrix. To avoid negative integration weights that cannot produce a stable time-stepping scheme, additional nodes have to be added to the element. Initial results for triangular elements can be found in (Cohen et al., 1993), (Cohen et al., 1995) and (Tordjman, 1995). The extension of this idea to triangular elements of still higher-order and to tetrahedra can be found in (Mulder, 1996a). There, a it was also shown by a comparison on a simple two-dimensional reflection problem that the higher-order FEM is more efficient than the FDM. A comparison between finite-element schemes of various orders revealed that the higher-order approximations are more efficient than the lower orders. This motivates the search for elements of still higher order.

New elements

So far, elements up to 5th order for triangles and 3rd order for tetrahedra have been found (in one space dimension, the Gauss-Lobatto points will provide suitable mass-lumped elements (Mulder, 1996b)). Here we continue the search for higher order elements in a systematic manner, using the theory on consistency conditions for symmetric integration rules (Keast and Diaz, 1983; Keast, 1986; Keast, 1987). Mass lumping without loss of accuracy is equivalent to numerical integration with certain weights on the element nodes. The numerical integration is accurate to a certain order if all polynomials up to a given degree are integrated exactly. This leads to a system of equations that is linear in the integration weights and polynomial in the parameters describing the node positions. Because this nonlinear system is, in general, difficult to solve, it helps to have conditions that guarantee the existence of a solution. The consistency conditions ensure that there is a sufficient number of nodes and node parameters to integrate the polynomials exactly and that the number of equations does not exceed the number of unknowns.

Table 1: A sixth-order triangular element with 30 nodes ($M = 5$, $M_f = 7$, $d = 10$).

nodes		weights	parameters
$(0, 0)$	3	0.7094239706792450E-03	-
$(\alpha_1, 0)$	6	0.6190565003676629E-02	0.3632980741536860E-00
$(\alpha_2, 0)$	6	0.3480578640489211E-02	0.1322645816327140E-00
(β_1, β_1)	3	0.3453043037728279E-01	0.4578368380791611E-00
(β_2, β_2)	3	0.4590123763076286E-01	0.2568591072619591E-00
(β_3, β_3)	3	0.1162613545961757E-01	0.5752768441141011E-01
(γ_1, δ_1)	6	0.2727857596999626E-01	0.7819258362551702E-01
			0.2210012187598900E-00

Although the conditions are neither necessary nor sufficient for general nonlinear systems, this approach turned out to be fruitful for the present problem. We have been able to find all known elements (Mulder, 1996a) and a few new ones, listed in Tables 1–3. For each element the

Table 2: A fourth-order tetrahedral element with 50 nodes ($M = 3$, $M_f = 5$, $M_i = 6$, $d = 7$).

nodes		weights	parameters
$(0, 0, 0)$	4	0.2143608668049743E-03	-
$(\alpha, 0, 0)$	12	0.8268179517797114E-03	0.2928294047674109E-00
$(\beta_1, \beta_1, 0)$	12	0.1840177904191860E-02	0.1972862280257976E-00
$(\beta_2, \beta_2, 0)$	12	0.1831324329245650E-02	0.4256461243139345E-00
(γ, γ, γ)	4	0.7542468904648131E-02	0.9503775858394107E-01
$(\delta, \delta, \frac{1}{2} - \delta)$	6	0.1360991755970793E-01	0.1252462362578136E-00

Table 3: A second fourth-order tetrahedral element with 50 nodes ($M = 3$, $M_f = 5$, $M_i = 6$, $d = 7$)

nodes		weights	parameters
$(0, 0, 0)$	4	0.2321968872348930E-03	-
$(\alpha, 0, 0)$	12	0.7328680241632055E-03	0.3052598756695660E-00
$(\beta_1, \beta_1, 0)$	12	0.2529792598144742E-02	0.4204599755540437E-00
$(\beta_2, \beta_2, 0)$	12	0.1564461923378417E-02	0.1480462980008327E-00
(γ, γ, γ)	4	0.7127911446564579E-02	0.1048645248917035E-00
$(\delta, \delta, \frac{1}{2} - \delta)$	6	0.1321679379720540E-01	0.1258796196682507E-00

nodes are listed. On each line, one node is given in normalised coordinates, followed by the number of symmetric nodes, the corresponding weight, and the parameter(s) describing the node position. The parameter M denotes the degree of the polynomials used for the basis functions. This space is enlarged in order to obtain positive weights. On triangles, the larger space consists of polynomials of degree $M_f \geq M$ that have a restriction of degree M on the edges. For tetrahedra, we include polynomials in the interior of degree $M_i \geq M_f \geq M$ that have a restriction of degree M_f on the faces, and polynomials of degree M_f on the faces that have a restriction of degree M on the edges. The vertices are always included. This approach guarantees conformity.

Comparison of orders in 2D

A comparison between finite differences and higher-order finite elements has been carried out in (Mulder, 1996a) for a simple reflection problem. It was shown that the finite-difference method is less efficient than the finite-

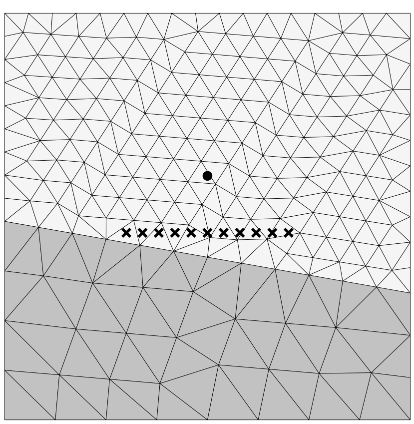

Fig. 1: Example of a FEM grid for a simple reflection problem. The top layer has a velocity of 1.5 km/s, the bottom of 3.0 km/s. The source is marked by a dot, the receivers by crosses.

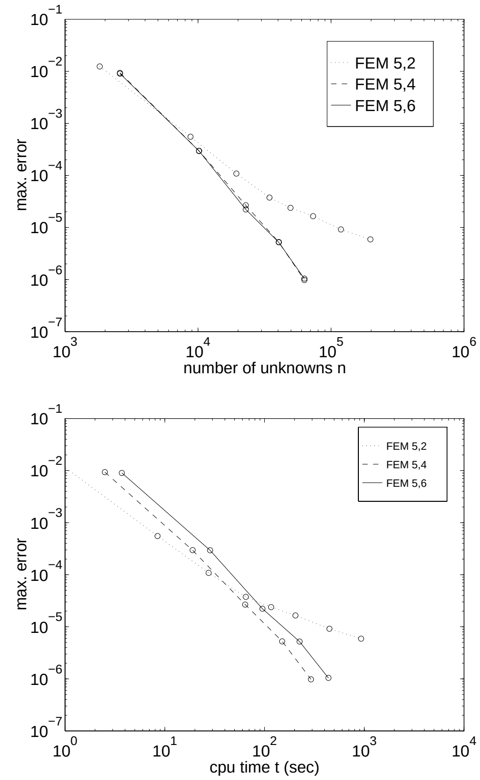

Fig. 2: Errors for the FEM (6th order in space) as a function of the number of degrees of freedom (left) and cpu-time (right) using a 2nd, 4th, and 6th order time-stepping scheme.

element method, despite the added complexity of the latter. The main reason is that the high-order finite-difference method loses its accuracy near sharp interfaces in the velocity model. Finite elements with edges that fit the interface maintain their accuracy.

Among the elements of various order, it turned out that higher-order elements are more efficient than the lower-order ones, at least up to 5th order. Here we extend the results of (Mulder, 1996a) by including the new 6th order element ($M = 5$). The main questions are: which temporal error is the most efficient for a given spatial error, and which spatial error is the most efficient?

Figure 1 shows the simple two-dimensional reflection problem and a finite-element grid that follows the sharp interface. The size of the domain is 1000×1000 m^2. The grid density has been scaled to the velocity. The traces (receiver data) have been recorded at the positions marked by crosses in Fig. 1.

The numerical solution has been compared to the exact solution for this simple problem, using trace data between 0.102 and 0.300 seconds at 2 ms intervals. The computation is initialised with the exact solution at 0.1 s, at which time the direct wave has not yet reached the interface.

Figure 2 shows the maximum error as a function of the number of degrees of freedom and cpu-time, respectively.

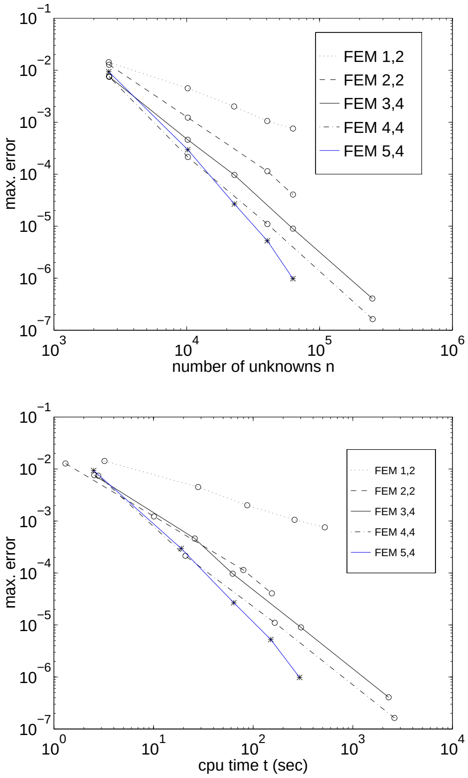

Fig. 3: Comparison of various orders

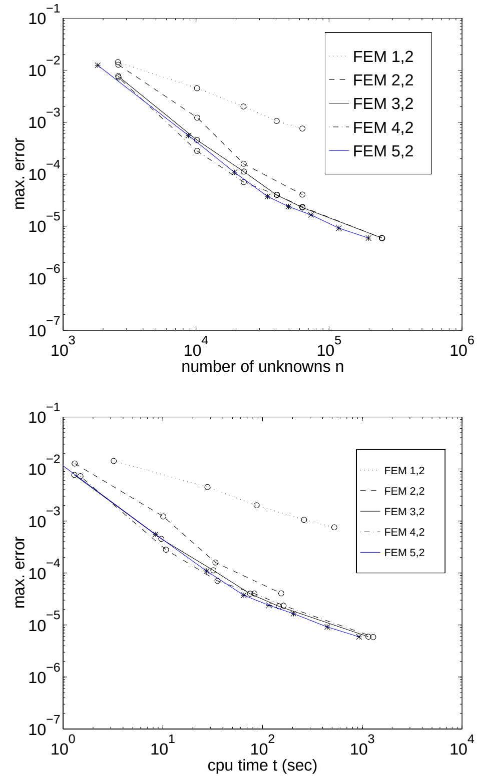

Fig. 4: Comparison of various orders with 2nd-order time-stepping

The computations where carried out on an IBM RS/6000 3AT with a program written in C, using double precision arithmetic. The time-stepping scheme is the same as in (Dablain, 1986). Here we considered 2nd, 4th, and 6th order in time. The time-step Δt was chosen such that $\Delta t\sqrt{n} \sim 0.022$.

In the left panel of Fig. 2 it can be seen that the scheme behaves as a sixth-order scheme for the larger errors. For smaller errors, the temporal error of the second-order time-stepping scheme starts to show up. For the 4th- and 6th-order time-stepping scheme, the errors are practically the same, showing that in those cases the spatial error dominates. Because the 6th-order time-stepping scheme involves about 1.5 times more operations than the 4th-order scheme, it appears as the less efficient one in the right panel of Fig. 2. For larger values of the error, the 2nd-order time-stepping scheme is more efficient.

Next we make a comparison among various orders (the spatial order is $M+1$ and $M = 1, \ldots, 5$). For each order, we choose the time-stepping scheme that appears to be the most efficient at the level of a maximum error around 10^{-5} (for the present problem, 1% accuracy corresponds to an error between 10^{-4} and 10^{-3}). The results are summarised in Fig. 3. The results for 2nd-order time-stepping are displayed in Fig. 4. These figures show that for moderate accuracy (10^{-4} or somewhat larger), 4th-order in space and 2nd-order in time is attractive. For high accuracy, 6th-order in space and 4th-order in time becomes the more efficient scheme.

This still leaves the question open, whether or not it pays to go to schemes of still higher spatial order.

Conclusions

We have followed a systematic approach for the construction of mass-lumped triangular and tetrahedral elements for solving the wave equation. The search has been restricted to elements that fulfill the following requirements: (i) conformity, (ii) symmetric arrangement of nodes, (iii) positive integration weights, (iv) same order of accuracy as elements without mass lumping. Using the theory of consistency conditions, we have been able to reproduce all known lower-order elements. A new 6th-order triangular element and a new 4th-order tetrahedral element have been found.

The computational efficiency of the new 6th-order triangular element has been tested by considering a simple two-dimensional seismic reflection problem. For this problem, it proved to be more efficient (with 4th-order time-stepping) than the lower-order elements if high accuracy is desired. For moderate accuracy, 4th-order in space and 2nd-order in time appears to be sufficient, at least for the simple short-time reflection problem considered here. The question remains if elements of still higher accuracy will be even more efficient when high accuracy is a requirement.

The efficiency of the mass-lumped tetrahedral elements is unknown, both in comparison to the finite-difference method, and in comparison to standard tetrahedral elements of higher order. In the last case, the use of fast iterative sparse-matrix solvers may produce a scheme that competes with the rather complex mass-lumped elements.

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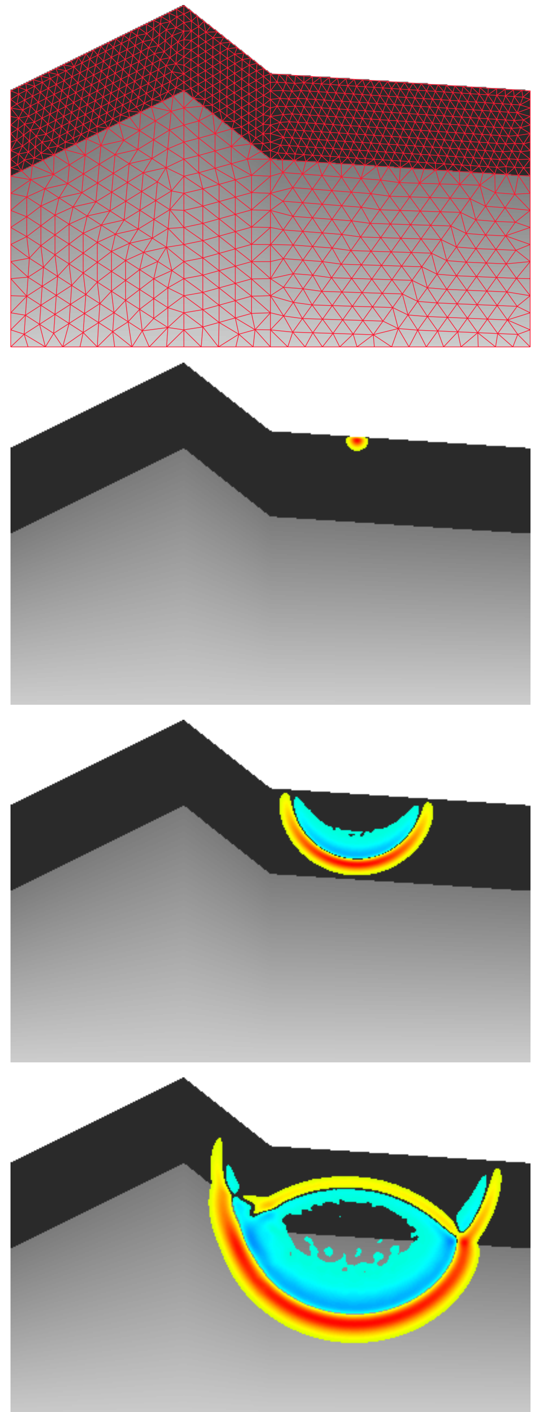

Fig. 5: Example of a model and snapshots at intervals of 0.2 seconds