Hard and Smooth Models in Seismic Imaging: Some Open Problems

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ABSTRACT

We discuss several aspects of raytracing solutions in smooth velocity models compared to hard interface models. The emphasis of this paper is to identify a fundamental open problem of describing wave propagation in a medium with singularities.

1 Introduction

Understanding of wave propagation is essential in geophysical imaging techniques. Most of these techniques require an a *priori* approximate model of the subsurface, the so-called background model $c_0(x)$. The true model $c(x)$ is considered as a linear perturbation on this background:

$$c(x) = c_0 + \delta c(x). \tag{1}$$

The data are considered as being caused by first order perturbations on this background and the imaging technique aims at determining these perturbations from the data. Apart from this linearization conventional imaging techniques involve another approximation namely a high frequency approximation to the solution of the wave equation in the background model. Thus a commonly used imaging formula is

$$\frac{\delta c(x)}{c_0^3} = \int drds\ B(x)\, p^{\mathrm{data}}(t = t_{sx} + t_{xr}) \tag{2}$$

where p^{data} is the data, $B(x)$ is an amplitude function and t is a traveltime function measuring the time from source to receiver via a reflection at x in the subsurface. Both the traveltime and amplitude functions are found from raytracing in the background model c_0. As a result of the high frequency approximation this imaging is not a true inverse. The reflectivity $\frac{\delta c(x)}{c_0^3}$ is determined up to smooth functions in x, that is, only the singular or rapid oscillatory components of the velocity are determined. It therefore seems an obvious choice to require the velocity model to be smooth: all reflection data are caused by discontinuities that can be considered as rough perturbations on the smooth background model. In its simplest form, the background is a piecewise constant model in which finite jumps occur across interfaces.

0-8194-2210-X/96/$6.00

Two aspects of this approach make the construction of a proper background model less obvious. First, frequencies are only measured in a finite band width. Secondly, variations of the medium parameters occur at several length scales. Large-scale fluctuations might be defined as variations that have a length scale larger than the dominant wavelengths in the data, where small-scale variations occur on length scales much smaller that the dominant wavelengths. Unfortunately, this distinction depends on the angle of incidence: for one range of angles the medium may appear smooth, whereas for other angles the same medium may appear discontinuous. As a consequence, it is difficult to separate those aspects from the model that may be considered smooth from those that may be viewed as perturbations.

2 Some Experiments

It is an open question how wave propagation in the high-frequency approximation in a smooth medium is related to the propagation in a discontinuous medium, except in one-dimensional media. In this talk we study the relation between smooth models and piecewise constant models (i.e., exhibiting velocity jumps at interfaces) by studying wave propagation in a one-parameter family of media that interpolate between the two extremes. We show evidence that the wave propagation in the limit from a smooth to a discontinuous model is *singular*. The reflection coefficient, however, has a smooth limit. This is in contrast with the situation in one dimension, where recent results show that propagation is non-singular in the limit from a smooth to a hard model.[1]

As an illustration, we consider a medium that consists of two half-spaces, with a constant velocity of 3.5 on the "left" and 1.5 on the "right". Figure 1 shows a two-dimensional finite-difference solution obtained in this medium. The wave front consists of several parts: the direct wave and transmitted wave, a reflected wave (part of a circle) and a refracted wave (a straight line). The refracted wave becomes tangent to the reflected wave at a given point, marked by the black dot in Fig. 2d.

Figure 2a–c show traces of ray-tracing solutions for smoothed media,

$$c_0^{-2} = A \tanh(z - z_0)/H + B, \quad A,\, B \text{ const.}, \tag{3}$$

superimposed on a 3D finite-difference solution in the hard medium. The amount of smoothing is characterised by a smoothing length H. Figure 2d shows the exact wave front. The wave front obtained by ray tracing can be seen to approach the wave front in the discontinuous case for decreasing H (Fig. 2a to 2c), except for one feature. The missing part corresponds to the reflected wave for smaller angles. This is the part of the trace data in Fig. 2d beyond the black dot (i.e., for later times). This suggests that the limit for vanishing smoothing length of the wave front is singular. In summary we draw the following conclusions:

- In a smooth medium, ray-tracing produces a *continuous* wave front.
- The various branches of the ray-tracing wave front can be identified with the direct wave, the transmitted wave, the refracted wave, and part of the reflected wave in the limit of vanishing smoothing length. This limit is singular.
- The part of the reflected wave that is captured by the ray-tracing in a smoothed medium is the part that lies between the critical angle and the point where the refraction wave becomes tangent to the reflected wave.

The theory needed to explain these observations exists only partially. Of course the theory of high frequency wave propagation in *smoothly* varying media is well developed. Propagation is described in terms of an operator assigning to the medium parameters the data, (i.e. the solution of the of the wave equation). It is known that the forward operator depends in a very nonlinear way on the medium parameters. This makes it hard to study this operator as a function of a decomposition of smooth and rough parts of the medium parameter. Recently, it was proven that the forward wave operator in a high frequency approximation is under mild conditions on the medium parameters a Fourier Integral Operator (FIO). There exists a rather complete calculus for such operators. With this calculus it is possible to associate to the Cauchy problem a so-called Lagarangian manifold in the phase space (containing the three dimensional physical space). The solution of the Cauchy problem requires the computation of this Lagrangian manifold, which may be done by applying the method of stationary phase.

If, however, the medium parameters have singularities, the Lagrangian manifolds are no longer simply connected and the construction by applying the method of stationary phase to compute the wave front set is not possible, partly due to the inherent diffraction effects occurring at the interface. At present it is unknown how the existing calculus of FIOs should be formulated to incorporate refraction effects at the interface. It is clear though that the theory of FIOs in media with singularities will provide much needed insight in the construction of the associated inverse problem. In one spatial dimension the problem has been studied in[1] using techniques applicable strictly in one dimension. Partial results combining FIOs with local calculations at an interface exist in higher dimensions[3] however, a complete theory is still lacking. There thus seems the following open, fundamental question:

How does the solution of the Cauchy problem in a smooth medium relates to the solution of the Cauchy problem in a discontinuous medium obtained from the smooth medium by a limiting process in which the medium becomes singular?

Related to the observations made above, we note two other apparent facts, which eventually need to be incorporated in a unifying theory of propagation in media with singularities, namely

- the traveltime branch that went through the focus, i.e. the reflection branch in this case, is more sensitive to the smoothing than the parts that did not go through the focus;
- the amplitude variation as measured by the density variation of rays is much more sensitive to the smoothing than the traveltime.

The second observation can be explained by considering the smooth model as a "perturbation" on a constant model:

$$c(x) = c_0[1 + \epsilon(x)], \quad |\epsilon(x)| \ll 1. \tag{4}$$

One may then derive a an analogous result as obtained in[2] for smooth random models, which implies that to first order the traveltime in the smooth model agrees with the constant model, but the amplitude along the wave front does not. The first observation above, may be explained by a similar generalization of the result in[2] to any smooth non-constant model. We may formulate this result as follows:

For any non-constant regular, smoothly varying velocity function in two dimensions, there will be a caustic point at a finite distance from almost any source position. Raytracing on scales larger than this distance is not uniformly valid, leading to substantial traveltime and amplitude errors.

Thus in a smooth model the wave front will eventually develop a caustic within a finite time in some direction of propagation (except for a small set of special source positions). Compare this with the result in a smooth, random model where a caustic will develop eventually in *any* direction from the source! We will make this result more precise in a forthcoming paper where we will also discuss a few practical consequences for raytracing in smooth backgrounds.

Acknowledgement We thank Shell International Exploration and Production B.V. for kind permission to publish this work.

3 REFERENCES

[1] J. J. Duistermaat *private communication. Prof. Duistermaat has informed us that this result is part of work in progress by D. Elton, Oxford University*

[2] B. White, *The stochastic caustic*, SIAM J. Appl. Math. **44**, 127-149, (1982).

[3] S. Hansen, Solution of a hyperbolic inverse problem by linearization, Comm. Partial Diff. Eq. **16**, 291-309, 1991.

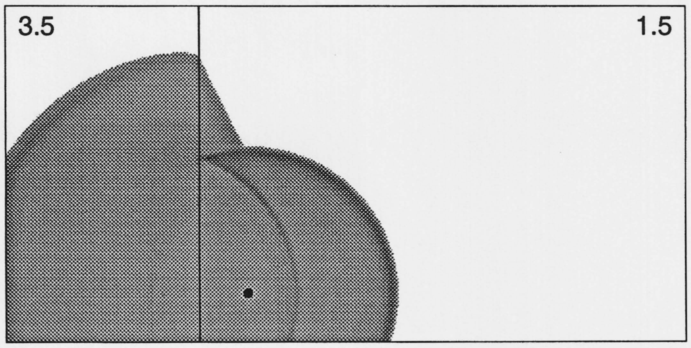

Fig. 1. 2D finite-difference solution obtained for a piecewise constant medium with a velocity of 3.5 at the left and 1.5 at the right. The position of the source is marked by a dot. The size of the domain is 14×7.

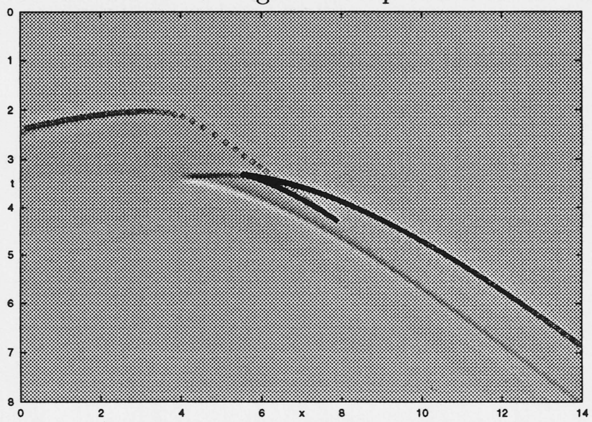

Fig. 2a. Ray-tracing result (diamonds) for a smooth model with $H = 0.5$. The gray-scale is obtained from a 3D finite-difference solution in the piecewise constant model ($H = 0$).

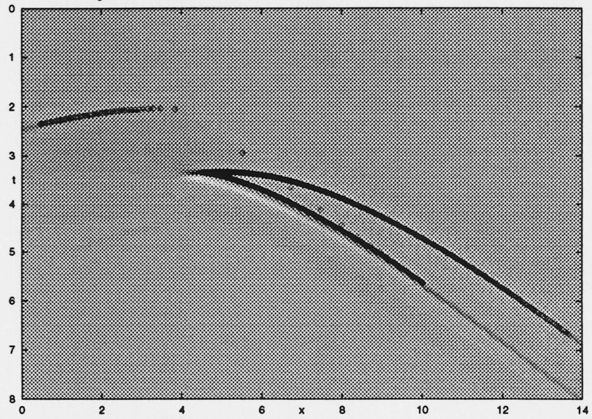

Fig. 2b. Ray-tracing result for a smooth model with $H = 0.2$ compared to a finite-difference solution with $H = 0$.

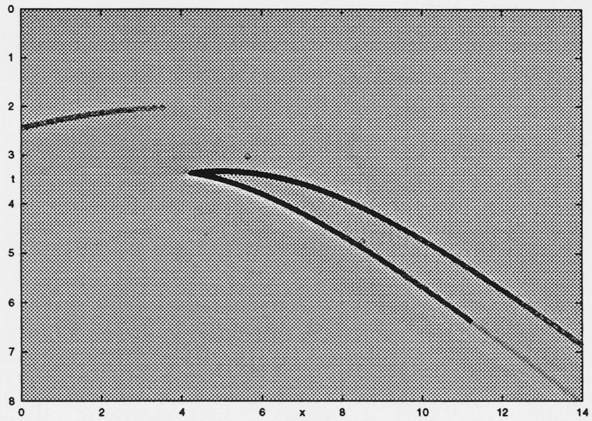

Fig. 2c. Ray-tracing result for a smooth model with $H = 0.1$ compared to a finite-difference solution with $H = 0$.

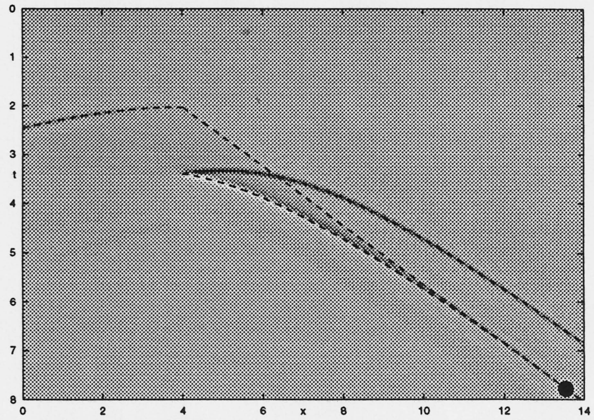

Fig. 2d. Exact ray-tracing solution (dashed lines) and finite-difference result.