FAST TRACK PAPER

The exact solution of the time-harmonic wave equation for a linear velocity profile

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SUMMARY

We present the exact solution of the acoustic wave equation in the frequency domain in two space dimensions for a velocity that increases linearly with depth and a density that is a power law in depth. The geometric properties of the solution are described by the bipolar coordinate system. The asymptotic behaviour for large frequencies matches the result obtained by ray theory. The 3-D solution was already known.

Key words: exploration seismology, seismic modelling, synthetic seismograms, wave propagation.

1 INTRODUCTION

Exact solutions of the acoustic and elastic wave equations are important for the qualitative analysis of various wave phenomena and for numerical applications. Examples are the construction of boundary element methods, testing the accuracy of modelling codes and determining the range of validity for approximate methods such as ray tracing.

Apart from the well-known solution for the homogeneous case, there are only a few solutions for inhomogeneous earth models. Pekeris (1946) solved the 3-D scalar wave equation for a vertical linear velocity profile $c(z) = c_0 + \alpha z$, where z denotes depth. This solution can be also found in the book by Brekhovskikh & Godin (1992). Beydoun & Ben-Menahem (1985) generalized this result to the elastodynamic case. Vrettos (1990a,b,c, 1991) considered horizontal shear waves in 2-D for a shear modulus increasing with $1 - \exp(-az)$, without and with a free surface. In 2-D, the equation for the SH waves has the same structure as the acoustic wave equation. Albertin (1992) obtained a solution of 2-D wave equation for a velocity profile of the form $c(z) = (c_0 - \beta z)^{-2}$ in terms of Airy functions. Bhattacharya (1970) lists vertical profiles for density and shear modulus for which the SH wave equation has standard transcendental functions as solutions. These solutions are only given in the horizontal wavenumber domain. Again, since the SH wave equation has the same structure as the 2-D acoustic wave equation, solutions of one can be used for the other. Ben-Menahem & Beydoun (1985) and Sánchez-Sesma *et al.* (2001) give approximate solutions for the 2-D wave equation with a linear vertical profile. Here, we present the exact solution.

In Section 2, we derive the Green function for the wave equation with the velocity profile of the form $c(z) = c_0 + \alpha z$ and a constant density. Next, formulae for the numerical evaluation and an example are presented. An asymptotic result is shown to agree with the high-frequency approximation made by ray theory in Section 2.3. The rays are parts of circles. The Green function has the same geometrical properties as described by the rays as well as by the bipolar coordinate system. Similar results for the 3-D case (Pekeris 1946) are included in Section 3 for completeness. Section 4 summarizes the straightforward extension to a density distribution that is a power law in depth.

2 TWO DIMENSIONS

2.1 Derivation

The wave equation for constant-density acoustics is given by

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\hat{p} = \hat{s}. \tag{1}$$

Here, $\hat{p}(x, z, t)$ is the pressure, $c(x, z)$ is the sound speed, and $\hat{s}(t, x, z)$ the source term. In two space dimensions, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$. We consider the case of a velocity that increases linearly with depth:

$$c(z) = c_0 + \alpha z, \tag{2}$$

where c_0 and α are positive constants. Since the sound velocity cannot be negative, the region where $z < -c_0/\alpha$ will not be considered. We take the Fourier transform of eq. (1) with respect to time t and horizontal coordinate x, using the conventions

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)e^{-ikx}dk, \quad \hat{g}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{+i\omega t}d\omega,$$

which are the conjugates of what is common in geophysics. Eq. (1) becomes

$$\left(\frac{\partial^2}{\partial z^2} - k^2 + \frac{\omega^2}{(c_0 + \alpha z)^2}\right)p(\omega, k, z) = -s(\omega, k, z), \tag{3}$$

where ω is the angular frequency, k the wavenumber corresponding to x, and $s(\omega, k, z)$ is the transformed source term.

Introducing the normalized variables $\hat{z} = z + c_0/\alpha$ and $\hat{\omega} = \omega/\alpha$, we obtain the equation for the Green function,

$$\left(\frac{\partial^2}{\partial \hat{z}^2} - k^2 + \frac{\hat{\omega}^2}{\hat{z}^2}\right) p_G(k, z) = -\delta(\hat{z} - \hat{z}_0), \tag{4}$$

where $\hat{z}_0$ is the source position. The general solution of eq. (4) is a linear combination of $\psi_1 = \hat{z}^{1/2} I_\nu(k\hat{z})$ and $\psi_2 = \hat{z}^{1/2} K_\nu(k\hat{z})$, where $I_\nu(k\hat{z})$ and $K_\nu(k\hat{z})$ are modified Bessel functions of the order

$$\nu = i\mu, \qquad \mu = \sqrt{\hat{\omega}^2 - 1/4}. \tag{5}$$

If $|\hat{\omega}| > 1/2$, then the order ν of the Bessel functions is purely imaginary. With a wavelength $\lambda = 2\pi c/\omega$, the above condition can be written as $\delta c/c < 4\pi \delta z/\lambda$. This corresponds to the case where significant changes in the velocity occur on a length scale that exceeds about one tenth of a wavelength, which is a typical situation in geophysical problems. For $|\hat{\omega}| \leq 1/2$, ν is real valued and the solution is not wave-like. The above solution is applicable to both cases.

We specify the boundary conditions as $p_G \to 0$ for $\hat{z} \to +\infty$ and $|p_G| < \infty$ for $0 \leq \hat{z} < \hat{z}_0$, that is, the wave amplitude is finite at depths less than the source depth, up to the the point where the local wave velocity vanishes. These boundary conditions are satisfied if $p_G(k, z) \propto \psi_1$ in the region $z < z_0$ and $p_G(k, z) \propto \psi_2$ in the region $z > z_0$. This can be combined into a single expression of the form $p_G(k, z) \propto \psi_1(k\hat{z}_<)\psi_2(k\hat{z}_>)$, where $z_< = \min(z, z_0)$ is the smaller and $z_> = \max(z, z_0)$ is the larger of the two coordinates z and z_0. The proportionality constant is found in the standard way by integrating the above function over the source. It is straightforward to check that this constant is equal to one, which results in

$$p_G(k, \hat{z}) = (\hat{z}_>\hat{z}_<)^{1/2}\, I_\nu(k\hat{z}_<) K_\nu(k\hat{z}_>). \tag{6}$$

Because the Green function is an even function of k, its inverse Fourier transform is $p_G(x, \hat{z}) = (1/\pi) \int_0^\infty p_G(k, z) \cos(kx)\, \mathrm{d}k$. This is a standard integral (Bateman & Erdélyi (1954) or 6.672.4 in Gradshteyn & Ryzhik (1965)) that exists for $\mathrm{Re}\, \hat{z}_> > |\mathrm{Re}\, \hat{z}_<|$ and $\mathrm{Re}\, \nu > -1/2$. Taking into account that both conditions are satisfied in our case, we find the Green function in the coordinate space,

$$p_G(x, \hat{z}) = \frac{1}{2\pi}\, Q_{\nu-1/2}(u). \tag{7}$$

Here $Q_{\nu-1/2}$ is the Legendre function and

$$u = \frac{x^2 + \hat{z}_0^2 + \hat{z}^2}{2\hat{z}_0\hat{z}} = 1 + \frac{x^2 + (\hat{z} - \hat{z}_0)^2}{2\hat{z}_0\hat{z}}. \tag{8}$$

Obviously, $u \geq 1$ everywhere, and $u = 1$ at the source.

2.2 Numerical formulae

For the numerical evaluation of the Legendre function, we use the representation

$$Q_{\nu-1/2}(\cosh\eta) = \sqrt{\pi}\, \frac{\Gamma(\frac{1}{2} + \nu)}{\Gamma(1 + \nu)} e^{-(1/2+\nu)\eta} \, F\left(\frac{1}{2}, \frac{1}{2} + \nu, 1 + \nu, e^{-2\eta}\right). \tag{9}$$

Here, Γ is the gamma-function and F is the hypergeometric function. Note that η is real as $u = \cosh\eta \geq 1$. The hypergeometric function can be calculated directly from its definition,

$$F(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{10}$$

where $(\ldots)_n$ are the Pochhammer symbols, $(a)_0 = 1$, $(a)_n = a(a+1)\ldots(a+n-1)$ etc. Series (10) converges slowly if z is close to unity. For $|z - 1| \ll 1$, we therefore use an alternative representation of the Legendre function which follows from the expression for the hypergeometric function with $c = a + b$:

$$Q_{\nu-1/2}(\cosh\eta) = e^{-(1/2+\nu)\eta} \times \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(n!)^2} \big[2\psi(1+n) - \psi(a+n) - \psi(b+n) - \ln(1 - e^{-2\eta})\big](1 - e^{-2\eta})^n. \tag{11}$$

Here, $a = 1/2$, $b = 1/2 + \nu$ and ψ is the digamma function, $\psi(z) = d \ln\Gamma(z)/dz$. The gamma and psi functions in the above equations have complex arguments and can be calculated using relations given in Abramowitz & Stegun (1964). From eq. (11), one can reconstruct the asymptotic behaviour of the function $Q_{\nu-1/2}(u)$ near the source,

$$Q_{\nu-1/2}(u) \to -\ln\left(\frac{2r}{z_0}\right) - \frac{i\pi}{2} \tanh(\pi\mu) + Const, \tag{12}$$

where μ is given by eq. (5), $r = [x^2 + (z - z_0)^2]^{1/2}$ and $Const = 2\psi(1) - \psi(1/2) - 2\ln 2 + \mathrm{Re}\,[2\psi(2\nu) - \psi(\nu)]$. The function $Q_{\nu-1/2}$ has a logarithmic singularity at $r \to 0$: $Q_{\nu-1/2} \to -\ln r$.

Fig. 1 shows results of calculations based on above formulae for a source at the origin. An example of time-domain traces is displayed in Fig. 2.

2.3 Ray theory and bipolar coordinates

It is well known that the rays in a linear velocity model are parts of circles, if they go downward (to $z > 0$) from a source at the origin. The centres of these circles are located at $z = -c_0/\alpha$ or $\hat{z} = 0$. They are given by

$$\left(x - \sqrt{r_c^2 - \hat{z}_0^2}\right)^2 + \hat{z}^2 = r_c^2,$$

when parametrized by their radius r_c. For a ray starting downward at the origin and travelling to some point (x, z), the traveltime is $\tau = \alpha^{-1} \operatorname{arccosh}(u(x, z))$ (Slotnick 1959; Červený 2001) and the rays are perpendicular to lines of constant u. The same geometry is described by bipolar coordinates, when the axes of that coordinate system are

$$\hat{z} = \frac{\hat{z}_0 \sinh\nu'}{\cosh\nu' - \cos\mu'}, \qquad x = \frac{\hat{z}_0 \sin\mu'}{\cosh\nu' - \cos\mu'},$$

with $\tanh\nu' = 1/u$ and $\sin\mu' = \hat{z}_0/r_c$.

In 2-D, the dynamic ray solution is

$$p_{\mathrm{ray}} = \frac{1}{2\pi} \frac{A}{\sqrt{i\omega}} e^{-i\omega\tau}, \qquad A = \sqrt{\frac{\pi\alpha}{2\sinh(\alpha\tau)}}, \tag{13}$$

with τ given above. The same result can be obtained by taking the argument of $Q_{i\mu-1/2}(u)$ for large μ. Using

$$Q_{\tilde{\nu}}(\cosh\eta) \sim \sqrt{\frac{\pi}{2\tilde{\nu}\sinh\eta}}\, e^{-(\tilde{\nu}+1/2)\eta}[1 + O(|\tilde{\nu}|^{-1})], \tag{14}$$

for large $|\tilde{\nu}|$, which is eq. (7.11.11) in Lebedev (1965), setting $\tilde{\nu} = \nu - 1/2$, noting that $\nu \sim i\omega/\alpha$ for $\omega \gg \alpha$ and inserting $\eta = \alpha\tau$, we find that $Q_{\nu-1/2}(u)/(2\pi) \sim p_{\mathrm{ray}}$ as given in eq. (13) in the high-frequency limit.

As mentioned before, the above assumes a Fourier representation of the form $\exp(i\omega t)$. If the convention used in geophysics is to be honoured, the complex conjugate should be taken everywhere.

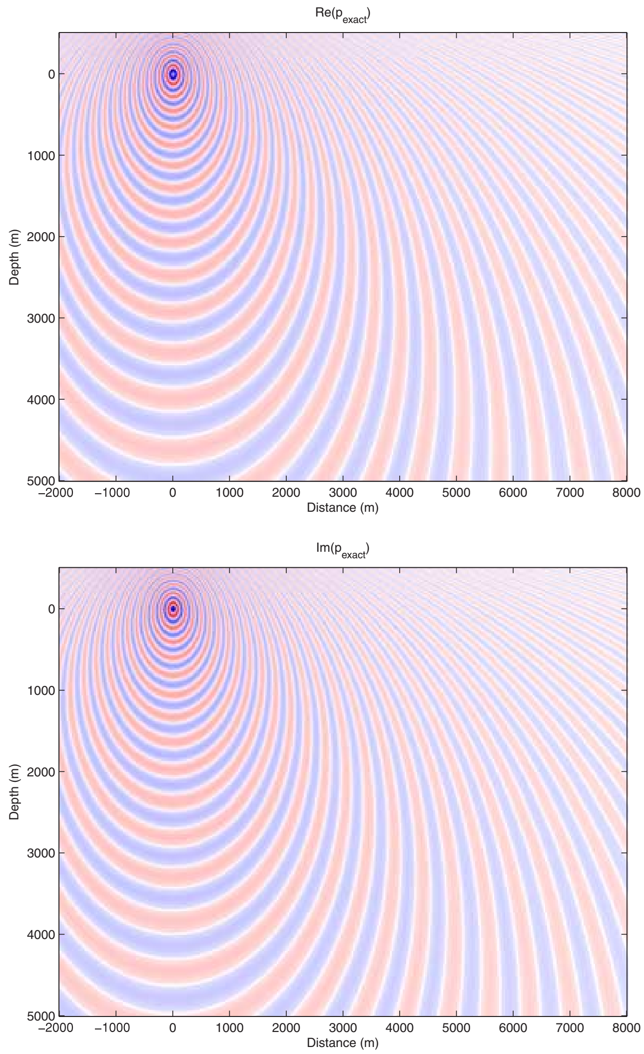

Figure 1. The top figure shows the real part and the bottom the imaginary part of the 2-D solution at 6 Hz for $c_0 = 800$ and $\alpha = 0.7$. The source is located at the origin.

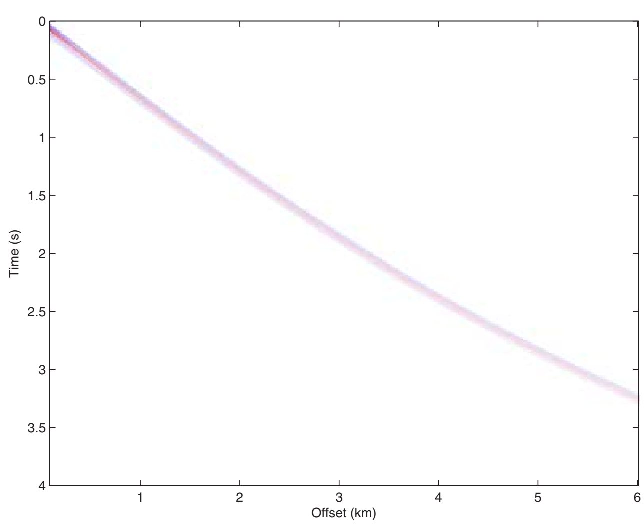

Figure 2. Traces obtained for a Ricker wavelet with a peak frequency at 15 Hz and offsets between 100 and 6000 m.

3 THREE DIMENSIONS

The derivation for the 3-D case leads to much simpler expressions and can be found in Pekeris (1946) and Brekhovskikh & Godin (1992). Here, it is summarized to show its relation to the 2-D case.

Applying a Hankel instead of a cosine transform to eq. (6), we find (Bateman & Erdélyi 1954)

$$p_G(R, \hat{z}) = -i(2\pi)^{-3/2}(\hat{z}_0\hat{z})^{-1/2}(\tilde{u}^2 - 1)^{-1/4} Q^{1/2}_{\nu-1/2}(\tilde{u}), \tag{15}$$

where

$$\tilde{u} = \frac{R^2 + \hat{z}^2 + \hat{z}_0^2}{2\hat{z}_0\hat{z}} = 1 + \frac{r^2}{2\hat{z}_0\hat{z}},$$

$$R = \sqrt{x^2 + y^2}, \quad r = \sqrt{R^2 + (\hat{z} - \hat{z}_0)^2}.$$

The associated Legendre function of the second kind is given by

$$Q^{1/2}_{\nu-1/2}(\tilde{u}) = i\sqrt{\frac{1}{2}\pi}\ (\tilde{u}^2 - 1)^{-1/4}\,[\tilde{u} + \sqrt{\tilde{u}^2 - 1}]^{-\nu}.$$

This leads to

$$p_G(R, z) = \frac{1}{4\pi}(\hat{z}_0\hat{z})^{-1/2}\,(\tilde{u}^2 - 1)^{-1/2}\,[\tilde{u} + \sqrt{\tilde{u}^2 - 1}]^{-\nu}, \tag{16}$$

which can also be expressed as

$$p_G(R, z) = \frac{\exp(-i\mu \operatorname{arccosh}\tilde{u})}{2\pi\sqrt{4cc_0\alpha^{-2}(\tilde{u}^2 - 1)}}.$$

If $\omega \le \frac{1}{2}\alpha$, $\mu = \sqrt{(\omega/\alpha)^2 - 1/4}$ is purely imaginary and the solution is no longer wave-like.

Near the source at the origin,

$$p_G(R, z) = \frac{1}{2}\left[\frac{1}{r} - \frac{\nu}{z_0} + O(r)\right], \quad r = \sqrt{R^2 + \hat{z}^2}.$$

Ray theory provides a solution $p_{\mathrm{ray}} = (A/2\pi)\,e^{-i\omega\tau}$, where

$$A = \frac{1}{2r\sqrt{1 + \frac{\alpha^2 r^2}{4cc_0}}} = \frac{1}{\sqrt{4cc_0\alpha^{-2}(\tilde{u}^2 - 1)}},$$

and

$$\tau = \alpha^{-1}\operatorname{arccosh}(\tilde{u}).$$

This matches p_G in the high-frequency limit, using $\mu \sim \omega/\alpha$ for $\omega \gg \alpha$.

4 VARIABLE DENSITY

Following Brekhovskikh & Godin (1992), the above can be generalized to the acoustic wave equation

$$-\frac{\omega^2}{\rho c^2}p - \nabla\cdot\frac{1}{\rho}\nabla p = s,$$

with a depth-dependent density of the form $\rho = \rho_0(\hat{z}/\hat{z}_0)^\gamma$. The solution p_G in 2-D or 3-D should then be replaced by $\rho_0(\hat{z}/\hat{z}_0)^{\gamma/2}p_G$ with $\mu = \nu/i = \sqrt{\hat{\omega}^2 - \frac{1}{4}(1+\gamma)^2}$.

5 CONCLUSIONS

The Green function was derived for a velocity profile that increases linearly with depth. In 2-D, the geometric properties of the solution can be described by the bipolar coordinate system. The asymptotic behaviour for large frequencies matches the result obtained by ray theory.

The results can be useful for migration in velocity models that are piecewise linear in depth, in the style of Albertin (1992). Integral equation methods and boundary element methods will benefit from the exact solutions in 2- and 3-D. They can also help in the analysis of waveform tomography for diving waves (Mulder & Plessix 2006).

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