SIAM J. SCI. STAT. COMPUT.
Vol. 11, No. 1, pp. 33–50, January 1990

MULTIGRID FOR THE ONE-DIMENSIONAL INVISCID BURGERS EQUATION*

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Abstract. A multigrid method for computing steady inviscid compressible flow is investigated for the one-dimensional scalar case. The discretisation in space is obtained by upwind differencing and has first- or second-order accuracy. Only relaxation schemes that affect the solution locally are examined. To obtain some insight into the convergence behaviour, two-level convergence analysis is carried out for the linear constant-coefficient case. The resulting two-grid convergence rates are compared to the asymptotic convergence rates observed in numerical experiments on the nonlinear one-dimensional inviscid Burgers equation.

For a test problem with a smooth steady solution, the observed asymptotic convergence rates agreed within $O(h)$ with the linear two-grid convergence rates. A discontinuous solution displayed slower convergence, due to the shock and the sonic point. Although the resulting convergence rate was still acceptable, it could be improved through local relaxation and regularisation of the shock. In this way, a first-order-accurate solution could be obtained in one F-cycle per grid, using damped Point-Jacobi relaxation and successive grid refinement. Second-order accuracy required about eight cycles, using the Defect Correction technique.

Key words. multigrid method, hyperbolic conservation laws, shock waves

AMS(MOS) subject classifications. 35L67, 65B99, 76A60

1. Introduction.

The multigrid technique is a powerful tool for the construction of $O(N)$ methods for a wide class of problems. Its application to the solution of elliptic partial differential equations has received much attention, and the technique is well established for these problems (cf. [3], [6], [20], and references therein). It took some time before any results for the computation of steady solutions to hyperbolic equations, specifically the Euler equations of gas dynamics, were obtained, but successful experiments are now available [1], [7]–[11], [13], [14], [16]. Theoretical estimates of multigrid convergence rates for hyperbolic equations are scarce and incomplete.

In this paper, the convergence towards the steady state by means of the multigrid method proposed in [13] is investigated for the one-dimensional scalar case. Steady solutions to a simple nonlinear one-dimensional hyperbolic equation, the inviscid Burgers equation, are considered. Of course, it does not make much sense to apply the multigrid technique to one-dimensional problems, because direct $O(N)$ methods are available (cf. [12]). Nevertheless, linear two-level analysis is carried out to estimate the two-grid convergence factor. The results are compared with the numerical experiments on the nonlinear equation. Special attention is paid to the effect of the sonic point and the shock on the convergence rate. The full system of (linearised) Euler equations in two dimensions is considered elsewhere [15].

Two periodic test problems, one with a continuous and one with a discontinuous steady solution, are described in §2.1. The spatial discretisation is obtained by upwind differencing, using the schemes of Godunov, Enquist-Osher, or Roe (cf. [23]). For the

*Received by the editors August 12, 1986; accepted for publication (in revised form) October 21, 1987. This work was supported by the Center for Large Scale Scientific Computing (CLaSSiC) Project at Stanford University, under Office of Naval Research contract N00014-82-K-0335.

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latter, the version that obeys the entropy condition is adopted (§2.2). The discretisation has first-order accuracy. The construction of second-order-accurate schemes is reviewed in §2.3. The singularity at the shock is examined in §2.4.

Two-grid convergence factors are estimated by two-level local-mode analysis [3] in §3. We only consider relaxation schemes that affect the solution locally, because the multigrid method should handle the global aspects of the solution. The analysis is carried out for a linear hyperbolic equation with a constant coefficient. We may expect the result to approximate the two-grid convergence rate in the variable coefficient and nonlinear case, as long as the convection speed varies gradually with x and does not change sign. This simplification certainly cannot explain the convergence behaviour at the shock and sonic point. The convergence rate at the shock is considered in §3.5, and at the sonic point in the Appendix.

Numerical experiments are described in §4. The multigrid method is implemented as a Full Approximation Storage (FAS) scheme, including successive grid refinement. This combination is also known as Full Multigrid (FMG). A comparison is made between predicted and observed convergence factors. Only damped Point-Jacobi (PJ) and one type of Red-Black relaxation (RB) are used, since these schemes emerge as the best in the previous section.

The main conclusions are summarised in §5.

2. Spatial discretisation.

2.1. The problem. We consider two types of steady solutions to the one-dimensional scalar hyperbolic equation with periodic boundary conditions

$$u_t + (\tfrac{1}{2}u^2)_x = s(x), \tag{2.1}$$

the first being smooth, the second discontinuous. The solutions are periodic in x on the interval $[0,1)$. Setting

$$u(x) = c_0 + \sin 2\pi(x-\xi), \qquad c_0 = 2, \tag{2.2a}$$

we are led to a source term

$$s(x) = 2\pi\ (c_0 + \sin 2\pi(x-\xi))\cos 2\pi(x-\xi). \tag{2.2b}$$

Although the steady state corresponding to this source term is continuous, a shock may occur during the evolution toward the steady state.

A discontinuous steady solution is found for

$$s(x) = \tfrac{1}{2}\pi \sin 2\pi(x-\xi), \tag{2.3a}$$

namely,

$$u(x) = \begin{cases} \sin\pi(x-\xi) & \text{for } 0 \le x < \xi + \frac{1}{2}, \\ -\sin\pi(x-\xi) & \text{for } \xi + \frac{1}{2} < x \le 1. \end{cases} \tag{2.3b}$$

For $0 < \xi < \frac{1}{2}$, this solution has a sonic point at $x = \xi$ and a shock at $x = \xi + \frac{1}{2}$ (cf. [23]). Numerical examples will be given for $\xi = 0.1$. It should be remarked that (2.1) must be interpreted as the limit for zero viscosity ϵ of the same equation with ϵu_{xx} added to the right-hand side. This is equivalent to imposing an entropy condition in addition to (2.1) and provides uniqueness of the solution. The average value of the

initial data is conserved and must, therefore, be the same for the initial data and the steady state.

2.2. Upwind differencing. The spatial discretisation of (2.1) is obtained by upwind differencing, using Godunov's [5], the Enquist–Osher (E–O) [4], [17], or Roe's scheme [18], [23]. The E–O scheme is identical to Flux-Vector Splitting (FVS) [19] for the one-dimensional scalar case.

A computational grid with N zones is defined by $x_k = (k + \frac{1}{2})h$, where the cell size $h = 1/N$ and $k = 0, \cdots, N-1$. Equation (2.1) is discretised in space by averaging per volume:

$$u_k^h = (I^h u)_k \equiv \frac{1}{h} \int_{x_k - h/2}^{x_k + h/2} u(x)\, dx. \tag{2.4}$$

The same discretisation operator I^h is applied to the residual

$$r(u, x) = s(x) - (\tfrac{1}{2}u^2)_x, \tag{2.5}$$

yielding the discrete first-order approximation

$$r_k = s_k - \frac{1}{h}(f_{k+1/2} - f_{k-1/2}). \tag{2.6}$$

The fluxes $f_{k\pm 1/2}$ are evaluated by upwind differencing. Godunov's scheme lets

$$f_{k+1/2} = f(u_k, u_{k+1}) = \tfrac{1}{2} \max\left[\max(0, u_k)^2, \min(0, u_{k+1})^2\right]. \tag{2.7}$$

The E–O scheme is equivalent to FVS when applied to Burgers' equation. For the latter we have

$$f(u_k, u_{k+1}) = f^+(u_k) + f^-(u_{k+1}), \tag{2.8a}$$

where

$$f^+(u) + f^-(u) = f(u), \qquad f^+(u) = \tfrac{1}{2}[\max(0, u)]^2. \tag{2.8b}$$

The E–O, or FVS scheme, differs from Godunov's only at the shock. Roe's scheme is identical to Godunov's, except at the sonic point, where the modification that suppresses expansion shocks lets

$$f_{k+1/2} = f(u_k, u_{k+1}) = \tfrac{1}{2} u_k u_{k+1} \quad \text{if } u_k \le 0 \le u_{k+1}. \tag{2.9}$$

With this flux, the solution at the sonic point is smooth, whereas Godunov's scheme sets $f_{k+1/2} = 0$, causing an $O(h)$ jump. However, if the stationary sonic point is positioned precisely at the cell interface, Roe's scheme (2.9) allows for two solutions: a smooth one and a discontinuous one. This problem will be ignored here by avoiding such a situation.

The numerical steady solution $\overline{u}_k^h$ obeys $r_k(\overline{u}_{k-1}, \overline{u}_k, \overline{u}_{k+1}) = 0$ for $i = 0, \cdots, N-1$. It is first-order accurate, i.e., $\|\overline{u}_k^h - u_{e\,k}^h\| = O(h)$, where $u_{e\,k}^h \equiv I^h u(x)$ is the average per cell of the exact solution. The norm is the ℓ_1 norm. Godunov's and Roe's scheme

are first-order accurate in the ℓ_∞ norm as well. The E–O, or FVS scheme, smears out the shock over two cells, thus causing a local $O(1)$ error.

2.3. Second-order accuracy. Second-order spatial accuracy can be obtained by assuming the solution to be piecewise linear rather than piecewise constant [22]. The idea is to write the solution as

$$u(x) = u_k + \left(\frac{x - x_k}{h}\right)\Delta_k, \qquad \frac{|x - x_k|}{h} < \frac{1}{2}. \tag{2.10a}$$

Here Δ_k/h is a discrete approximation to $I^h \partial u/\partial x$, with

$$\Delta_k = \text{ave}(u_k - u_{k-1},\, u_{k+1} - u_k). \tag{2.10b}$$

In smooth regions we want $\text{ave}(\Delta_-, \Delta_+) = \frac{1}{2}(\Delta_- + \Delta_+)$, whereas limiting to the smaller of Δ_- and Δ_+ is required near discontinuities to preserve monotonicity (i.e., to avoid local under- or overshoots). We use an averaging-limiting procedure from [21]:

$$\text{ave}(\Delta_-, \Delta_+) = \frac{(\Delta_+^2 + \epsilon_a^2)\Delta_- + (\Delta_-^2 + \epsilon_a^2)\Delta_+}{\Delta_-^2 + \Delta_+^2 + 2\epsilon_a^2}. \tag{2.11}$$

The bias ϵ_a prevents division by zero. Clipping of smooth extrema is avoided if ϵ_a approximately equals the average value of $|u_k - u_{k-1}|$ in smooth regions of the flow. For the numerical examples mentioned above we adopt $\epsilon_a = 4\,h$ for the smooth problem (2.2) and $\epsilon_a = 2\,h$ for the discontinuous problem (2.3). Once the Δ_k are computed, the fluxes at $k + \frac{1}{2}$ are evaluated by $f(u_k + \frac{1}{2}\Delta_k, u_{k+1} - \frac{1}{2}\Delta_{k+1})$, using one of the schemes mentioned in §2.2.

2.4. The singularity at the shock. The numerical problem of determining the steady state to (2.1) can be written as

$$r_k(u_{k-p}, \cdots, u_{k+p}) = 0 \quad \text{for } k = 0, \cdots, N-1. \tag{2.12}$$

Here $p = 1$ for a first-order scheme and $p = 2$ for a second-order scheme. For the example given in (2.3), (2.12) is ill posed, as the position of the shock depends on the initial data. To obtain a unique steady solution, we need the additional requirement that

$$\frac{1}{N}\sum_{k=0}^{N-1} u_k = C \quad \text{(a constant)}. \tag{2.13}$$

The singularity at the shock is appropriate, because the original differential equation is singular. In the latter, the singularity is regularised by augmenting the differential equation with the jump relation across the shock. This is implicit in the conservation form of the equation. Here, however, conservation in time is abandoned to obtain fast convergence, and (2.13) must be imposed to obtain a unique stationary solution.

To describe the singularity in more detail, we need to know the structure of stationary shocks [23, §5]. Assume that (2.12) is obtained by Godunov's or Roe's scheme, and that the source term is absent. Let $u_L > 0$ be the state left of the shock,

and $u_R = -u_L < 0$ at the right. Between these two states there is a cell M containing the shock: $u_L \geq u_M \geq u_R$. The corresponding residual for the first-order scheme is

$$r_M = -(\tfrac{1}{2}u_R^2 - \tfrac{1}{2}u_L^2), \tag{2.14}$$

which is independent of u_M. Thus, the state u_M never enters the discrete equations (2.12) and can be chosen arbitrarily, as long as $u_L > u_M > u_R$. We thus have N equations in $N-1$ unknowns. Condition (2.13) is required to determine a unique value for u_M.

If the E–O, or FVS scheme, is used, a steady shock is represented by a sequence of four cells: u_L, u_M, u_N, and u_R, where $u_L \geq u_M \geq 0 \geq u_N \geq u_R$ and $u_L = -u_R$. The two residuals,

$$\begin{aligned} r_M &= -\left[(\tfrac{1}{2}u_M^2 + \tfrac{1}{2}u_N^2) - \tfrac{1}{2}u_L^2\right], \\ r_N &= -\left[\tfrac{1}{2}u_R^2 - (\tfrac{1}{2}u_M^2 + \tfrac{1}{2}u_N^2)\right], \end{aligned} \tag{2.15}$$

show that both u_M and u_N enter the discrete equations, but only as the combination $\frac{1}{2}u_M^2 + \frac{1}{2}u_N^2$, thus leaving the individual values of u_M and u_N undetermined. We again have N equations in $N-1$ unknowns, and (2.13) must be imposed as an extra condition to obtain uniqueness.

The second-order-accurate discretisation is singular as well. However, the structure of stationary shocks is more complicated and will not be considered here.

3. Convergence factors.

3.1. Preliminaries. Two-grid convergence factors for a number of relaxation schemes will be estimated by means of two-level analysis in Fourier space [6]. Here we consider the linear equation $u_t + au_x = 0$, where a is a positive constant. This is, of course, a major simplification. Let the iteration error $v^h \equiv \overline{u}^h - u^h$, where $\overline{u}^h$ is the steady state. The discrete linear operator is

$$L^h = \frac{a}{h}(I - T^{-1}). \tag{3.1}$$

Here I is the identity and the shift operator T acts according to $T^s v_k = v_{k+s}$. The discrete Fourier transform of v_k^h will be denoted by $\hat{v}_l^h$. The shift operator in Fourier space is

$$\hat{T}(\theta) = \exp(i\theta), \qquad \theta \equiv 2\pi l/N, \qquad l = -(\tfrac{1}{2}N - 1), \cdots, \tfrac{1}{2}N. \tag{3.2}$$

The high frequencies, which cannot be represented on the next coarser grid, lie in the range $\frac{1}{2}\pi \leq |\theta| \leq \pi$. The coarse-grid correction (CGC) operator, and also the RB relaxation operator, couples the frequencies θ and $\theta + \pi$. Therefore, define

$$\hat{V}_l^h \equiv \begin{pmatrix} \hat{v}_l \\ \hat{v}_{l+N/2} \end{pmatrix}. \tag{3.3}$$

A 2×2 matrix $\hat{Z}$, operating on $\hat{V}_l^h$, must have the property

$$\hat{z}_{21}(\theta) = \hat{z}_{12}(\theta + \pi), \qquad \hat{z}_{22}(\theta) = \hat{z}_{11}(\theta + \pi). \tag{3.4}$$

Note that $\hat{T}(\theta + \pi) = -\hat{T}(\theta)$. The residual operator $\hat{L}^h$ vanishes for the longest wave ($\theta = 0$), which reflects the fact that the solution is determined up to a constant. This wave will be ignored.

3.2. Coarse-grid correction (CGC) operator. The CGC operator describes the result of (i) restriction of the fine-grid residual to the next coarser grid, (ii) exact solution of the coarse-grid equations, and (iii) prolongation of the CGC to the fine grid. We choose a restriction operator I_h^{2h} that averages the values in two neighbouring cells $(2k, 2k+1)$, where $k = 0, 1, \cdots, \frac{1}{2}N - 1$. The prolongation operator I_{2h}^h describes piecewise constant interpolation from the coarse to the fine grid. Both are first-order operators. The CGC operator,

$$K = I - I_{2h}^h (L^{2h})^{-1} I_h^{2h} L^h, \tag{3.5}$$

has a Fourier representation

$$\hat{K} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1+\hat{T}^{-1} \\ 1-\hat{T}^{-1} \end{pmatrix} \left(\hat{L}^{2h}\right)^{-1} \frac{1}{2} (1+\hat{T} \quad 1-\hat{T}) \begin{pmatrix} \hat{L}^h(\theta) & 0 \\ 0 & \hat{L}^h(\theta+\pi) \end{pmatrix}. \tag{3.6a}$$

$\hat{K}$ is understood to operate on $\hat{V}^h$. Using (3.1) and $\hat{L}^{2h} = (a/2h)(1-\hat{T}^{-2})$, we obtain

$$\hat{K} = \frac{1}{2} \begin{pmatrix} 1-\hat{T} & 1+\hat{T} \\ 1-\hat{T} & 1+\hat{T} \end{pmatrix}. \tag{3.6b}$$

This matrix obeys property (3.4). In physical space, the CGC operator lets

$$\left.\begin{aligned} \tilde{v}_{2k} &= v_{2k} - v_{2k+1} \\ \tilde{v}_{2k+1} &= 0 \end{aligned}\right\} \quad k = 0, 1, \cdots, \tfrac{1}{2}N - 1, \tag{3.6c}$$

where v denotes the iteration error before the CGC, and $\tilde{v}$ denotes the iteration error after.

3.3. Relaxation. A relaxation scheme provides an approximate solution $\tilde{u}^h$ to the nonlinear problem $r^h = 0$. Here we will discuss relaxation schemes by starting with a time-accurate integration of the equations and then making simplifications. The resulting relaxation schemes are not time-accurate.

An example is the explicit scheme

$$\left[\frac{1}{\Delta t}\right](\tilde{u}^h - u^h) = r^h, \tag{3.7}$$

or the implicit scheme

$$\left[\frac{1}{\Delta t} - \alpha\left(\frac{dr}{du}\right)^h\right](\tilde{u}^h - u^h) = r^h. \tag{3.8}$$

The residual r^h is computed from the solution u^h, and $\tilde{u}^h$ is the solution after applying the explicit or implicit scheme. Both are based on the time-dependent problem and provide a steady state by a time-accurate integration of the differential equation. The choice $\alpha = 1$ yields a "backward Euler" scheme. If the time-step Δt becomes large, the latter reduces to Newton's method. This property is exploited by the Switch Evolution/Relaxation (SER) scheme, proposed in [12] and [24], where $\Delta t \propto 1/\|r^h\|$. After some initial time-accurate searching while the residuals are large, this scheme automatically switches to Newton's method when the solution approaches the steady

state. The finite $1/\Delta t$ term allows for the solution of the linear system (3.8) even if $(dr/du)^h$ is singular.

Note that the implicit scheme (3.8) is not conservative in time, except for $\alpha = 0$. For other values of α, we still have *global* conservation in time on a periodic grid. For a residual of the form (2.6), and a periodic grid with N points, we obtain, after summation of the N equations represented by (3.8),

$$\sum_{i=1}^{N} \frac{\tilde{u}_i - u_i}{\Delta t} = \sum_{i=1}^{N} s_i, \tag{3.9}$$

which is independent of α. The same result is obtained for the explicit scheme (3.7). The test problems considered here have $\sum_{i=1}^{N} s_i = 0$.

The linear system (3.8) for the one-dimensional problem requires the solution of a periodic tridiagonal matrix. This can be done directly, as in [12]. Here we will consider schemes based on a PJ approximation to the left-hand side of (3.8), i.e., the periodic tridiagonal system is replaced by a diagonal one.

Now consider the linear operator L^h. The relaxation operator S^h updates the error according to $\tilde{v}^h = S^h v^h$, where

$$S^h = I - \left(\tilde{L}^h\right)^{-1} L^h, \tag{3.10a}$$

and $\tilde{L}^h$ is an approximation to L^h that can be inverted easily. For the one-dimensional scalar problem studied here, we obtain

$$S^h = I - \beta(I - T^{-1}). \tag{3.10b}$$

For the explicit scheme (3.7), β equals the local Courant-Friedrichs-Lewy number σ:

$$\beta = \sigma \equiv \Delta t|a|/h, \tag{3.11a}$$

whereas for the PJ approximation to the implicit scheme (3.8)

$$\beta = \frac{\sigma}{1 + \alpha\sigma}. \tag{3.11b}$$

In both cases β can be considered as a relaxation parameter, which can be optimised to provide the best convergence factor.

We will now derive $\hat{S}^h$ for various relaxation schemes, in a form that obeys property (3.4). First consider schemes that do not couple frequencies, i.e., $\hat{s}_{21} = \hat{s}_{12} = 0$. The simplest is the PJ relaxation:

$$\hat{s}_{11}^{PJ} = 1 - \beta(1 - \hat{T}^{-1}). \tag{3.12a}$$

Stability requires

$$|\hat{s}_{11}^{PJ}(\theta)|^2 = 1 - 2\beta(1 - \beta)(1 - \cos\theta) \leq 1, \qquad 0 \leq |\theta| \leq \pi, \tag{3.12b}$$

implying

$$0 \leq \beta \leq 1. \tag{3.12c}$$

At the stability limit, $\beta = 1$ and $\hat{s}_{11}^{PJ} = \hat{T}^{-1}$. In that case, an explicit scheme runs at the maximum local time-step, and errors are convected over a distance h, the cell size. For a single-grid scheme, this choice is attractive, as errors are convected as fast as possible toward the boundaries (if present), where they leave the computational domain, or toward a shock, where they are absorbed by the dissipation in the shock.

The second relaxation scheme is a Multi-Stage (MS) Runge-Kutta scheme. Here we consider only two stages. The first stage is a PJ step with $\beta = \beta_1$, advancing the solution u to u'. The second stage resembles PJ, but now $\beta = \beta_2$ and the solution is advanced from u to $\tilde{u}$, using the residual $r' = r(u')$. The growth factor for one MS step is

$$\hat{s}_{11}^{MS} = 1 - \beta_2(1 - \hat{T}^{-1})[1 - \beta_1(1 - \hat{T}^{-1})], \tag{3.13a}$$

which is stable for

$$|\beta_1| \leq \tfrac{1}{2}, \qquad 0 \leq \beta_2 \leq 1 + 2\beta_1. \tag{3.13b}$$

At the stability limit $\beta_1 = \frac{1}{2}$, $\beta_2 = 2$, and $\mu = \hat{T}^{-2}$.

As a third relaxation scheme, we consider checkerboard or RB relaxation. It consists of a PJ sweep on the even points ($k = 0, 2, \cdots, N-2$), an update of the solution and residuals, followed by a similar operation on the odd points ($k = 1, 3, \cdots, N-1$):

$$\left.\begin{aligned} \tilde{v}_{2k} &= (1-\beta)v_{2k} + \beta v_{2k-1} \\ \tilde{v}_{2k+1} &= (1-\beta)v_{2k+1} + \beta\tilde{v}_{2k} \end{aligned}\right\} \quad k = 0, 1, \cdots, \tfrac{1}{2}N - 1. \tag{3.14}$$

Because of the asymmetric form of the CGC operator that is apparent in (3.6c), it is necessary to distinguish between two variants of RB. The distinction is based on the ordering with respect to the coarse-grid cell and the direction of the flow. The first variant will be noted by RB1 and corresponds to the (*even, odd*) ordering described in (3.14), i.e., first the cells with an even index and then those with an odd index are relaxed, assuming that $a > 0$ and that the restriction operator combines the cells $(2k, 2k+1)$. For negative a, first the odd and then the even cells are relaxed. The second version, RB2, has the opposite ordering. In other words, RB1 follows the flow, relative to the coarse-grid cell, and RB2 goes against the flow. We might call RB1 *Convective Red-Black* relaxation. For a single-grid scheme this distinction disappears.

In Fourier space we have, for $a > 0$,

$$\begin{aligned} \hat{s}_{11}^{RB1}(\hat{T}) &= 1 - \beta(1 - \hat{T}^{-1}) - \hat{s}_{12}^{RB1}(-\hat{T}), \\ \hat{s}_{12}^{RB1}(\hat{T}) &= -\tfrac{1}{2}\beta^2\hat{T}^{-1}(1 + \hat{T}^{-1}). \end{aligned} \tag{3.15a}$$

The other variant has

$$\hat{s}_{11}^{RB2}(\hat{T}) = \hat{s}_{11}^{RB1}(\hat{T}), \quad \hat{s}_{12}^{RB2}(\hat{T}) = -\hat{s}_{12}^{RB1}(\hat{T}). \tag{3.15b}$$

Both variants are stable for $0 \leq \beta \leq 1$.

Similar expressions can be derived for a second-order-accurate residual. In that case

$$\hat{L}^h = \frac{a}{h}(1 - \hat{T}^{-1})(1 + \tfrac{1}{4}(\hat{T} - \hat{T}^{-1})) \quad \text{for } a > 0. \tag{3.16}$$

With this residual, PJ as a single-grid relaxation scheme is unstable for all $\beta \neq 0$. The MS scheme is stable for $0 \leq \beta_1 \leq \frac{1}{2}$ and $0 \leq \beta_2 \leq 2\beta_1$, and RB for $0 \leq \beta \leq 1$. We can consider a multigrid scheme with either first-order or second-order accuracy on the coarser grids. In the latter case, an expression for the CGC operator $\hat{K}$ is obtained that is slightly more complicated than (3.6b). The values in Table 1 for second-order accuracy are based on such an operator.

3.4. Two-grid convergence factors. The performance of the two-grid method can be described by the smoothing rate of the relaxation scheme and by the two-grid convergence rate. The first measures how well the relaxation scheme removes the high frequency part of the error, and can be easily evaluated. The second measures convergence across the entire spectrum, under the assumptions that the coarser grid is solved exactly. Both rates will be considered in this section.

The smoothing rate is defined by

$$\overline{\mu} \equiv \max_{\pi/2 \leq |\theta| \leq \pi} \hat{s}_{11}(\theta), \tag{3.17}$$

for schemes without coupling, i.e., if $\hat{s}_{12} = \hat{s}_{21} = 0$. It describes how well the relaxation scheme removes the part of the spectrum that cannot be represented on the coarser grid. For schemes with coupling, such as RB, we still can use (3.17), assuming that there are no low frequencies, i.e., $\hat{v}(\theta) = 0$ for $0 \leq |\theta| < \pi/2$ (see [3, (3.2)]).

The two-grid convergence factor is

$$\overline{\lambda} \equiv \max_{0 \leq |\theta| \leq \pi} \rho(\hat{S}^{\nu_2} \hat{K} \hat{S}^{\nu_1}), \tag{3.18}$$

where $\rho(\cdot)$ is the spectral radius. It describes the result of applying ν_1 prerelaxation sweeps, a CGC, and ν_2 postrelaxation sweeps.

Table 1 lists smoothing rates and two-grid convergence rates for the three relaxation schemes of §3.3. The number of relaxation sweeps $\nu = \nu_1 + \nu_2 = 1$. The results have been obtained by evaluating (3.17) and (3.18) analytically or numerically. Minimum values with respect to the relaxation parameter(s) for a given relaxation scheme are marked by an asterisk. They have been computed by analytical or numerical optimisation of $\overline{\mu}$ and $\overline{\lambda}$, respectively, as a function of the relaxation parameter(s).

For the first-order discretisation, PJ with $\beta = \frac{1}{2}$ is obviously the best choice. Both the MS scheme and RB relaxation require more operations per sweep. The MS scheme reduces to PJ relaxation when optimised for $\overline{\lambda}$. It is clear that *optimising the smoothing rate does not necessarily imply a good two-grid convergence factor.* RB relaxation becomes dependent on the ordering when coarser grids are involved. For $\beta = 1$, the version that follows the flow (RB1) has a zero convergence factor, whereas RB2 is unstable.

It should be noted that PJ for a first-order-accurate residual and $\beta = \frac{1}{2}$ lets

$$\hat{K}\hat{S}^{PJ} = -\tfrac{1}{4}(\hat{T} - \hat{T}^{-1}) \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \tag{3.19}$$

resulting in a spectral radius zero. However, if PJ is applied more than once ($\nu > 1$), different results are obtained. For instance, $\overline{\lambda} = \frac{1}{2}$ if $\nu = 2$.

For Convective RB relaxation with $\beta = 1$ we have a stronger result than for PJ:

$$\hat{K}\hat{S}^{RB1} = \hat{S}^{RB1}\hat{K} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.20a}$$

TABLE 1

Smoothing rate $\overline{\mu}$ and two-grid convergence factor $\overline{\lambda}$ for various relaxation schemes in the linear constant-coefficient case. The amount of underrelaxation is given by β. Minimum values with respect to the relaxation parameter(s) are denoted by an asterisk. The results are obtained for one relaxation step ($\nu = 1$). Second-order PJ *is unstable as a single-grid scheme.* RB1 *follows the flow, as seen from the coarse-grid cell, whereas* RB2 *acts in the opposite direction.*

Accuracy	First-order			Second-order		
Scheme	β	$\overline{\mu}$	$\overline{\lambda}$	β	$\overline{\mu}$	$\overline{\lambda}$
PJ	$\frac{1}{2}$	0.707*	0 *	$\frac{1}{5}$	0.949*	0.717
	$\frac{1}{2}$	0.707	0 *	0.145	0.953	0.711*
MS	$\frac{1}{3}$, 1	0.333*	0.333	0.350, 0.592	0.645*	0.677
	0, $\frac{1}{2}$	0.707	0 *	0.295, 0.588	0.719	0.517*
RB1	0.682	0.457*	0.101	0.510	0.806*	0.552
	1	0.707	0 *	0.928	1.153	0.386*
	$\frac{1}{2}$	0.530	0.25	0.405	0.821	0.596
RB2	0.682	0.457*	0.830	0.510	0.806*	0.676
	1	0.707	2	0.928	1.153	3.829
	$\frac{1}{2}$	0.530	0.25 *	0.405	0.821	0.636*

To obtain this performance in an actual computer code, we have to adapt the ordering to the direction of the flow. This can be avoided by choosing $\beta = \frac{1}{2}$, resulting in a convergence factor $\overline{\lambda} = \frac{1}{4}$. Alternatively, two relaxation sweeps with $\beta = 1$ can be applied, one with RB1 and one with RB2. Then

$$\hat{S}^{RB1}\hat{K}\hat{S}^{RB2} = \hat{S}^{RB2}\hat{K}\hat{S}^{RB1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.20b}$$

The behaviour of RB for $\beta = 1$ is better understood by considering (3.6c) and (3.14), assuming $a > 0$. After the CGC, $v_{2k-1} = 0$ and the first RB1 sweep at cell $(2k)$ convects this error from cell $(2k-1)$ to $(2k)$, thus making $\tilde{v}_{2k} = 0$. The second RB1 sweep at cell $(2k+1)$ is actually superfluous. On the other hand, RB2 selects $v_{2k} \neq 0$ and convects this error, resulting in the amplification of the shorter waves.

The fact that the second sweep of RB1 is superfluous can be exploited to construct a faster scheme, which we call Semi-Red-Black (SRB) relaxation. SRB only carries out the first step of RB1. Obviously, $\overline{\lambda} = 0$ for this scheme. However, for solutions with a sonic point or shock, SRB is likely to be less robust than RB1.

The most attractive scheme for a first-order-accurate discretisation is PJ with $\beta = \frac{1}{2}$ and $\nu = 1$. The convergence factors for second-order accuracy are not so good, as can be seen from Table 1. For a second-order discretisation, PJ is unstable as a single-grid scheme but still provides acceptable two-grid convergence factors.

In the numerical experiments of §4 only PJ relaxation with $\beta = \frac{1}{2}$ and RB1 are considered. Second-order-accurate solutions are computed by the Defect Correction technique [6, (14.3.1)]. The corresponding two-level operator is given by

$$I - (I - S^{\nu_2} K S^{\nu_1})(L^h_{p=1})^{-1} L^h_{p=2}, \tag{3.21}$$

where $L^h_{p=1}$ denotes the discrete residual operator of first-order accuracy (3.1) and $L^h_{p=2}$ of second-order accuracy (3.16). For PJ with $\beta = \frac{1}{2}$ and $\nu = \nu_1 + \nu_2 = 1$, we find $\overline{\lambda} = \frac{1}{2}\sqrt{3} = 0.866$, whereas RB1 with $\beta = 1$ provides $\overline{\lambda} = \frac{1}{2}$.

3.5. The singularity at the shock. The two-level analysis presented above describes the linear constant coefficient case. We might expect approximately the same two-grid convergence rates in the variable coefficient and nonlinear case, as long as the convection speed remains fairly constant with varying x and does not change sign. We will study this by numerical experiments for the nonlinear case in §4.2. The convection speed changes sign at a sonic point or shock. Convergence rates at the sonic point are considered in the Appendix, under the simplifying assumption that the convection speed is linear in x. Here we will examine the effect of the singularity at the shock on the convergence rate.

For Godunov's or Roe's scheme, which are identical around the shock, we have the following: if there is a shock, the two-grid convergence factor for a first-order scheme is in general $\overline{\lambda} \geq \frac{1}{2}$. To assert this, assume that we have two grids, a fine grid m with cell size h, and a coarse $m-1$ with cell size $2h$. Also assume that the coarse-grid problem is solved exactly, and that the residuals are small everywhere, even at the shock. The special situation of a steady shock positioned at the cell interface is excluded. In that case the errors v_k^h will be small, except at the shock, where v_M^h can still be large. Without loss of generality we choose an even index M for the cell that contains the singularity. The coarse-grid error, after restriction, is $v_{\frac{1}{2}M}^H = \frac{1}{2}(v_M^h + v_{M+1}^h) \simeq \frac{1}{2}v_M^h$. The exact solution of the coarse grid is prolongated back to the fine grid, resulting in

$$\begin{aligned} \tilde{v}_M^h &= v_M^h - v_{\frac{1}{2}M}^H \simeq \tfrac{1}{2}v_M^h, \\ \tilde{v}_{M+1}^h &= v_{M+1}^h - v_{\frac{1}{2}M}^H \simeq -\tfrac{1}{2}v_M^h. \end{aligned} \tag{3.22}$$

If the new $\tilde{u}_M^h$ still obeys

$$\tilde{u}_{M-1}^h > \tilde{u}_M^h > \tilde{u}_{M+1}^h \quad \text{with } \tilde{u}_{M-1}^h > 0 > \tilde{u}_{M+1}^h, \tag{3.23}$$

then only $\tilde{u}_{M+1}^h$ will result in fairly large residuals at M and $M+1$. An ideal relaxation scheme such as RB1 with $\beta = 1$ will first bring $\tilde{v}_{M+1}^h$ down to practically zero, causing the residuals at M and $M+1$ to vanish. The error $\tilde{v}_M^h \simeq \frac{1}{2}v_M^h$ will be unaffected. Thus the two-grid convergence factor $\overline{\lambda} = \frac{1}{2}$ in this case. In general, we expect $\overline{\lambda} \geq \frac{1}{2}$.

We can just accept the slower convergence at the shock. However, after every prolongation, there will be fairly large residuals near the shock. If these are not sufficiently damped by the relaxation scheme, convergence may be lost. PJ is likely to be good enough to obtain convergence. This will be further investigated by numerical experiments (§4).

The problem is that the singularity shows up during relaxation. As a remedy, we propose to use *local relaxation* (cf. [2]), with *local conservation* imposed as an extra condition to remove the singularity at the shock. In practice this is accomplished by selecting, say, four cells, the first having an even index (this determines its position with respect to the coarse-grid cell), and the second or third containing the shock. Let the indices of these cells be elements of the set $\mathcal{L}$. The SER method (3.8) is applied with $\alpha = 1$ and

$$\frac{1}{\Delta t} = \max_{k \in \mathcal{L}} (|r_k| / a_k^*). \tag{3.24}$$

Here a_k^* is an numerical approximation to the convection speed, to be specified in §4. The linear system associated with the SER scheme is singular at the shock for

$\Delta t \to \infty$. At the shock, we replace the equation for which $\partial r_M/\partial u_k = 0$, $k \in \mathcal{L}$, by local conservation:

$$\sum_{k\in\mathcal{L}} (\tilde{u}_k - u_k) = 0, \tag{3.25}$$

even for finite Δt. In the numerical experiments described in §4.3, it appeared to be sufficient to apply this once after every prolongation on every grid.

The condition of local conservation (3.25) can be justified as follows. Consider a sequence of cells u_l, $l = 1, \cdots, L$, which is a renumbered subset of the solution on the entire grid. Let one of these cells with index M, $1 < M < L$, contain the shock, and let u_l be positive left from the shock, and negative right from the shock. Here we have assumed that Godunov's or Roe's scheme is used. In addition, we assume that L is even and that cells $l = 1$ and $l = 2$ coincide with a cell on the coarser grid. The same is then true for the other pairs of cells corresponding to higher values of l. We may expect the average value of the solution directly after prolongation to be converged: $\sum_{l=1}^{L} u_l \simeq \sum_{l=1}^{L} \overline{u}_l$. Given this expression, (3.25) follows by requiring $\tilde{u}_l$ to equal the desired result $\overline{u}_l$.

If we want to avoid the replacement of the singular equation at the shock by (3.25), the SER scheme can be used without modification. This is due to the global conservation property (3.9). For the explicit scheme (3.7) and the implicit scheme (3.8) we have

$$\frac{1}{\Delta t}\sum_{l=1}^{L}(\tilde{u}_l - u_l) = \left(\sum_{l=1}^{L} s_l\right) - \tfrac{1}{2}(u_{L+1}^2 - u_0^2). \tag{3.26}$$

Note that u_0 and u_{L+1} are taken as fixed boundary values here, and that u_l is positive left and negative right from the shock. After prolongation, the right-hand side of (3.26) will have appeared on coarser grids as the restriction of the fine-grid residuals, or as the sum of restricted residuals. Convergence on coarser grids causes the sum of these residuals to vanish, at least approximately, which implies that directly after prolongation (3.26) is approximately the same as (3.25).

If the E–O, or FVS scheme, is used, local relaxation with local conservation can be carried out in a similar way.

4. Numerical experiments.

4.1. Technical details. The numerical experiments are carried out in a FMG setting. The steady state on a grid with two cells is computed directly from the nonlinear equations, subject to (2.13), and then interpolated to the next finer grid to obtain a good initial guess. The corresponding interpolation operator I_{2h}^h lets

$$u_{2k}^h = u_k^{2h} - \tfrac{1}{4}\Delta_k^{2h}, \qquad u_{2k+1}^h = u_k^{2h} + \tfrac{1}{4}\Delta_k^{2h}, \tag{4.1}$$

and is third-order-accurate if the solution is smooth. At the shock, however, an $O(1)$ error results. The numerical experiments described below show that this can be reduced to $O(h^p)$ for Godunov's or Roe's scheme by using local relaxation. The E–O, or FVS scheme, maintains an $O(1)$ error at the shock, both for $p = 1$ and $p = 2$. Still, local relaxation is carried out after grid refinement.

After interpolation by I_{2h}^h (and local relaxation, if desired) the stationary solution is computed by a FAS scheme. As the multigrid strategy, we use three multigrid cycles with respect to each of the coarser grids to simulate a two-grid algorithm. For

convergence down to the truncation error, an F-cycle [20] is chosen. In all cases, one postrelaxation sweep is carried out ($\nu_1 = 0$, $\nu_2 = 1$). On the coarsest grid ($N = 2$), the nonlinear equations are solved directly.

Damped PJ and RB1 are used as relaxation schemes. The first is implemented as

$$\tilde{u}_k = u_k + \beta(a_k^*)^{-1} r_k, \tag{4.2}$$

with $\beta = \frac{1}{2}$. The two steps of RB1 are carried out in the same way with $\beta = 1$. The nonlinear residual is updated after each of the two steps. We consider the following approximations to the convection speed:

$$a_k^* = -\frac{\partial r_k}{\partial u_k}, \tag{4.3a}$$

$$a_k^* = \max_l\left(-\frac{\partial r_k}{\partial u_k}, \frac{\partial r_k}{\partial u_l}\right), \tag{4.3b}$$

$$a_k^* = \tfrac{1}{2}\left(-\tfrac{\partial r_k}{\partial u_k} + \textstyle\sum_{l\neq k} \tfrac{\partial r_k}{\partial u_l}\right). \tag{4.3c}$$

The first choice (4.3a) cannot be applied with Godunov's or Roe's scheme, because of the singularity at the shock. The E–O, or FVS scheme, does not suffer from this drawback. The second and third choice can be used at the singularity.

If Godunov's, the E–O, or the FVS scheme is used, a_k^* may become small around the sonic point or at the shock, resulting in too large changes of the solution. To avoid this, a_k^* can be replaced by

$$\tilde{a}_k^* = a_k^* + |r_k|/a_k^*, \tag{4.4}$$

which is the equivalent of a SER scheme with a local time-step.

Local relaxation, if used, is carried out on a subset of four or six cells. These are determined by finding the index of the maximum residual and positioning it in the middle of the subset, in such a way that the first cell has an even index. If a singular equation is detected, it is replaced by the condition of local conservation (3.25).

4.2. Results for the smooth test problem. We have used the smooth problem (2.2) to make a comparison between two-grid convergence factors for the nonlinear equation and the linear two-level results. On coarser grids, three multigrid cycles were used to simulate a two-level algorithm with an exact solution of the coarse-grid equations [6, §2.5]. No local relaxation was applied. The experiments showed an $O(h)$ difference between the observed asymptotic convergence factors and the two-level results for the linear constant-coefficient case listed in Table 1. It is likely that close to the solution and away from sonic points and shocks, the nonlinear problem behaves as a linear one with a variable coefficient. The convection speed then has an $O(h)$ variation with x, which probably accounts for the observed $O(h)$ difference in the convergence factors.

The predicted convergence factors for the Defect Correction technique, given at the end of §3.4, were confirmed by the numerical experiments. Here the observed values tended to be slightly better.

The results were insensitive to the choices of a_k^* (4.3) or $\tilde{a}_k^*$ (4.4). There is no distinction between Godunov's, Roe's, and the E–O scheme for the current test problem, as u_k is positive on the entire grid.

For practical purposes, convergence down to the truncation error is sufficient. In that case, the experiments showed that a first-order solution could be obtained in one F-cycle per grid, using damped PJ or RB1 as part of a FMG scheme. A second-order solution computed with the Defect Correction technique required about four F-cycles per grid with damped PJ as relaxation scheme. A few more cycles were required for computing the steady state on the coarser grids ($N > 64$). RB1 required three F-cycles on the coarser, and two F-cycles on the finer grids (down to $N = 512$).

4.3. The discontinuous case. Several issues will be discussed for the discontinuous case (2.3): the choice of the discretisation, the effect of the various a_k^* (4.3) and $\tilde{a}_k^*$ (4.4), the singularity at the shock, and defect correction. We start with the discretisation scheme.

The experiments indicate that Godunov's scheme does not always provide proper convergence rates around the sonic point. This is partly explained by the analysis presented in the Appendix, which shows divergence for certain positions of the sonic point. Another complication occurs if $\overline{u}_k > 0$ is the steady state just right of the sonic point, and the initial guess $u_k < 0$. In that case u_k has to go through zero to reach the steady state. In more detail, the following happens.

Let the stationary sonic point lie between $\overline{u}_{k-1} < 0$ and $\overline{u}_k > 0$. Allowing for an initial guess $u_k < 0$, the residual $r_k = s_k - u_k|u_k|/(2h) = (\overline{u}_k^2 - u_k|u_k|)/(2h)$, and the corresponding value of $a_k^* = |u_k|/h$ by (4.3a) and (4.3b). This value is modified according to (4.4), and used in the iterative method (4.2). For $\beta = 1$ the right-hand side of (4.2) becomes $2u_k^3/(1 + 3u_k^2)$ if $u_k < 0$ and $\overline{u}_k = 1$. Thus, convergence is obtained to $u_k = 0$ rather than $\overline{u}_k = 1$. Once u_k has become zero, the iterations continue in a different way towards $\overline{u}_k = 1$.

As a whole, this process will take too long. For this reason, and because of the instability found in the Appendix, we abandon the use of Godunov's scheme in favour of Roe's (2.9) at the sonic point. In the following discussion, we will consider Roe's scheme (scheme I), and a variant of the E–O, or FVS scheme, that uses (2.9) at the sonic point (scheme II). Local relaxation is always applied several times after grid refinement. For scheme I, we use four cells for local relaxation. For scheme II, the experiments showed that four cells did not provide satisfactory results, but six cells allowed us to obtain convergence results comparable to scheme I.

If damped PJ without local relaxation was used, the convergence towards the steady state was dominated by the shock. For scheme I (Roe's) we observed a convergence factor 0.5 for the choices (4.3b) and (4.3c) of a^*, with or without the modification (4.4). This is in agreement with the discussion of §3.5. The two-grid convergence factor was obtained for a simulated two-grid algorithm, using three multigrid cycles on each of the coarser grids.

Scheme II provided values between 0.3 and 0.7, depending on the choice of a^* and the position of the shock. The choice (4.3a) did not work unless modified by (4.4), and even then the convergence was not very good. For (4.3b) with or without (4.4), we observed convergence factors between 0.3 and 0.6, and for (4.3c) between 0.4 and 0.5, depending on the position of the shock. Given the lower accuracy of scheme II, not much is gained by the sometimes slightly faster convergence.

If local relaxation was used once after every prolongation, the convergence rate was dominated by the slower convergence at the sonic point.

Next we consider convergence down to the truncation error. If no local relaxation was applied after prolongation, but only after grid refinement, both schemes I and II required two F-cycles with (4.3b) and three F-cycles with (4.3c). It turned out that

with local relaxation, just one F-cycle was sufficient to obtain the steady state for both scheme I and II with damped PJ.

Second-order accurate solutions can be computed by the Defect Correction technique. For schemes I and II this required about eight F-cycles when damped PJ was used. Convergence for (4.3b) was slightly better than for (4.3c). For RB1 with (4.3b) and $\beta = 1$, a stationary solution was obtained in five F-cycles. Without local relaxation after prolongation, about 15 F-cycles were required for both schemes I and II. Here local relaxation was applied only after grid refinement. Thus, convergence was about a factor two slower if local relaxation after prolongation was omitted.

5. Conclusions. Convergence towards the steady state of the one-dimensional Burgers equation by means of a multigrid method has been studied. In order to exploit the global character of multigrid relaxation, only relaxation schemes that affect the solution locally have been considered. This excludes a global scheme such as line relaxation, and also Gauss–Seidel relaxation which uses only local data but has a global effect.

Linear two-level analysis has been carried out for a one-dimensional upwind differenced convection equation with a constant coefficient. Both smoothing rates and two-grid convergence factors have been derived. It turns out that optimising the smoothing rate does not necessarily imply a good two-grid convergence factor.

Experiments on the nonlinear inviscid Burgers equation show an $O(h)$ difference between the observed convergence factors for a simulated two-grid algorithm and the two-grid convergence factors for the linear constant-coefficient case, if there are no shocks or sonic points. The linear constant-coefficient case cannot explain what happens if the convection speed changes sign, i.e., at the sonic point and the shock. Convergence at the sonic point can be analysed separately.

At the shock the discrete nonlinear steady-state equations, obtained by upwind differencing, become singular. This reflects the singularity of the differential equation, which is overcome by invoking the jump condition across a discontinuity. The jump condition is automatically satisfied if the equation is solved in conservation form. In the present steady-state calculations, only conservation in space is retained. Conservation in time is dropped to obtain fast convergence. This introduces the singularity at the shock, and makes the steady state nonunique. The obvious way to regularise the discrete equations is by requiring global conservation in time, which is imposed as an additional condition on the coarsest grid. In this way, the multigrid method converges towards the correct steady state.

The linear two-level analysis provides a zero two-grid convergence factor for damped PJ. The numerical experiments confirm this, within $O(h)$, for a smooth test problem. At the shock, the convergence rate is about 0.5, which can be improved by local relaxation using a variant of Newton's method. Here the additional condition of local conservation is imposed to remove the singularity. If local relaxation was applied once after every prolongation, the steady state for the discontinuous test problem could be computed in only one F-cycle, given an initial guess from the coarser grid.

RB relaxation, if used as a single-grid scheme, is independent of the colouring, i.e., the convergence rate is the same if first the red and then the black cells are relaxed, or the other way around. This property is lost if RB is used as part of a multigrid scheme. The multigrid convergence rate depends on the colouring relative to the coarse-grid cells and the direction of the flow. If the order in which first the red and then the black cells are relaxed follows the flow as seen from each coarse-grid cell, then RB has a convergence rate zero in the linear constant-coefficient case; otherwise,

it is unstable. A damped version of RB does not suffer from the instability.

Steady solutions with second-order accuracy have been computed by the Defect Correction technique. Convergence is considerably slower than in the first-order case.

An extension of the present work to two dimensions can be found in [15]. Nonlinear singularities of the discrete equations are not considered in that paper. In two or three dimensions, such singularities are less likely to occur, because the coupling between the state variables is increased by having more than one direction. Still, a regularisation is required if the nonlinear steady-state problem is singular. The regularisation chosen here is based on global conservation in time of the initial states. However, this approach cannot be used if singularities occur in more than one cell. In [13], the isenthalpic Euler equations were solved, and some time accuracy was maintained, which avoids possible problems with singularities. This may not be sufficient in all cases. An alternative regularisation can be obtained through additional viscous terms, at the expense of the spatial accuracy. Brandt proposes the use of double discretisation [3, §10.2] to overcome this problem, but it remains to be seen if a robust method can be obtained by this technique.

Appendix. Convergence rates at the sonic point. Two-grid convergence rates at the sonic point are estimated by assuming the numerical steady state to be linear in x, and by linearising the discrete nonlinear equations. We consider six cells and a numerical steady state

$$\text{(A1)} \qquad \overline{u}_k = a_k = a_2 + (k-2)ha_2' + O(h^2) \quad \text{with } a_2' > 0, \qquad k = 0, \cdots, 5.$$

There are two cases, corresponding to two different positions of the sonic point with respect to the coarse-grid cells:

$$\text{(A2a)} \qquad \text{Case 1}: \qquad a_2 = \alpha h a_2',$$

$$\text{(A2b)} \qquad \text{Case 2}: \qquad a_2 = -\alpha h a_2',$$

where $0 \le \alpha \le 1$. The numerical sonic point lies between cell one and two in Case 1, and between two and three in Case 2. To obtain a linear discrete operator, we evaluate the nonlinear residuals, rewrite them in terms of the iteration error $v_k = \overline{u}_k - u_k$, and ignore terms of $O(v_k^2)$. The upwind differencing of the nonlinear equation is based on $\overline{u}_k$, not on u_k. Note that these two choices *cannot* be justified by assuming v_k to be small. Their only justification is the simplification of the analysis.

We will not present all the details of the derivation of the two-grid convergence factors, but merely mention some of the intermediate results for Case 1 when using Roe's scheme. In that case, we have, for instance,

$$\text{(A3a)} \qquad \begin{aligned} r_0 &= s_0 - \frac{1}{2h}(u_1^2 - u_0^2) = \frac{1}{2h}(\overline{u}_1^2 - \overline{u}_0^2) - \frac{1}{2h}\left((\overline{u}_1^2 - v_1)^2 - (\overline{u}_0 - v_0)^2\right) \\ &\simeq (a_1 v_1 - a_0 v_0)/h, \end{aligned}$$

and

$$\text{(A3b)} \qquad \begin{aligned} r_2 &= s_2 - \frac{1}{2h}(u_2^2 - u_1 u_2) \\ &= \frac{1}{2h}(\overline{u}_2^2 - \overline{u}_1 \overline{u}_2) - \frac{1}{2h}\left((\overline{u}_2 - v_2)^2 - (\overline{u}_1 - v_1)(\overline{u}_2 - v_2)\right) \\ &\simeq \left(a_2 v_2 - \tfrac{1}{2}(a_1 v_2 + a_2 v_1)\right)/h. \end{aligned}$$

The other residuals follow in a similar way by using Roe's scheme and choosing the upwind direction from $\overline{u}$. In this way a linear operator L^h is obtained, which can be written as a 6×6 matrix acting on $(v_0, v_1, \cdots, v_5)^T$.

Only damped PJ is considered as a relaxation operator. For the choice (4.3a) of a_k^*, this relaxation operator becomes

$$S^{PJ} = I - \tfrac{1}{2} N^{-1} L^h, \tag{A4}$$

where N is a diagonal matrix with its diagonal elements equal to those of L^h. The restriction and prolongation operators are chosen as in §3.2. For the CGC operator, we need L^H, which can be obtained by Galerkin coarsening ($L^H = I_h^H L^h I_H^h$), or evaluated directly from the coarse-grid equations, using $\overline{u}_k^H = a_k^H = \frac{1}{2}(a_{2k}^h + a_{2k+1}^h)$ for $k = 0, 1, 2$. Only the last option is considered here. In Case 1 with Roe's scheme, we obtain

$$L^H = \frac{1}{2h} \begin{pmatrix} \frac{1}{2}a_1^H - a_0^H & \frac{1}{2}a_0^H & 0 \\ -\frac{1}{2}a_1^H & a_1^H - \frac{1}{2}a_0^H & 0 \\ 0 & -a_1^H & a_2^H \end{pmatrix}. \tag{A5}$$

Similar results are obtained for Case 2, and for Godunov's scheme. It should be noted that in Case 2 we obtain operators that are different for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

Once the relaxation operator S^{PJ} and the CGC operator K have been obtained, the two-grid convergence rate follows from

$$\overline{\lambda}(\alpha) = \rho(KS^{PJ}). \tag{A6}$$

Table 2 lists the smallest and largest values of the two-grid convergence factor, i.e., $\min_\alpha \overline{\lambda}(\alpha)$ and $\max_\alpha \overline{\lambda}(\alpha)$. Different results are obtained for Godunov's and Roe's scheme, and for the choices of a_k^* listed in (4.3). The entry ∞ is due to a vanishing coarse-grid convection speed a_1^H for $\alpha = \frac{1}{2}$ in Case 2. This problem does not occur for Roe's scheme.

TABLE 2

Two-grid convergence factors around the sonic point. The values shown are the minimum and maximum of $\overline{\lambda}(\alpha)$ over α, which determines the position of the sonic point relative to the grid. The choices of a_k^* are listed in (4.3). *Damped* PJ ($\beta = 1/2$) *is used as relaxation scheme.*

Scheme	a_k^*	Case 1		Case 2	
Godunov	a, b	0.354	0.500	0.500	∞
	c	0.500	0.556	0.375	0.556
Roe	a, b	0.500	0.500	0.318	0.359
	c	0.249	0.312	0.375	0.614

REFERENCES

[1] W. K. ANDERSON, *Implicit multigrid algorithms for the three-dimensional flux split Euler equations*, Ph. D. thesis, Mississippi State University, Mississippi State, MS, 1986.

[2] D. BAI AND A. BRANDT, *Local mesh refinement multilevel techniques*, SIAM J. Sci. Statist. Comput., 8 (1986), pp. 109-134.

[3] A. BRANDT, *Guide to multigrid development*, Lecture Notes in Math., 960 (1981), pp. 220-312.

[4] B. ENGQUIST AND S. OSHER, *Stable and entropy satisfying approximations for transonic flow calculations*, Math. Comp., 34 (1980), pp. 45-75.

[5] S. K. GODUNOV, *Finite-difference method for numerical computation of discontinuous solutions of the equations of fluid dynamics*, Mat. USSR Sb., 47 (1959), pp. 271-306.

[6] W. HACKBUSCH, *Multi-Grid Methods and Applications*, Springer Series in Computational Mathematics, 4, Springer-Verlag, Berlin, Heidelberg, New York, 1985.

[7] P. W. HEMKER AND S. P. SPEKREIJSE, *Multiple grid and Osher's scheme for the efficient solution of the steady Euler equations*, Appl. Numer. Math., 2 (1986), pp. 475-493.

[8] P. W. HEMKER, *Defect correction and higher order schemes for the multigrid solution of the steady Euler equations*, Lecture Notes in Math., 1228 (1986), pp. 149-165.

[9] A. JAMESON, *Solution of the Euler equations for two-dimensional transonic flow by a multigrid method*, Appl. Math. Comput., 13 (1983), pp. 327-356.

[10] D. C. JESPERSEN, *Design and implementation of a multigrid code for the Euler equations*, Appl. Math. Comput., 13 (1983), pp. 357-374.

[11] B. KOREN, *Defect correction and multigrid for an efficient and accurate computation of airfoil flows*, J. Comput. Phys., 77 (1988), pp. 183-206.

[12] W. A. MULDER AND B. VAN LEER, *Experiments with implicit upwind methods for the Euler equations*, J. Comput. Phys., 59 (1985), pp. 232-246.

[13] W. A. MULDER, *Multigrid relaxation for the Euler equations*, J. Comput. Phys., 60 (1985), pp. 235-252.

[14] ———, *Computation of the quasi-steady gas flow in a spiral galaxy by means of a multigrid method*, Astron. and Astrophys., 156 (1986), pp. 354-380.

[15] ———, *Analysis of a multigrid method for the Euler equations of gas dynamics in two dimensions*, in Multigrid Methods: Theory, Applications, and Supercomputing, S. McCormick, ed., Marcel Dekker, New York, 1988, pp. 467-489.

[16] R. H. NI, *A multiple grid scheme for solving the Euler equations*, AIAA J., 20 (1982), pp. 1565-1571.

[17] S. OSHER AND F. SOLOMON, *Upwind difference schemes for hyperbolic systems of conservation laws*, Math. Comput., 38 (1982), pp. 339-374.

[18] P. L. ROE, *Approximate Riemann solvers, parameter vectors, and difference schemes*, J. Comput. Phys., 43 (1981), pp. 357-372.

[19] J. STEGER AND R. WARMING, *Flux vector splitting of the inviscid gas dynamics equations with applications to finite-difference methods*, J. Comput. Phys., 40 (1981), pp. 263-293.

[20] K. STÜBEN AND U. TROTTENBERG, *Multigrid methods: Fundamental algorithms, model problem analysis and applications*, Lecture Notes in Math., 960 (1981), pp. 1-176.

[21] G. D. VAN ALBADA, B. VAN LEER, AND W. W. ROBERTS, *A comparative study of computational methods in cosmic gas dynamics*, Astron. and Astrophys., 108 (1982), pp. 76-84.

[22] B. VAN LEER, *Towards the ultimate conservative difference scheme.* IV. *A new approach to numerical convection*, J. Comput. Phys., 23 (1975), pp. 276-299.

[23] ———, *On the relation between the upwind-differencing schemes of Godunov, Enquist–Osher, and Roe*, SIAM J. Sci. Statist. Comput., 5 (1984), pp. 1-20.

[24] B. VAN LEER AND W. A. MULDER, *Relaxation methods for hyperbolic conservation laws*, in Numerical Methods for the Euler Equations of Fluid Dynamics, F. Angrand, A. Dervieux, J. A. Desideri, and R. Glowinski, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1985, pp. 312-333.