Lecture notes on locally compact quantum groups

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Contents

1 Introduction

These notes contain an introduction to the theory of *locally* compact quantum group. In particular we focus on the problems/novelties involved in defining such objects compared to *compact* quantum groups. It turns out that there are different approaches to the theory, but here we will mainly focus on the approach taken by J. Kustermans and S. Vaes [22], [23] which was settled successfully around the year 2000 and is nowadays the most applied approach.

These lecture notes are set up in the following way.

- In the **Introduction** we will cover some background material: what is the proper notion of a non-compact quantum group? What is the history behind their development? How are operator algebras involved? And in particular, what is the advantage of a *von Neumann*-algebraic approach above a *C^*-algebra*ic approach and the other way around. We will also give references and the reader should see these notes as a guideline to start reading in these references, which can be technically quite demanding.
- In **Section 2** we give some background on von Neumann algebras. We assume that the reader is familiar with some general C*-algebra theory; essentially what is needed to treat compact quantum groups. Section 2 serves as a quick reminder of some essential ingredients and notation we need for the further sections.
- In **Section 3** we treat the Kustermans–Vaes approach to locally compact quantum groups. We state their definition and give basic examples.
- **Section 4** treats Pontrjagin-duality (or Pontrjagin-Van Kampen duality). We define the left regular representation for quantum groups and show how a dual quantum group can be constructed from it. The construction of the left regular representation can be considered as the most crucial and essential part of the theory. We also show the relations between the various objects defined so far. In particular, we focus on the relations between a quantum group and its dual.
- **Section 5** treats examples of locally compact quantum groups. It should immediately be said that examples are very sparse and the construction of them is a non-trivial task in itself.
- **Section 6** gives a very brief treatment of cocycle twisting. The section is certainly a bit too much on the concise side, as for a suitable treatment we should have introduced crossed products first.

1.1 What is a good definition of a locally compact quantum group?

Recall that if G is a locally compact abelian group, then the dual group $\widehat{\mathsf{G}}$ is defined as the group of all irreducible unitary strongly continuous representations of G. Since irreducible representations are necessarily one dimensional (as G is abelian) we can multiply representations as functions and hence $\widehat{\mathsf{G}}$ carries a group structure. The Pontrjagin duality theorem states that $\widehat{\widehat{\mathsf{G}}}$ is isomorphic as a group to G.

One of the main motivating questions behind the development of operator quantum groups is if the Pontrjagin duality theorem can be extended beyond the category of abelian groups. That is, one searches for a category in which the objects incorporate at least all locally compact groups. But since for an arbitrary locally compact group the irreducible representations are not necessarily 1-dimensional anymore it is not expected that the dual will have the structure of a group anymore. That is, the category we are looking for will include more objects (in particular it turns out that group von Neumann algebras will be in there).

The other motivating question is that in the 1980's work of V. Drinfeld and S.L. Woronowicz produced examples of structures admitting nice representation theory. The typical examples are $SU_q(n)$ and its corresponding q-deformed universal enveloping Lie algebra. The former one is due to Woronowicz, the latter one due to Drinfeld. These turn out to give examples of compact quantum groups (see the other mini-courses of this summer school). But also – on a purely algebraic level – notions of $SU_q(n, m)$ have been found. These give certain series of examples that should somehow be part of a conceptual category that we would like to call locally compact quantum groups.

Summarizing the previous paragraphs a good notion of a locally compact quantum group should at least:

- Feature a Pontrjagin duality theorem;
- Incorporate all compact quantum groups;
- Incorporate all locally compact groups;
- Incorporate known q-deformed examples of (non-compact) Lie groups;
- Be a concise and workable definition.

So which approach we take? In 1987 Woronowicz suitably settled the theory of compact quantum groups [36]. We recall his definition here:

Definition 1.1. A compact quantum group consists of a pair (A, Δ) where A is a unital C*-algebra and $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ is a unital $*$-homomorphism such that $(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta$. Moreover, the cancellation laws hold:

$$\overline{\Delta(\mathsf{A})(\mathsf{A} \otimes 1)} = \mathsf{A} \otimes \mathsf{A} = \overline{\Delta(\mathsf{A})(1 \otimes \mathsf{A})}.$$

Here the tensor product is the minimal tensor product of C*-algebras and the overline indicates the norm closure.

These conditions are sufficient to derive the existence of a Haar weight on the quantum group. It turns out that simply generalizing this definition to a 'non-compact' definition is insufficient. Such a generalization would mean that we drop the unitality condition on A and let Δ be a map from A to $M(\mathsf{A} \otimes \mathsf{A})$; the multiplier algebra of $\mathsf{A} \otimes \mathsf{A}$. The classical case would then correspond to $\mathsf{A} = C_0(\mathsf{G})$. Unfortunately there is no (known) reasonable set of extra assumptions on A that allows us to derive the existence of the Haar weight(s) in this case.

Another approach to the theory (mainly due to Baaj–Skandalis and Woronowicz from the 1990's) is through multiplicative unitaries. Roughly, for classical groups this approach postulates the existence of the left regular representation. The approach is partly successful, also nowadays (see for example [25]), but again proving the existence of Haar weights with a reasonable set of extra assumptions has not been possible. The reader interested in this approach is referred to [30] and references there. In the current notes multiplicative unitaries will be part of our theory as we shall be able to construct them.

So what else can we do? We approach the theory by postulating the existence of Haar weights. There is one drawback to this approach: namely that the Haar weights are postulated to exist whereas in the case of locally compact groups one is able to construct them. There are many advantages though: namely one can define locally compact quantum groups in a way that satisfies *all* of the bullet points above. Moreover, in examples the Haar weights are usually easy to construct. Or better: they are easy to find, as they are unique up to scaling.

We state the precise definition of a locally compact quantum group in Section 3. This definition is proposed by Johan Kustermans and Stefaan Vaes as they have successfully established this approach in [22], [23]. The Kustermans–Vaes theory of locally compact quantum groups comes in two versions: a von Neumann algebraic one (the measurable side) and a C*-algebraic one (the continuous side). The original paper [22] is written in the C*-algebraic language but uses von Neumann algebraic techniques in the proofs. A purely von Neumann algebraic definition is given in [23] but the proofs rely on [22]. A self-contained approach to the Von Neumann algebraic theory was later also given by Van Daele [35].

The main importance of the von Neumann algebraic approach is that one is able to use Tomita-Takesaki theory. We do not expect that the reader is familiar with Tomita-Takesaki theory, but let us say some words about it. An integral on a matrix algebra is as a mapping $\varphi : M_n(\mathbb{C})^+ \to [0, \infty]$ that preserves convex combinations. The integral is called a trace if $\varphi(xy) = \varphi(yx)$. It turns out that in many application this trace property is of extreme importance. So what if φ is not a trace? Well, then there is still a way to suitably treat this theory: Tomita-Takesaki theory. There exists a one–parameter series of automorphisms $\sigma : \mathbb{R} \to \mathrm{Aut}(M_n(\mathbb{C}))$ such that we have the skew relation $\varphi(xy) = \varphi(y\sigma_{-i}(x))$, where σ_{-i} is an analytic extension of $t \mapsto \sigma_t$. This is a (main) consequence of Tomita-Takesaki theory and it is remarkably effective for von Neumann algebra theory. In fact classification of hyperfinite von Neumann algebras can fully be described in terms of such automorphism groups. There is no general C*-algebraic version of such theory (it in fact becomes a new definition, which you might encounter as KMS-state) and this is what makes von Neumann algebras important in the theory of quantum groups.

So what about C*-algebras? Operator algebraic quantum groups need C*-algebras because of their representation theory. They should not be regarded as an alternative approach to the theory, but they go side-by-side with the von Neumann algebraic theory. They enhance the von Neumann algebraic theory and vice-versa. A beautiful example of this is Kasprzak's approach to quantum homogeneous spaces [18]. Also, classically the representations of a group G correspond to representations of its universal group C*-algebra. In quantum group theory, this approach goes completely parallel and was developed by Kustermans [24].

Finally, does the definition of locally compact quantum groups – being more general – make the definition of compact quantum groups redundant? The answer is definitely *no.* In almost any research paper dealing only with the compact case, the compact definition is used. The reason is (in my opinion/understanding) that this definition is more comprehensible (C*-algebras are accessible to a larger audience) and it is for sure more authentical.

1.2 Further literature

In these notes we introduce the basic concepts of the theory of locally compact quantum groups and omit almost all technical details. We focus on constructions instead of proofs, giving the reader a good starting point to look further in the literature. In particular, we explain how the Pontrjagin dual quantum groups is constructed as well as many other features as the antipode, its polar decomposition and the basic treatment of some examples.

Entering the complete theory requires certainly some time and a basic acquintance with C*- and von Neumann algebra theory, but there are good sources available. Accessible references are Stefaan Vaes his thesis [33] and Alfons van Daele his paper [35]. These contain a concise and well-explained treatment of general locally compact quantum groups. Then there are the original papers [22] and [23] of which [22] contains all the essential proofs. The book [30] also contains a treatment of operator algebraic quantum groups, but for proofs refers again to [22].

Finally, each of these references require a reasonable background in weight theory for von Neumann algebras and in particular Tomita-Takesaki theory. For these we refer the reader to the final section of [26] and of course the standard reference [28].

1.3 Some preliminaries and notation

Basic notation. We will basically take over notation from [28]. The symbol $\otimes$ means several different things: it is an algebraic tensor product of two elements. For C*-algebras it is the minimal tensor product (see [27]), for von Neumann algebras it is is the closure in the strong operator topology of the algebraic tensor product (which is the usual von Neumann algebraic tensor product). The symbol ι denotes the identity mapping.

Slicing and tensoring linear maps. If A and B are C*-algebras (resp. von Neumann algebras) and $\pi : \mathsf{A} \to \mathsf{A}$ is a (resp. normal) $*$-homomorphism, then there is a unique (resp. normal) $*$-homomorphism $(\pi \otimes \iota) : \mathsf{A} \otimes \mathsf{B} \to \mathsf{A} \otimes \mathsf{B}$ that maps the algebraic tensor product $a \otimes b$ to $\pi(a) \otimes b$. Similarly, if $\omega : \mathsf{A} \to \mathbb{C}$ is a bounded (resp. normal) functional then there exists a bounded (resp. normal) map $(\omega \otimes \iota) : \mathsf{A} \otimes \mathsf{B} \to \mathsf{B}$ sending $a \otimes b$ to $\omega(a)b$.

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Section 7 of the current notes is almost entirely taken from earlier lecture notes written together with Erik Koelink for a winter school in Bizerte, 2010 [4]. The author is indebted to Erik Koelink for allowing him to copy these notes. These results are published in [19] (see also [16]).

2 Von Neumann algebras

In this section we recall some notions from von Neumann algebra theory. We shall also briefly say something about Tomita-Takesaki theory which is important in the theory of locally compact quantum groups (in fact already for compact quantum groups, it is an important tool). We assume that the reader is familiar with notions from C*-algebra theory such as the definition of C*-algebras, GNS-constructions and the Gelfand-Naimark theorem.

2.1 Von Neumann algebras

Von Neumann algebras are special kinds of C*-algebras and by definition they are represented as bounded operators on a Hilbert space. In order to define them more precisely, recall that the strong operator topology on $B(\mathcal{H})$ is the topology induced by the semi-norms $x \mapsto \|x\xi\|$, where $\xi \in \mathcal{H}$.

Definition 2.1. A von Neumann algebra M is a unital $*$-algebra of bounded operators on some Hilbert space $\mathcal{H}$ that is closed in the strong operator topology.

Von Neumann's famous double commutant theorem states that a von Neumann algebra may alternatively be defined through its double commutant.

Theorem 2.2. *Let $A \subseteq B(\mathcal{H})$ be a unital $*$-algebra. A is a von Neumann algebra if and only if A is equal to its double commutant $(A')'$. Here the commutant of an algebra $B \subseteq B(\mathcal{H})$ is defined as*

$$B' = \{x \in B(\mathcal{H}) \mid \forall y \in B : xy = yx\}.$$

Example 2.3. Every abelian von Neumann algebra is of the form $L^\infty(X)$ where X is some measure space. Other examples are $L^\infty(X) \otimes B(\mathcal{H})$ (strong closure of the algebraic tensor product) and direct sums of von Neumann algebras. In fact these exhaust all von Neumann algebras of type I (see [27] for details). Most of the known examples of non-compact quantum groups have a type I von Neumann algebra. In particular the von Neumann algebra of quantum $SU_q(1,1)$ is a type I algebra, see [5].

We let M_* be the space of all (bounded) functionals $\mathsf{M} \to \mathbb{C}$ that are continuous for the strong operator topology on bounded sets. There is a natural pairing,

$$\langle \,\cdot\,,\,\cdot\,\rangle : \mathsf{M}_* \times \mathsf{M} \to \mathbb{C} : (\omega, x) \mapsto \omega(x),$$

and through this pairing $(\mathsf{M}_*)^* = \mathsf{M}$. Therefore M_* is called the predual of M. In fact M_* is the *unique* (up to isomorphism) Banach space X such that M is the dual of X. Also, a unital C*-algebra A is a von Neumann algebra if and only if it is the dual of a banach space X. The topology induced on M by M_* is called the σ-weak topology (sometimes also called the ultraweak topology or the $\sigma(\mathsf{M}, \mathsf{M}_*)$-topology).

Suppose that $(x_i)_i$ is a bounded and increasing net of positive elements in M. Then there exists a (unique) element $x \in \mathsf{M}^+$ that is the supremum of $(x_i)_i$. That is, $\forall i : x_i \leq x$ and whenever for some $y \in \mathsf{M}^+$ we have $\forall i : x_i \leq y$ then $x \leq y$. We denote $\sup x_i$ for this x. The following definition should be considered as an unbounded version of a state.

Definition 2.4. A weight $\varphi : \mathsf{M}^+ \to [0, \infty]$ is a map that preserves convex combintations. We call a weight φ faithful if $\varphi(x^*x) = 0$ implies that $x = 0$. A weight φ is called normal if $\varphi(\sup x_i) = \sup \varphi(x_i)$. Set

$$\mathfrak{n}_\varphi = \{x \in \mathsf{M} \mid \varphi(x^*x) < \infty\}.$$

The weight φ is called semi-finite if the set $\mathfrak{n}_\varphi$ is σ-weakly dense in M.

Whereas not every von Neumann algebra possesses a normal, faithful state, every von Neumann algebra does posess a normal, faithful semi-finite weight. We shall briefly write nsf weight for normal, semi-finite, faithful weight (and this is occassionally done so in the literature). Now fix a nsf weight φ on a von Neumann algebra M. We shall do a GNS-construction as in the C*-algebra case. Firstly, because of the inequality,

$$\varphi(x^*y^*yx) \leq \|y\|^2 \varphi(x^*x), \qquad x, y \in \mathsf{M}, \tag{2.1}$$

we find that $\mathfrak{n}_\varphi$ is a left-ideal. Now we equip $\mathfrak{n}_\varphi$ with the following inner-product:

$$\langle x, x \rangle = \varphi(x^*x), \qquad x \in \mathfrak{n}_\varphi,$$

which can be extended to an inner product $\mathfrak{n}_\varphi \times \mathfrak{n}_\varphi \to \mathbb{C}$ through the polarization identity

$$\langle x, y \rangle = \sum_{k=0}^{3} i^k \langle x + i^k y, x + i^k y \rangle, \tag{2.2}$$

and in fact we shall write as well $\varphi(y^*x)$ for this quantity. So letting $\mathfrak{m}_\varphi$ be the linear span of $\mathfrak{n}_\varphi^* \mathfrak{n}_\varphi$ we have properly defined φ on $\mathfrak{m}_\varphi$. Since φ is faithful (2.2) defines a non-degenerate inner-product and we complete $\mathfrak{n}_\varphi$ into a Hilbert space $\mathcal{H}_\varphi$. We shall distinguish elements in $\mathfrak{n}_\varphi$ viewed as a subset of M from elements in $\mathfrak{n}_\varphi$ viewed as an element of the Hilbert space $\mathcal{H}_\varphi$. That is, for $x \in \mathfrak{n}_\varphi \subseteq \mathsf{M}$ we shall write $\Lambda_\varphi(x)$ for the corresponding element in $\mathcal{H}_\varphi$. So in particular $\langle \Lambda_\varphi(x), \Lambda_\varphi(y) \rangle = \varphi(y^*x)$. Λ_φ turns out to be closed in the σ-weak/norm topology (equivalently in the σ-weak/weak topology). The inequality (2.1) shows that for $y \in \mathsf{M}$ there is a bounded operator

$$\pi_\varphi(y) : \Lambda_\varphi(x) \mapsto \Lambda_\varphi(yx).$$

Using that φ is faithful, one can show that the representation π_φ is faithful. Using that φ is normal, one can prove that π_φ is continuous in the σ-weak topology. The triple $(\mathcal{H}_\varphi, \pi_\varphi, \Lambda_\varphi)$ is called the GNS-representation of φ (sometimes it is also called the cyclic GNS-representation). Since in these notes φ will be fixed (namely the left Haar weight) we will simply write $(\mathcal{H}, \pi, \Lambda)$.

Remark 2.5. If φ is a nsf *state*, so $\varphi(1) = 1$. Then $\mathcal{H}_\varphi$ possesses a cyclic and separating vector Ω_φ, namely $\Lambda_\varphi(1)$. In this case it is also common to write $\Lambda_\varphi(x) = x\Omega_\varphi$.

2.2 Tomita–Takesaki theory

Let M be a von Neumann algebra with nsf weight φ. We may assume that M is represented on its GNS-space $\mathcal{H}_\varphi$ for which we shall simply write $\mathcal{H}$. This way we may drop the GNS-map π_φ in our notation. Consider the anti-linear mapping,

$$S : \Lambda_\varphi(x) \mapsto \Lambda_\varphi(x^*), \qquad x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*.$$

The mapping is closed and generally unbounded. Let $S = J\nabla^{1/2}$ be its polar decomposition. Here $J : \mathcal{H} \to \mathcal{H}$ is an anti-unitary map and $\nabla^{1/2}$ is a closed map with $\mathrm{Dom}(\nabla^{1/2}) = \mathrm{Dom}(S)$. By Tomita-Takesaki theory:

$$J\mathsf{M}J = \mathsf{M}', \qquad \nabla^{it}\mathsf{M}\nabla^{-it} = \mathsf{M}, \qquad t \in \mathbb{R}.$$

In case S and hence ∇ is a bounded operator, the latter fact admits a short proof which can be found in [1]. Otherwise the proof is much more involved and can be found in [28]. We define the modular automorphism group as:

$$\sigma^\varphi : \mathbb{R} \to \mathrm{Aut}(\mathsf{M}) : t \mapsto \left(x \mapsto \nabla^{it} x \nabla^{-it}\right).$$

We set:

$$\mathcal{T}_\varphi = \{x \in \mathsf{M} \mid x \text{ is analytic for } \sigma^\varphi \text{ and } \forall z \in \mathbb{C} : \sigma_z^\varphi(x) \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*\}.$$

The set $\mathcal{T}_\varphi$ is sometimes referred to as the Tomita algebra, though stricly speaking this terminology is not correct. It is the Hilbert space identification $\Lambda(\mathcal{T}_\varphi)$ that forms a Tomita algebra in the sense of [28]. We have the following lemma, of which we include a proof as it involves a standard technique in quantum group theory. Note that in the proof we use that $\varphi \circ \sigma_t^\varphi = \varphi$ (which can easily be derived from the definitions) and that consequently $\mathfrak{n}_\varphi$ is invariant under σ^φ.

Lemma 2.6. *$\mathcal{T}_\varphi$ is σ-weakly dense in M.*

Proof. Since φ is semi-finite, by definition $\mathfrak{n}_\varphi$ is dense in M. Since $\mathfrak{n}_\varphi$ is a left-ideal we see that $\mathfrak{n}_\varphi^*$ is a right ideal. It follows that $\mathfrak{n}_\varphi^*\mathfrak{n}_\varphi$ is contained in $\mathfrak{n}_\varphi^* \cap \mathfrak{n}_\varphi$. Moreover, we claim that $\mathfrak{n}_\varphi^*\mathfrak{n}_\varphi$ is σ-weakly dense in M. Indeed, let $(e_j)_{j\in J}$ be a net in $\mathfrak{n}_\varphi$ converging σ-weakly to 1 (which exists by σ-weak density of $\mathfrak{n}_\varphi$ in M), then for every $x \in \mathfrak{n}_\varphi$ the net $(e_j^* x)_{j\in J}$ converges to x in the σ-weak topology. As $\mathfrak{n}_\varphi$ is σ-weakly dense in M, this implies that $\mathfrak{n}_\varphi^*\mathfrak{n}_\varphi$ is σ-weakly dense in M. Summarizing we proved that $\mathfrak{n}_\varphi^* \cap \mathfrak{n}_\varphi$ is σ-weakly dense in M. Now take $x \in \mathfrak{n}_\varphi^* \cap \mathfrak{n}_\varphi$. Consider

$$x_n = \sqrt{\frac{\pi}{n}} \int_{-\infty}^{\infty} e^{-nt^2} \sigma_t^\varphi(x) dt,$$

We have $x_n \in \mathcal{T}_\varphi$ with $\sigma_z^\varphi(x) = \sqrt{\frac{\pi}{n}} \int_{-\infty}^{\infty} e^{-n(t-z)^2} \sigma_t^\varphi(x) dt \in \mathfrak{n}_\varphi^* \cap \mathfrak{n}_\varphi$. Moreover $x_n \to x$ in the σ-weak topology. $\square$

Exercise 2.7. Prove that $\Lambda(\mathcal{T}_\varphi^2)$ is dense in $\mathcal{H}_\varphi$. Here $\mathcal{T}_\varphi^2$ is the product of $\mathcal{T}_\varphi$ with itself.

Exercise 2.8. Consider the von Neumann algebra $\mathsf{M} = M_n(\mathbb{C})$ and let $\varphi(x) = \mathrm{Tr}(xA)$ for some positive matrix $A \in M_n(\mathbb{C})$. Determine J and ∇.

Remark 2.9. The results mentioned in this section are in fact *consequences* of a bigger theory: Tomita-Takesaki theory (we often mentioned it without really explaing what it is). The complete theory proceeds through the notion of Hilbert algebras and can be found in either [28] or [26]. The general theory is in fact essential in the construction of the Haar weights on a (dual) quantum group, but we rather give a reference at a later point than recalling the complete theory here.

3 Locally compact quantum groups

In this section we explain the notion of locally compact quantum groups and give classical and compact examples.

3.1 The definition

The following definition is due to Johan Kustermans and Stefaan Vaes and can be found in [23]. See also [22].

Definition 3.1. A locally compact quantum group $\mathbb{G} = (\mathsf{M}, \Delta, \varphi, \psi)$ is a 4-tuple consisting of:

- A von Neumann algebra M;
- A comultiplication $\Delta : \mathsf{M} \to \mathsf{M} \otimes \mathsf{M}$, which is a normal, unital $*$-homomorphism satisfying the coassociativity relation:

$$(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta;$$

- Two normal, semi-finite, faithful weights $\varphi : \mathsf{M}^+ \to [0, \infty]$ and $\psi : \mathsf{M}^+ \to [0, \infty]$ satisfying:

$$\begin{aligned} \varphi((\omega \otimes \iota) \circ \Delta(x)) &= \varphi(x), \qquad x \in \mathsf{M}^+, \omega \in \mathsf{M}_*^+, \\ \psi((\iota \otimes \omega) \circ \Delta(x)) &= \psi(x), \qquad x \in \mathsf{M}^+, \omega \in \mathsf{M}_*^+. \end{aligned} \tag{3.1}$$

Remark 3.2. The weight φ in Definition 3.1 is also called the *left Haar weight* whereas the weight ψ is called the *right Haar weight*. It is common to write $\mathbb{G} = (\mathsf{M}, \Delta)$ instead of the 4-tuple $(\mathsf{M}, \Delta, \varphi, \psi)$ and hence suppress the Haar weights in the notation. In fact if the Haar weights exist then they must be unique up to multiplication with a positive scalar. The conditions (3.1) are called *left invariance* (of φ) and *right invariance* (of ψ). Of course these conditions are in the classical case (see Example 3.3) equivalent to the left and right invariance of the Haar weights on a locally compact group.

Example 3.3. Let G be a locally compact group. Set $\mathsf{M} = L^\infty(\mathsf{G})$ and let $\Delta_{\mathsf{G}} : \mathsf{M} \to \mathsf{M} \otimes \mathsf{M}$ be the pull-back of the multiplication. So,

$$\Delta_{\mathsf{G}}(f)(s,t) = f(st), \qquad s, t \in \mathsf{G},$$

which is well-defined as $\mathsf{M} \otimes \mathsf{M} \simeq L^\infty(\mathsf{G} \times \mathsf{G})$. Let $\varphi(f) = \int_{\mathsf{G}} f(s) ds, f \in \mathsf{M}^+$ be integration against the left Haar measure and let $\psi(f) = \int_{\mathsf{G}} f(s^{-1}) ds, f \in \mathsf{M}^+$ be integration against the right Haar measure. Then the 4-tuple $\mathbb{G} = (\mathsf{M}, \Delta_{\mathsf{G}}, \varphi, \psi)$ forms a locally compact quantum group.

Exercise 3.4. Verify that Example 3.3 determines a locally compact quantum group.

Example 3.5. Let G be a locally compact group and let

$$\widehat{\mathsf{M}} := \left\{ \lambda(f) \mid f \in L^1(\mathsf{G}) \right\}'',$$

be its group von Neumann algebra. Here $\lambda(f) = \int_{\mathsf{G}} f(s)\lambda_s ds$ is the left regular representation, where $\lambda_s, s \in \mathsf{G}$ is acting on $L^2(\mathsf{G})$ by means of left translation: $(\lambda_s\xi)(t) = \xi(s^{-1}t)$ for $\xi \in L^2(\mathsf{G})$. We have (that is, one can prove that) $\lambda_s \in \widehat{\mathsf{M}}, s \in \mathsf{G}$ and there exists a unique normal, unital $*$-homomorphism that is determined by,

$$\widehat{\Delta} : \widehat{\mathsf{M}} \to \widehat{\mathsf{M}} \otimes \widehat{\mathsf{M}} : \lambda_s \mapsto \lambda_s \otimes \lambda_s.$$

For $x \in \widehat{\mathsf{M}}$ (so in particular it acts on $L^2(\mathsf{G})$) we set $\widehat{\varphi}(x^*x) = \|f\|^2_{L^2(\mathsf{G})}$ in case there exists $f \in L^2(\mathsf{G})$ such that $xg = f * g, g \in L^2(\mathsf{G})$. We set $\widehat{\varphi}(x^*x) = \infty$ otherwise. Similarly, for $f \in \mathsf{G} \to \mathbb{C}$ we set $f^\vee(s) = f(s^{-1})\Delta_{\mathsf{G}}^{-1}(s)$. For $x \in \widehat{\mathsf{M}}$ we set $\widehat{\psi}(x^*x) = \|f^\vee\|^2_{L^2(\mathsf{G})}$ in case there exists $f \in L^2(\mathsf{G})$ such that $xg = f * g, g \in L^2(\mathsf{G})$. We set $\widehat{\psi}(x^*x) = \infty$ otherwise. The tuple $(\widehat{\mathsf{M}}, \widehat{\Delta}, \widehat{\varphi}, \widehat{\psi})$ forms a locally compact quantum group.

Exercise 3.6. Verify that Example 3.5 determines a locally compact quantum group. You may assume that $\widehat{\Delta}$ indeed extends as a normal $*$-homomorphism.

Remark 3.7. Let G be a locally compact abelian group. Let $\widehat{\mathsf{G}}_0$ be the set of all strongly continuous irreducible unitary representations $\pi : \mathsf{G} \to B(\mathcal{H}_\pi)$ on a Hilbert space $\mathcal{H}_\pi$. Two representations $\pi_1, \pi_2 \in \widehat{\mathsf{G}}_0$ are called unitarily equivalent if there exists a unitary map $U : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ intertwining the representations π_1 and π_2. Let $\widehat{\mathsf{G}}$ be the set $\widehat{\mathsf{G}}_0$ modulo unitary equivalence. Since irreducible representations of an abelian group are 1-dimensional $\widehat{\mathsf{G}}$ carries a multiplication, which is just pointwise multiplication of functions on G. In fact $\widehat{\mathsf{G}}$ is a group called the *Pontrjagin dual group.*

We now relate the Examples 3.3 and 3.5 for abelian groups: we claim that Example 3.5 constructed from G is isometrically isomorphic with the locally compact quantum group associated with $\widehat{\mathsf{G}}$ as constructed in Example 3.3. So let $\mathbb{G} = (\mathsf{M}, \Delta_{\mathsf{G}}, \varphi, \psi)$ and $\widehat{\mathbb{G}} = (\widehat{\mathsf{M}}, \widehat{\Delta}, \widehat{\varphi}, \widehat{\psi})$ be locally compact quantum groups constructed in Examples 3.3 and 3.5 starting from G. Consider the Fourier transform:

$$\mathcal{F}_2 : L^2(\mathsf{G}) \to L^2(\widehat{\mathsf{G}}) : f \mapsto \widehat{f},$$

where $\widehat{f}(\pi) = \int_{\mathsf{G}} f(s)\pi(s)ds$ in case $f \in L^1(\mathsf{G}) \cap L^2(\mathsf{G})$. $\mathcal{F}_2$ is a unitary transformation. Furthermore, we get the following spatial identification,

$$\mathcal{F}_2\widehat{\mathsf{M}}\mathcal{F}_2^{-1} = L^\infty(\widehat{\mathsf{G}}). \tag{3.2}$$

In fact under the mapping $x \mapsto \mathcal{F}_2 x \mathcal{F}_2^{-1}$ it turns out that the comultiplication $\widehat{\Delta}$ and Haar weights $\widehat{\varphi}$ and $\widehat{\psi}$ on $\widehat{\mathsf{M}}$ are transfered into the comultiplication $\Delta_{\widehat{\mathsf{G}}}$ and left and right Haar measures on $L^\infty(\widehat{\mathsf{G}})$.

The conclusion is that Example 3.5 should be considered as the Pontrjagin dual quantum group of Example 3.3. We shall come back to Pontjagin duality for arbitrary locally compact quantum groups in Section 4. The discussion above shows that in the abelian case the identification of the Pontrjagin duals is established by the Fourier transform. An important consequence is that Fourier theory for quantum groups partially trivializes, being part of its construction, see [2, Section 5].

Exercise 3.8. Verify that $\mathbb{Z}$ and $\mathbb{T}$ (the torus) are the Pontrjagin dual group to eachother.

Exercise 3.9. Verify (3.2).

Example 3.10. A locally compact quantum group $\mathbb{G} = (\mathsf{M}, \Delta, \varphi, \psi)$ is called compact in case φ and ψ are states and it is called unimodular in case $\varphi = \psi$. Compact quantum groups are always unimodular. To every compact quantum group in the sense of Definition 3.1 there is a canonical way of constructing a compact quantum group in the sense of Woronowicz (see Definition 1.1). Essentially this is done by finding a suitable C*-subalgebra A of M (being dense in M in the strong operator topology) to which Δ restricts as a comultiplication satisfying the cancellation law. Indeed, one can construct A as follows. We have to jump a bit ahead to Section 4 where the left multiplicative W was constructed. Then let A be the norm-closed linear span of all matrix coefficients,

$$\left\{ (\iota \otimes \omega)(W) \mid \omega \in B(L^2(\mathbb{G}))_* \right\}.$$

One can indeed show that A is a unital C*-algebra (the argument involves manipulations with W as in Remark 4.5) which is contained in M. Moreover Δ restricts to a map $\mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ witnessing that (A, Δ) is a compact quantum group in the sense of Woronowicz.

3.2 Corepresentation theory

Representation theory for groups translate into corepresentation theory for locally compact quantum groups. A unitary *corepresentation* of a locally compact quantum group $\mathbb{G}$ is an operator

$$U \in \mathsf{M} \otimes B(\mathcal{H}_U),$$

satisfying the identity

$$(\Delta \otimes \iota)(U) = U_{13} U_{23}. \tag{3.3}$$

Example 3.11. If $\pi : \mathsf{G} \to B(\mathcal{H}_\pi)$ is a strongly continuous representation of a locally compact group G on a Hilbert space $\mathcal{H}_\pi$, then we may regard π as an element of $L^\infty(\mathsf{G}, B(\mathcal{H}_\pi)) \simeq L^\infty(\mathsf{G}) \otimes B(\mathcal{H}_\pi)$. Let U_π be the element in the latter von Neumann algebra corresponding to π. Then U_π is a corepresentation. Indeed, to see this regard U_π again as function on G with values in $B(\mathcal{H}_\pi)$, so the function $s \mapsto \pi(s)$. Then, $(\Delta \otimes \iota)(U_\pi)$ is a two variable function on G with values in $B(\mathcal{H}_\pi)$, namely $(s,t) \mapsto \pi(st)$, by definition of the comultiplication. Then,

$$(\Delta \otimes \iota)(U_\pi)(s,t) = U_\pi(st) = U_\pi(s) U_\pi(t),$$

which corresponds to the identity (3.3).

If $U_1 \in \mathsf{M} \otimes B(\mathcal{H}_1)$ and $U_2 \in \mathsf{M} \otimes B(\mathcal{H}_2)$ are corepresentations then we may form the direct sum corepresentation $U_1 \oplus U_2$ acting on $\mathcal{H}_1 \oplus \mathcal{H}_2$ by taking the direct sum of the operators. A corepresentation is called irreducible if it cannot be written as the direct sum of two irreducible operators. A map $T : \mathcal{H}_1 \to \mathcal{H}_2$ is called an intertwiner of U_1 and U_2 if $(1 \otimes T)U_1 = (1 \otimes T)U_2$. As for groups, locally compact quantum groups admit a Schur lemma: that is U_1 is irreducible if and only if the space of intertwiners is 1-dimensional.

3.3 The antipode and its polar decomposition

The construction of the anti-pode for locally compact quantum groups is a delicate task and is done hand-in-hand with the construction of the multiplicative unitary to which we come back in Section 4. The following theorem yields the existence of the antipode and its polar decomposition as a map on the von Neumann algebra. For the details of its construction we refer to [22] and [35].

Theorem 3.12. *We have the following:*

1. *There exists a unique σ-strongly closed operator $S : Dom(S) \to \mathsf{M}$ on a σ-strongly dense domain $Dom(S) \subseteq \mathsf{M}$ that has $\{(\iota \otimes \omega)(W) \mid \omega \in B(L^2(\mathbb{G}))_*\}$ as a σ-strong core on which it is determined by:*

$$S(\iota \otimes \omega)(W) = (\iota \otimes \omega)(W^*).$$

2. *There exists uniquely a $*$-anti-homomorphism $R : \mathsf{M} \to \mathsf{M}$ and a strongly continuous 1-parameter group of automorphisms $(\tau_t)_{t\in\mathbb{R}}$ of M such that $S = R \circ \tau_{-i/2}$.*

3.4 Example: von Neumann algebraic quantum $SU_q(2)$

Presuming that the example of $SU_q(2)$ is treated in one or more of the other mini-courses, we give a quick account of its von Neumann algebraic version. Let $-1 < q < 1$. Recall that algebraically $SU_q(2)$ is defined as the $*$-algebra generated by operators α and γ that satisfy the following relations:

$$\alpha^*\alpha + \gamma^*\gamma = 1, \;\; \alpha\alpha^* + q^2\gamma\gamma^* = 1, \;\; \gamma\gamma^* = \gamma^*\gamma, \;\; q\gamma\alpha = \alpha\gamma, \;\; q\gamma^*\alpha = \alpha\gamma^*.$$

We shall define an explicit representation for these operators. Set the Hilbert space $\mathcal{H} = L^2(\mathbb{N}) \otimes L^2(\mathbb{Z})$ with canonical basis $e_n \otimes f_k, n \in \mathbb{N}, k \in \mathbb{Z}$. And define operators,

$$\alpha e_n \otimes f_k = \sqrt{1 - q^{2n}} e_{n-1} \otimes f_k, \qquad \gamma e_n \otimes f_k = q^n e_n \otimes f_{k+1}.$$

Let M be the von Neumann algebra generated by α and γ. A (not too hard) exercise shows that it equals $\mathsf{M} \simeq B(L^2(\mathbb{N})) \otimes L^\infty(\mathbb{T})$. The comultiplication is determined by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma,$$

and extends uniquely to a normal, unital $*$-homomorphism $\Delta : \mathsf{M} \to \mathsf{M} \otimes \mathsf{M}$. The Haar weights $\varphi = \psi$ are given by the following formula. Take $x = x(t) \in L^\infty(\mathbb{T}, B(L^2(\mathbb{N}))$, then:

$$\varphi(x) = \frac{1-q^2}{2\pi} \int_{\mathbb{T}} \sum_{i=0}^{\infty} q^{2i} \langle x(t)e_i, e_i \rangle dt.$$

4 Pontrjagin duality

In Remark 3.7 we already explained how duality is involved in the examples we have seen so far. In this section we construct the Pontrjagin dual of an arbitrary quantum group. Pontrjagin duality is one of the main motivations of quantum group theory. The construction of the dual quantum group proceeds via an important operator: the multiplicative unitary.

4.1 The left multiplicative unitary

As observed in Section 3.2 representation theory of groups translates in corepresentation theory for quantum groups. In particular to any unitary representation π of a group G on a Hilbert space $\mathcal{H}_\pi$ one may associated a corepresentation, which is a unitary operator $U_\pi \in L^\infty(\mathsf{G}) \otimes B(\mathcal{H}_\pi)$. See Section 3.2 for details. In case π is the left-regular representation the associated corepresentation U_π is normally denoted by W and for arbitrary locally compact quantum groups is called the *(left) multiplicative unitary.* We shall give its construction for arbitrary compact quantum groups below. It plays the most crucial role in the theory: the single operator W turns out to contain *all* the data of $\mathbb{G}$ (comultiplication, von Neumann algebra, Haar weights and more) as well as of the dual quantum group $\widehat{\mathbb{G}}$ that we shall also construct in this section.

Theorem-Definition 4.1. Let $\mathbb{G}$ be a locally compact quantum group. There exists a unique unitary operator $W \in B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ that is determined by:

$$W^* : L^2(\mathbb{G}) \otimes L^2(\mathbb{G}) \to L^2(\mathbb{G}) \otimes L^2(\mathbb{G}) : \Lambda(x) \otimes \Lambda(y) \mapsto (\Lambda \otimes \Lambda)\, \Delta(y)(x \otimes 1).$$

Remark 4.1. The proof showing that W in Theorem-Definition 4.1 is isometric relies on the left invariance of the Haar weight φ. The more difficult part is to show that W is actually surjective. The proof of this fact relies on the existence of *both* the left and right Haar weights of $\mathbb{G}$ even though W needs only the left Haar weight to be defined. We refer to [35] for part of the proof and [22] for the complete proof of Theorem-Definition 4.1

Remark 4.2. Let G be a locally compact group. Consider the left regular representation $\lambda : \mathsf{G} \to B(L^2(\mathsf{G}))$. We may regard λ as an element of $L^\infty(\mathsf{G}, B(L^2(\mathsf{G}))) \simeq L^\infty(\mathsf{G}) \otimes B(L^2(\mathsf{G}))$. The corresponding element in the latter von Neumann algebra is exactly W of Theorem–Definition 4.1.

An important feature of W is that it satisfies the *pentagon equation*:

$$W_{12}W_{13}W_{23} = W_{23}W_{12}, \tag{4.1}$$

which is an equation of operators in $B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$. Here we have used again the leg-numbering notation $W_{12} = W \otimes 1, W_{23} = 1 \otimes W$ et cetera. We shall now explain how W encodes the structure of $\mathbb{G} := (\mathsf{M}, \Delta, \varphi, \psi)$. Firstly, we have

$$\mathsf{M} = \{(\mathrm{id} \otimes \omega)(W) \mid \omega \in B(L^2(\mathbb{G}))_*\}''.$$

Secondly the comultiplication may be written as

$$\Delta(x) = W^*(1 \otimes x)W.$$

Also the Haar weights can be recovered from W. However, the construction is less direct and for the sake of exposition it is better to construct the dual Haar weights instead.

Remark 4.3. The name left multiplicative unitary suggests that there exists also a right multiplicative unitary. Indeed such a unitary can be constructed similarly but we will not do it here.

4.2 The dual quantum group: the von Neumann algebra and comultiplication

Let $\mathbb{G}$ be a locally compact quantum group and let W be its multiplicative unitary. We shall construct a Pontrjagin dual quantum group $\widehat{\mathbb{G}}$ in the following way. Firstly, set

$$\widehat{\mathsf{M}} := \{(\omega \otimes \iota)(W) \mid \omega \in \mathsf{M}_*\}''. \tag{4.2}$$

Remark 4.4. It is not directly clear that Equation (4.2) indeed determines a von Neumann algebra: one needs to check that $\{(\omega \otimes \iota)(W) \mid \omega \in \mathsf{M}_*\}$ is closed under taking adjoints. This follows from the fact that the set defined in (4.4) is dense in M_*.

Next we define a comultiplication $\widehat{\Delta} : \widehat{\mathsf{M}} \to \widehat{\mathsf{M}} \otimes \widehat{\mathsf{M}}$ by setting,

$$\widehat{\Delta}(x) = W(x \otimes 1)W^*.$$

Remark 4.5. Note that clearly $\widehat{\Delta}$ is a normal $*$-homomorphism. Its image lands in $\widehat{\mathsf{M}} \otimes \widehat{\mathsf{M}}$ (we omit the proof). We can prove coassociativity from the Pentagon equation (and we should at least incorporate one such proof in these notes to see why the Pentagon equation is useful). Indeed, let $x = (\omega \otimes \iota)(W)$ be an element of $\widehat{\mathsf{M}}$. Then,

$$\begin{aligned}
&(\widehat{\Delta} \otimes \iota)\widehat{\Delta}(x) = (\widehat{\Delta} \otimes \mathrm{id})(\Sigma W(x \otimes 1)W^*\Sigma) \\
=&\Sigma_{12}W_{12}\Sigma_{13}W_{13}(x \otimes 1 \otimes 1)W_{13}^*\Sigma_{13}W_{12}^*\Sigma_{12} \\
=&(\omega \otimes \iota \otimes \iota \otimes \iota)(\Sigma_{12}W_{12}\Sigma_{13}W_{13}W_{01}W_{13}^*\Sigma_{13}W_{12}^*\Sigma_{12})
\end{aligned}$$

Using the pentagon equation twice and a couple of elementary commutations we get the following (using the 0 in the leg numbering to denote the 1st tensor, the 1 for the 2nd et cetera),

$$\begin{aligned}
&\Sigma_{12}W_{12}\Sigma_{13}W_{13}W_{01}W_{13}^*\Sigma_{13}W_{12}^*\Sigma_{12} \\
=&\Sigma_{12}W_{12}\Sigma_{13}W_{01}W_{03}\Sigma_{13}W_{12}^*\Sigma_{12} \\
=&\Sigma_{12}W_{12}W_{03}W_{01}W_{12}^*\Sigma_{12} \\
=&\Sigma_{12}W_{03}W_{12}W_{01}W_{12}^*\Sigma_{12} \\
=&\Sigma_{12}W_{03}W_{01}W_{02}\Sigma_{12} \\
=&W_{03}W_{02}W_{01}.
\end{aligned}$$

The previous equalities yield that

$$(\widehat{\Delta} \otimes \iota)\widehat{\Delta}(x) = (\omega \otimes \iota \otimes \iota \otimes \iota)(W_{03}W_{02}W_{01}).$$

As similar argument yields that also,

$$(\iota \otimes \widehat{\Delta})\widehat{\Delta}(x) = (\omega \otimes \iota \otimes \iota \otimes \iota)(W_{03}W_{02}W_{01}),$$

which concludes the claim.

4.3 Dual Haar weights

Here we show how the dual Haar weight $\widehat{\varphi}$ is constructed. Its existence follows by constructing a left Hilbert algebra that by general theory [28] gives a weight on $\widehat{\mathsf{M}}$. We shall explicitly construct the left Hilbert algebra and refer to [28] for the abstract construction. We end the section by giving a characterization of this weight.

Firstly we define the convolution product on M_* as follows,

$$\omega * \theta = (\omega \otimes \theta) \circ \Delta, \qquad \omega, \theta \in \mathsf{M}_*.$$

In the classical case, see Example 3.3, we have $\mathsf{M}_* = L^1(\mathsf{G})$ and $*$ corresponds to the convolution product $(f * g)(s) = \int_{\mathsf{G}} f(t)g(t^{-1}s)dt$. This turns M_* into a Banach algebra. We define the set,

$$\mathcal{I} := \{\omega \in \mathcal{M}_* \mid \text{There exists a bounded map } L^2(\mathbb{G}) \to \mathbb{C} : \Lambda(x) \mapsto \omega(x^*)\}.$$

Recall from the Riesz representation theorem that every bounded functional on a Hilbert space $\mathcal{H}$ is given by an inner product map $\eta \mapsto \langle \eta, \xi \rangle$ for some $\xi \in \mathcal{H}$. Therefore, for every $\omega \in \mathcal{I}$ there exists a vector $\xi(\omega)$ such that

$$\langle \xi(\omega), \Lambda(x) \rangle = \omega(x^*).$$

It is also useful to introduce the notation

$$\lambda(\omega) = (\omega \otimes \iota)(W),$$

and to note that $\lambda : \mathsf{M}_* \to \widehat{\mathsf{M}}$ into an injective map. We now have the following results. Lemma 4.7 in particular implies that $\mathcal{I}$ is an algebra.

Lemma 4.6. *$\mathcal{I}$ is dense in $\mathcal{M}_*$.*

Proof. Let $a, b \in \mathcal{T}_\varphi$. Then,

$$\varphi(ax^*b) = \varphi(x^*b\sigma^\varphi_{-i}(a)) = \langle \Lambda(b\sigma^\varphi_{-i}(a)), \Lambda(x) \rangle, \tag{4.3}$$

and hence $\varphi(a \cdot b)$ is in $\mathcal{I}$. Since $\mathcal{I}$ is clearly linear, hence convex it now suffices to show that if $\varphi(axb) = 0$ for all $a, b \in \mathcal{T}_\varphi$ then $x = 0$. But this follows from (4.3) and the fact that $\Lambda(\mathcal{T}_\varphi^2)$ is dense in $L^2(\mathbb{G})$, see Exercise 2.7. $\square$

Lemma 4.7. *$\mathcal{I}$ is a left ideal in $\mathcal{M}_*$.*

Proof. Let $\theta \in \mathcal{I}$ and let $\omega \in \mathcal{M}_*$. Then,

$$\begin{aligned}(\omega * \theta)(x^*) = (\omega * \theta)\Delta(x^*) = \theta((\omega \otimes \iota)\Delta(x^*)) = \theta((\overline{\omega} \otimes \iota)(\Delta(x))^*) \\ = \langle \xi(\theta), \Lambda((\overline{\omega} \otimes \iota)\Delta(x))\rangle = \langle \xi(\theta), (\overline{\omega} \otimes \iota)(W^*)\Lambda(x)\rangle = \langle (\omega \otimes \iota)(W)\xi(\theta), \Lambda(x)\rangle.\end{aligned}$$

This implies that $\omega * \theta \in \mathcal{I}$ and $\xi(\omega * \theta) = \lambda(\omega)\xi(\theta)$. $\square$

Next we consider the set,

$$\mathsf{M}_*^\sharp = \{\omega \in \mathsf{M}_* \mid \text{There exists } \theta \in \mathsf{M}_* \text{ with } (\theta \otimes \iota)(W) = (\omega \otimes \iota)(W)^*\}. \tag{4.4}$$

Note that given $\omega \in \mathsf{M}_*$, if a θ as in (4.4) exists, then it is necessarily unique. From this point we shall denote it by ω^*. So,

$$(\omega^* \otimes \iota)(W) = (\omega \otimes \iota)(W), \qquad \omega \in \mathsf{M}_*^\sharp. \tag{4.5}$$

In fact in many applications the following characterization is useful. A proof can be found in [24].

Lemma 4.8. *Let $\omega \in \mathsf{M}_*$. The following are equivalent:*

1. *$\omega \in \mathsf{M}_*^\sharp$, i.e. there exists a (unique) functional ω^* satisfying* (2)*;*
2. *The mapping $Dom(S) \to \mathbb{C} : x \mapsto \overline{\omega(S(x)^*)}$ extends boundedly to a functional $\omega^* : \mathsf{M} \to \mathbb{C}$.*

Moreover the notation is consistent in the sense that ω^* of (1) *and* (2) *agree.*

Now let $\mathcal{J} = \mathcal{I} \cap \mathsf{M}_*^\sharp \cap (\mathsf{M}_*^\sharp)^*$ and set $\mathcal{U} = \Lambda(\mathcal{J})$. $\mathcal{U}$ is a subspace of $L^2(\mathbb{G})$ and hence carries an inner product. We can define an involution $\sharp$ and algebra structure $\cdot$ on $\mathcal{U}$ by setting:

$$\begin{aligned}\xi(\omega) \cdot \xi(\theta) :=& \xi(\omega * \theta), \qquad \omega, \eta \in \mathcal{J} \\ \xi(\omega)^\sharp :=& \xi(\omega^*), \qquad \omega \in \mathcal{J}\end{aligned}$$

For the notion of left Hilbert algbras we refer to [28]. The proof of the next lemma is not terribly hard. The claim that N equals $L^\infty(\widehat{\mathbb{G}})$ is a consequence of Lemma 4.7. In order to check that $\mathcal{U}$ is a left Hilbert algebra, the main issue is to show that $\mathcal{U}^2$ is dense in $\mathcal{U}$ (which relies on a couple of standard approximation technqiues).

Lemma 4.9. *$\mathcal{U}$ has the structure of a left Hilbert algebra. Moreover, let N be the von Neumann algebra generated by $\mathcal{U}$, i.e. the strong closure of operators,*

$$\xi(\omega) \mapsto \xi(\theta * \omega), \qquad \theta \in \mathcal{J}. \tag{4.6}$$

Then N is isomorphic to $L^\infty(\widehat{\mathbb{G}})$.

By Hilbert algebra theory [28] the previous lemma implies that there exists a normal, semi-finite, faithful weight $\hat{\varphi}$ on $L^\infty(\widehat{\mathbb{G}})$ with the property

$$\langle \xi(\omega), \xi(\omega)\rangle = \hat{\varphi}(\lambda(\omega)^* \lambda(\omega)). \tag{4.7}$$

Summarizing we state the following theorem explicitly defining $\widehat{\varphi}$ in a unique way, which usually suffices for all practical purposes.

Theorem 4.10. *There exists a unique normal, semi-finite, faithful weight $\hat{\varphi}$ on $L^\infty(\widehat{\mathbb{G}})$ such that $\lambda(\mathcal{I}) \in \mathfrak{n}_{\hat{\varphi}}$, such that* (4.7) *holds and finally, such that $\lambda(\mathcal{I})$ is a σ-weak/norm core for the GNS-construction of $\hat{\varphi}$.*

The dual right Haar weight $\hat{\varphi}$ on $L^\infty(\hat{\mathbb{G}})$ is constructed through the dual unitary anti-pode. It is determined by a similar theorem.

Theorem 4.11. *Let $\mathcal{I}_\psi$ be the set $\{\overline{\omega \circ R} \mid \omega \in \mathcal{I}\}$. There exists a unique normal, semi-finite, faithful weight $\hat{\psi}$ on $L^\infty(\widehat{\mathbb{G}})$ such that $\lambda(\mathcal{I}_\psi) \in \mathfrak{n}_{\hat{\psi}}$, such that*

$$\hat{\psi}(\lambda(\omega)^*\lambda(\omega)) = \langle \xi(\overline{\omega \circ R}), \xi(\overline{\omega \circ R})\rangle.$$

holds and finally, such that $\lambda(\mathcal{I}_\psi)$ is a σ-weak/norm core for the GNS-construction of $\hat{\psi}$.

We conclude this section by stating two main theorems.

Theorem 4.12. *The 4-tuple $\widehat{\mathbb{G}} := (\widehat{\mathsf{M}}, \widehat{\Delta}, \widehat{\varphi}, \widehat{\psi})$ is a locally compact quantum group.*

Theorem 4.13. *We have $\widehat{\widehat{\mathbb{G}}} = \mathbb{G}$.*

It deserves to be emphasized that in Theorem 4.13 we really mean equality and not just an isomorphism. This is a consquence of the fact that we construct the dual of a quantum group on the same (GNS-)Hilbert space and the fact that Fourier theory is implicitly included in the construction of quantum groups, see Remark 3.7.

4.4 Relations between the quantum group and its dual

So far we have defined the dual quantum group in the current section. From this point one can in principle proceed and construct the anti-pode, the unitary anti-pode et cetera. There are various relations between these objects and their duals and we use this section to summarize them.

Relations involving the anti-pode. The following relations hold, where $t \in \mathbb{R}$ and Σ is the flip map,

$$\begin{aligned}
(R \otimes \hat{R})(W) =& W^*,\\
(\tau_t \otimes \hat{\tau}_t)(W) =& W,\\
\Delta \circ R =& \Sigma(R \otimes R) \circ \Delta,\\
(\tau_t \circ \tau_t) \circ \Delta =& \Delta \circ \tau_t,\\
R(x) =& \hat{J} x^* \hat{J},\\
\tau_t(x) =& \hat{\nabla}^{it} x \hat{\nabla}^{-it}.
\end{aligned}$$

Relations involving modular operators and weights. Firstly, the left and right Haar weight of a quantum group are related. That is, there exists a positive self-adjoint operator δ that is affiliated with $L^\infty(\mathbb{G})$ (meaning that all finite spectral projections of δ are contained in $L^\infty(\mathbb{G})$) such that formally:

$$\varphi(x) = \psi(\delta^{\frac{1}{2}} x \delta^{\frac{1}{2}}), \qquad x \in L^\infty(\mathbb{G}). \tag{4.8}$$

For a classical locally compact group G we have either $\delta = \Delta_{\mathsf{G}}$ or $\delta = \Delta_{\mathsf{G}}^{-1}$ depending on how the modular function Δ_{G} is defined (most texts use the definition with the inverse). The formal meaning of (4.8) can be found in [31]. Also from [31] it turns out that there is a *scaling constant* $\nu > 0$ such that,

$$\varphi \circ \tau_t = \nu^{-t} \varphi, \qquad t \in \mathbb{R}.$$

We now have the following relations, see also [23],

$$\begin{aligned}
&\hat{\nabla}^{it} \nabla^{is} = \nu^{ist} \nabla^{is} \hat{\nabla}^{it}, \\
&\hat{\nabla}^{it}_{\hat{\psi}} \nabla^{is}_{\psi} = \nu^{ist} \nabla^{is}_{\hat{\psi}} \hat{\nabla}^{it}_{\hat{\psi}}, \\
&\hat{J} \nabla \hat{J} = \nabla_{\psi}, J \nabla J = \nabla^{-1}, \nabla^{is} \nabla^{it}_{\psi} = \nabla^{it}_{\psi} \nabla^{is}, \\
&\nabla^{is} \delta^{it} = \nu^{ist} \delta^{it} \nabla^{is}, \nabla^{is}_{\psi} \delta^{it} = \nu^{ist} \delta^{it} \nabla^{is}_{\psi}, \\
&\hat{\nabla}^{is} \delta^{it} = \delta^{it} \hat{\nabla}^{is}, \hat{\nabla}^{is}_{\psi} \delta^{it} = \delta^{it} \hat{\nabla}^{is}_{\psi}.
\end{aligned}$$

5 Examples

There are plenty of examples of *compact* quantum groups. The reason is twofold:

- Firstly, there are a couple of tools that allow us to construct compact quantum groups. For example Tannaka-Krein duality provides a very strong tool (see the lectures on easy quantum groups/partition quantum groups).

- The second reason – being most important for this section – is that compact quantum groups can normally be understood on an algebraic level from which it is possible to construct C*-algebraic quantum groups (see [21]). In the non-compact case these technicalities are much more involved: the generators of an algebraic quantum group are normally represented as unbounded operators. But not only the technicalities are more involved: sometimes it is *impossible* to construct a locally compact quantum group from purely algebraic relations. We shall come back to this in Theorem 5.1.

Examples of non-compact quantum groups are very sparse and constructing them is an important problem in the theory of non-compact quantum groups. The most important available examples at this moment are the following ones:

- The quantum $E(2)$ group, also $E_q(2)$. This example is basically due to Woronowicz [37] and its von Neumann algebraic version is completely worked out in Jacobs his thesis [17].

- A quantum analogue of $SU(1,1)$, originally denoted $\widetilde{SU_q(1,1)}$ as a quantum analogue of the normalizer of $SU(1,1)$ in $SL(2,\mathbb{C})$ (the interpretation as normalizer is only partially correct and therefore the quantum group sometimes appears under the name extended quantum $SU_q(1,1)$ [10], [3]). This example has been defined in the operator algebra context by Koelink and Kustermans [19]. Another way of defining this quantum group was given by De Commer [10].

- In [8] De Commer gives an (abstract) method of constructing quantum groups through Galois co-objects. The method is sometimes called "generalized twisitng" and is concretized in special cases, see [10] and [11]. The method yields new abstract quantum groups that sometimes can be identified with/interpreted as other quantum groups. For examples this method gives a way of defining $SU_q(1,1)$. In [12] series of quantum groups have been found as well.

- There are a couple of abstract constructions that yield new quantum groups. The most important ones are twisting (see above) and the bicrossed product (see [32]).

In this section we shall explain the construction of the operator algebraic version of quantum $SU(1,1)$. We follow the approach of Koelink and Kustermans [19]. I am indepted to Erik Koelink for providing me these notes.

Warning. In this section $\mathbb{N} = \{1,2,3,\cdots\}$, and $\mathbb{N}_0 = \{0,1,2,3\cdots\}$.

5.1 $SU_q(1,1)$ on the Hopf $*$-algebra level

We first study the q-deformed version of $SU(1,1)$ on the algebraic level and elaborate on why it is difficult to construct proper operator algebras associated to it. We first recall that,

$$SU(1,1) = \left\{ g \in SL(2,\mathbb{C}) \mid g^* J g = J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a & c \\ \bar{c} & \bar{a} \end{pmatrix} \mid a, c \in \mathbb{C},\ |a|^2 - |c|^2 = 1 \right\}.$$

We let $0 < q < 1$ be fixed. The associated Hopf $*$-algebra is $\tilde{A}$, the unital $*$-algebra generated by elements α_0, γ_0 and relations

$$\alpha_0^\dagger \alpha_0 - \gamma_0^\dagger \gamma_0 = 1, \qquad \alpha_0 \alpha_0^\dagger - q^2\, \gamma_0^\dagger \gamma_0 = 1,$$
$$\gamma_0^\dagger \gamma_0 = \gamma_0\, \gamma_0^\dagger, \qquad \alpha_0\, \gamma_0 = q\, \gamma_0\, \alpha_0, \qquad \alpha_0\, \gamma_0^\dagger = q\, \gamma_0^\dagger \alpha_0$$

where † denotes the $*$-operation on $\tilde{A}$ (in order to distinguish this kind of adjoint with the adjoints of possibly unbounded operators in Hilbert spaces). Then the comultiplication $\Delta_0 \colon \tilde{A} \to \tilde{A} \odot \tilde{A}$ is given by

$$\Delta_0(\alpha_0) = \alpha_0 \otimes \alpha_0 + q\, \gamma_0^\dagger \otimes \gamma_0 \qquad \Delta_0(\gamma_0) = \gamma_0 \otimes \alpha_0 + \alpha_0^\dagger \otimes \gamma_0$$

and counit ε_0 and antipode S_0 are determined by

$$S_0(\alpha_0) = \alpha_0^\dagger, \qquad S_0(\alpha_0^\dagger) = \alpha_0, \qquad S_0(\gamma_0) = -q\, \gamma_0, \qquad S_0(\gamma_0^\dagger) = -\frac{1}{q}\, \gamma_0^\dagger,$$
$$\varepsilon_0(\alpha_0) = 1, \qquad \varepsilon_0(\gamma_0) = 0.$$

In order to perform the harmonic analysis we need to represent these relations, and it turns out that one needs unbounded operators on a Hilbert space. However, it turned out that it is impossible to realise the comultiplication on the level of operators.

Theorem 5.1 (Woronowicz (1991))**.** *For (α^1, γ^1), resp. (α^2, γ^2), closed operators on an infinite dimensional Hilbert space H^1, resp. H^2, representing the relations above, there exist no closed operators α, γ acting on $H^1 \otimes H^2$ representing the relations and extending $\alpha^1 \otimes \alpha^2 + q\,(\gamma^1)^* \otimes \gamma^2$, $\gamma^1 \otimes \alpha^2 + (\alpha^1)^* \otimes \gamma^2$, such that α^*, γ^* extend $(\alpha^1)^* \otimes (\alpha^2)^* + q\,\gamma^1 \otimes (\gamma^2)^*$, $(\gamma^1)^* \otimes (\alpha^2)^* + \alpha^1 \otimes (\gamma^2)^*$.*

Remark 5.2. The obstruction of Woronowicz' result lies in the extension of symmetric operators to self-adjoint operators. Recall that an unbounded operator x on a Hilbert space H with dense domain $\mathrm{Dom}(x)$ is called symmetric if for every $v, w \in \mathrm{Dom}(x)$ we have

$$\langle xv, w \rangle = \langle v, xw \rangle. \tag{5.1}$$

We have $\mathrm{Dom}(x^*) = \{w \in H \mid \mathrm{Dom}(x) \ni v \mapsto \langle xv, w\rangle \text{ is bounded}\}$ and for $w \in \mathrm{Dom}(x^*)$ the vector $x^* w$ is (by the Riesz representation theorem) the unique vector ξ such that $\langle v, \xi \rangle = \langle xv, w \rangle$. We have $\mathrm{Dom}(x) \subseteq \mathrm{Dom}(x^*)$ and if these domains are equal then x

is called self-adjoint, notation: $x = x^*$. A closed symmetric operator need not be self-adjoint, but it may have a self-adjoint extension (i.e. its domain can be extended in such a way that the extension becomes self-adjoint). Von Neumann precisely determined when such a self-adjoint extension exists (and how much choice there is). Namely consider the deficiency spaces

$$H_+ = \ker(x^* - i), \qquad H_- = \ker(x^* + i),$$

and let $n_+ = \dim(H_+), n_- = \dim(H_-)$ be the deficiency indices. Then x has a self-adjoint extension if and only if $n_+ = n_-$. The problem for $SU_q(1,1)$ lies in finding a self-adjoint extension of $\Delta(\gamma_0^\dagger\gamma_0)$, of which Woronowicz showed that it does not exist. However, there is a remedy. If x is an unbounded operator on H with deficiency indices n_+^x and n_-^x and y is an unbounded operator on K with deficiency indices n_+^y and n_-^y. Then $x \oplus y$ has deficiency indices $n_+^x + n_+^y$ and $n_-^x + n_-^y$. If x is given we may choose a suitable y such that $x \oplus y$ has equal deficiency indices. The self-adjoint extension of $x \oplus y$ *must* then mix the spaces H an K (i.e. these spaces cannot be invariant subspaces anymore). This trick is used to define $SU_q(1,1)$, i.e. it can be defined by introducing extra operators. The space H will be $L^2(\mathbb{T}) \otimes L^2(q^{\mathbb{Z}})$ whereas $K = L^2(\mathbb{T}) \otimes L^2(-q^{\mathbb{N}})$. Details will follow now.

So, Woronowicz's theorem [37] says that it is impossible to realise the comultiplication on the level of operators on a Hilbert space. However, it was pointed out by Korogodsky [20] in 1994 that we can adapt the Hopf $*$-algebra to represent the functions on the normaliser in $SL(2,\mathbb{C})$ of $SU(1,1)$.

So, define $\mathcal{A}_q$ to be the unital $*$-algebra generated by elements $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and $\boldsymbol{e}$ and relations

$$\begin{aligned}
&\boldsymbol{\alpha}^\dagger\boldsymbol{\alpha} - \boldsymbol{\gamma}^\dagger\boldsymbol{\gamma} = \boldsymbol{e}, \qquad \boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger - q^2\,\boldsymbol{\gamma}^\dagger\boldsymbol{\gamma} = \boldsymbol{e}, \qquad \boldsymbol{\gamma}^\dagger\boldsymbol{\gamma} = \boldsymbol{\gamma}\,\boldsymbol{\gamma}^\dagger,\\
&\boldsymbol{\alpha}\,\boldsymbol{\gamma} = q\,\boldsymbol{\gamma}\,\boldsymbol{\alpha}, \qquad \boldsymbol{\alpha}\,\boldsymbol{\gamma}^\dagger = q\,\boldsymbol{\gamma}^\dagger\boldsymbol{\alpha},\\
&\boldsymbol{e}^\dagger = \boldsymbol{e}, \qquad \boldsymbol{e}^2 = 1, \qquad \boldsymbol{\alpha}\,\boldsymbol{e} = \boldsymbol{e}\,\boldsymbol{\alpha}, \qquad \boldsymbol{\gamma}\,\boldsymbol{e} = \boldsymbol{e}\,\boldsymbol{\gamma},
\end{aligned} \tag{5.2}$$

where $\dagger$ denotes the $*$-operation on $\mathcal{A}_q$ (again to distinguish this kind of adjoint with the adjoints of possibly unbounded operators in Hilbert spaces). In case we take $\boldsymbol{e} = 1$ in (5.2) we obtain the $*$-algebra $\tilde{A}$ described above. The additional generator $\boldsymbol{e}$ has been introduced by Korogodsky.

For completeness we give the Hopf $*$-algebra structure on $\mathcal{A}_q$. By $\mathcal{A}_q \odot \mathcal{A}_q$ we denote the algebraic tensor product. There exists a unique unital $*$-homomorphism $\boldsymbol{\Delta}\colon \mathcal{A}_q \to \mathcal{A}_q \odot \mathcal{A}_q$ such that

$$\boldsymbol{\Delta}(\boldsymbol{\alpha}) = \boldsymbol{\alpha} \otimes \boldsymbol{\alpha} + q\,(\boldsymbol{e}\,\boldsymbol{\gamma}^\dagger) \otimes \boldsymbol{\gamma} \qquad \boldsymbol{\Delta}(\boldsymbol{\gamma}) = \boldsymbol{\gamma} \otimes \boldsymbol{\alpha} + (\boldsymbol{e}\boldsymbol{\alpha}^\dagger) \otimes \boldsymbol{\gamma} \qquad \boldsymbol{\Delta}(\boldsymbol{e}) = \boldsymbol{e} \otimes \boldsymbol{e} \tag{5.3}$$

The counit $\boldsymbol{\varepsilon}\colon \mathcal{A}_q \to \mathcal{A}_q$ and antipode $\boldsymbol{S}\colon \mathcal{A}_q \to \mathcal{A}_q$ are given by

$$\begin{aligned}
&\boldsymbol{S}(\boldsymbol{\alpha}) = \boldsymbol{e}\,\boldsymbol{\alpha}^\dagger \qquad \boldsymbol{S}(\boldsymbol{\alpha}^\dagger) = \boldsymbol{e}\,\boldsymbol{\alpha} \qquad \boldsymbol{S}(\boldsymbol{\gamma}) = -q\,\boldsymbol{\gamma} \qquad \boldsymbol{S}(\boldsymbol{\gamma}^\dagger) = -\frac{1}{q}\,\boldsymbol{\gamma}^\dagger \qquad \boldsymbol{S}(\boldsymbol{e}) = \boldsymbol{e}\\
&\boldsymbol{\varepsilon}(\boldsymbol{\alpha}) = 1 \qquad \boldsymbol{\varepsilon}(\boldsymbol{\gamma}) = 0 \qquad \boldsymbol{\varepsilon}(\boldsymbol{e}) = 1
\end{aligned} \tag{5.4}$$

This makes $\mathcal{A}_q$ into a Hopf $*$-algebra.

Exercise 5.3. Verify that $\mathcal{A}_q$ is a Hopf $*$-algebra.

We can represent the commutation relations (5.2) by unbounded operators acting on the Hilbert space $H = L^2(\mathbb{T}) \otimes L^2(I_q)$, where $I_q = -q^{\mathbb{N}} \cup q^{\mathbb{Z}}$ is equipped with the counting measure. Here $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ denotes the unit circle, $\mathbb{N} = \{1, 2, \cdots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$. If $p \in I_q$, we define $\delta_p(x) = \delta_{x,p}$ for all $x \in I_q$, so the family $\{\delta_p \mid p \in I_q\}$ is the natural orthonormal basis of $L^2(I_q)$. For $L^2(\mathbb{T})$ we have the natural orthonormal basis $\{\zeta^m \mid m \in \mathbb{Z}\}$, with ζ the identity function on $\mathbb{T}$. Then $\{\zeta^m \otimes \delta_p \mid m \in \mathbb{Z}, p \in I_q\}$ is an orthonormal basis for H. Define linear operators α_0, γ_0, e_0 on the space E of finite linear combinations of $\zeta^m \otimes \delta_p$ by

$$\begin{aligned} &\alpha_0(\zeta^m \otimes \delta_p) = \sqrt{\mathrm{sgn}(p) + p^{-2}}\ \zeta^m \otimes \delta_{qp}, \\ &\gamma_0(\zeta^m \otimes \delta_p) = p^{-1}\ \zeta^{m+1} \otimes \delta_p, \qquad e_0(\zeta^m \otimes \delta_p) = \mathrm{sgn}(p)\ \zeta^m \otimes \delta_p. \end{aligned} \tag{5.5}$$

for all $p \in I_q$, $m \in \mathbb{Z}$. The actions of $\alpha_0^\dagger$ and $\gamma_0^\dagger$ on E can be given in a similar fashion by taking formal adjoints, and these satisfy the relations (5.2), and give a faithful representation of the algebra $\mathcal{A}_q$.

Exercise 5.4. (i) Write down the expressions for $\alpha_0^\dagger$ and $\gamma_0^\dagger$ on E, and show that α_0, γ_0, e_0, $\alpha_0^\dagger$, $\gamma_0^\dagger$ do satisfy the relations (5.2).
(ii) Prove that the representation on E obtained in this way is faithful.
(iii) Check that the operators α_0, γ_0, e_0, $\alpha_0^\dagger$, $\gamma_0^\dagger$ are closable as unbounded operators on H. Note that E is a dense subspace of H. Denote the closures of α_0, γ_0 by α, γ.
(iv) Check that $\alpha_0^\dagger \subset \alpha^*$, $\gamma_0^\dagger \subset \gamma^*$.
(v) Show that the adjoints α^* and γ^* are the closures of $\alpha_0^\dagger$, $\gamma_0^\dagger$.

So the operators α_0, γ_0 are closable with densely defined closed unbounded operators α, γ as their closure. Moreover, the adjoints α^* and γ^* are the closures of $\alpha_0^\dagger$, $\gamma_0^\dagger$. Let e be the closure of e_0, then e is a bounded linear self-adjoint operator on H. Consider the linear map $T\colon \zeta^m \otimes \delta_p \mapsto \zeta^m \otimes \delta_{-p}$, $T \in B(H)$, where we take $\delta_p = 0$ in case $p \notin I_q$, and let u be its partial isometry (in the polar decomposition; note that u is the operator mixing the spaces of Remark 5.2).

Definition 5.5. M is the von Neumann algebra in $B(H)$ generated by α, γ, e and u.

To give meaning to a von Neumann algebra generated by unbounded operators we use the following definition. For $T_1, \ldots, T_n$ closed, densely defined (possibly unbounded) linear operators acting on a Hilbert space H we define the von Neumann algebra

$$N = \{x \in B(H) \mid xT_i \subseteq T_i x, \text{ and } xT_i^* \subseteq T_i^* x \ \forall\, i\}'.$$

Then N is the smallest von Neumann algebra so that $T_1, \ldots, T_n$ are affiliated to N, and we call N the von Neumann algebra generated by $T_1, \ldots, T_n$. Note that in particular, α and γ are affiliated to M.

5.2 Special functions related to the multiplicative unitary

In order to understand how the definition of the multiplicative unitary W in this case can be obtained, we recall

$$\Delta(x) \,=\, W^*(1 \otimes x)W, \qquad x \in M.$$

To determine W note that this formula says that W should map the eigenspaces of (a self-adjoint extension of) $\gamma_0^\dagger\gamma_0$ to (the self-adjoint extension of) $\Delta(\gamma_0^\dagger\gamma_0)$. It is easy to determine the spectral decomposition of $\gamma_0^\dagger\gamma_0$. For $\Delta(\gamma_0^\dagger\gamma_0)$ this is certainly more involved, but there are surely a couple of techniques from the theory of orthogonal polynomials/special functions that are useful. See the following remark.

Remark 5.6. Let μ be a finite positive measure on the interval $[-1,1]$. It leads to a (degenerate) inner product $\langle f, g\rangle = \int f(s)\overline{g}(s)ds$. Then one may construct orthogonal polynomials by applying a Gramm-Schmidt orthogonalisation procedure to the initial basis $1, x, x^2, x^3, \ldots$ of polynomials on $[-1,1]$. This leads to orthonormal polynomials $P_0, P_1, P_2, \ldots$. Here P_i is of degree i. Moreover (a small exercise shows that) these polynomials satisfy a recurrence relation

$$sP_i(s) = a_iP_{i+1}(s) + b_iP_i(s) + a_{i-1}P_{i-1}(s).$$

Now consider the general operator $x : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ determined by $\delta_i \mapsto a_i\delta_{i+1} + b_i\delta_i + a_{i-1}\delta_{i-1}$. The spectral decomposition of this operator can be given as follows. One defines a unitary map $U : \ell^2(\mathbb{N}) \to L^2([-1,1],\mu)$ by sending $\delta_i \mapsto P_i$. Then UxU^* is the multiplication operator on $L^2([-1,1],\mu)$ with the function $s \mapsto s$. So the spectrum of x is the support of μ. In fact this shows that there is a 1-1 correspondence between orthogonal polynomials and 3-terms recurrence relations (in case the operator x associated to this recurrence is bounded at least). For our analysis the analogue of this idea is applied for unbounded 3-terms recurrence relations which requires more delicate (non-obvious!) analysis.

So the strategy is as follows now: one first determines $A := \Delta(\gamma_0^\dagger\gamma_0)$ algebraically and shows that the eigenvalue problem of this operator leads to a certain three terms recurrence relation. Then we are able to do the spectral analysis of this 3-term recurrence operator.

Now checking the representation (5.5) of (5.2), we see that the element $\gamma_0^\dagger\gamma_0$ acts diagonally in these representations, so the idea is to study the operator corresponding to $\boldsymbol{\Delta}(\boldsymbol{\gamma}^\dagger\boldsymbol{\gamma})$ acting initially on the dense domain $E \odot E \subset H \otimes H$, and we denote this operator by $\Delta_0(\gamma_0^\dagger\gamma_0)$.

Exercise 5.7. Calculate the action of $\Delta_0(\gamma_0^\dagger\gamma_0)$ on $\zeta^m \otimes \delta_p$ explicitly using (5.2), (5.3), (5.5).

For use now, and later use as well, we introduce the following notation, which will be used throughout the section.

Definition 5.8. (i) $\chi\colon -q^{\mathbb{Z}} \cup q^{\mathbb{Z}} \to \mathbb{Z}$ such that $\chi(x) = \log_q(|x|)$ for all $x \in -q^{\mathbb{Z}} \cup q^{\mathbb{Z}}$;
(ii) $\kappa\colon \mathbb{R} \to \mathbb{R}$ such that $\kappa(x) = \operatorname{sgn}(x)\, x^2$ for all $x \in \mathbb{R}$;
(iii) $\nu\colon -q^{\mathbb{Z}} \cup q^{\mathbb{Z}} \to \mathbb{R}^+$ such that $\nu(t) = q^{\frac{1}{2}(\chi(t)-1)(\chi(t)-2)}$ for all $t \in -q^{\mathbb{Z}} \cup q^{\mathbb{Z}}$;
(iv) $s\colon \mathbb{R}_0 \times \mathbb{R}_0 \to \{-1, 1\}$ is defined such that

$$s(x,y) = \begin{cases} -1 & \text{if } x > 0 \text{ and } y < 0 \\ 1 & \text{if } x < 0 \text{ or } y > 0 \end{cases}$$

for all $x, y \in \mathbb{R}_0 = \mathbb{R}\backslash\{0\}$.

In order to study the eigenvalues and eigenvectors of $\Delta_0(\gamma_0^\dagger \gamma_0)$, we need a suitable self-adjoint operator extending $\Delta_0(\gamma_0^\dagger \gamma_0)$ with domain $E \odot E$. Define a linear map $L\colon \mathcal{F}(\mathbb{T} \times I_q \times \mathbb{T} \times I_q) \to \mathcal{F}(\mathbb{T} \times I_q \times \mathbb{T} \times I_q)$ by

$$\begin{aligned}(Lf)(\lambda, x, \mu, y) \;=\; & [\, x^{-2}(\operatorname{sgn}(y) + y^{-2}) + (\operatorname{sgn}(x) + q^2\, x^{-2})\, y^{-2}\,]\, f(\lambda, x, \mu, y) \\ & + \operatorname{sgn}(x)\, q^{-1}\, \bar{\lambda}\mu\, x^{-1} y^{-1} \sqrt{(\operatorname{sgn}(x) + x^{-2})(\operatorname{sgn}(y) + y^{-2})}\, f(\lambda, qx, \mu, qy) \\ & + \operatorname{sgn}(x)\, q\, \lambda\bar{\mu}\, x^{-1} y^{-1} \sqrt{(\operatorname{sgn}(x) + q^2 x^{-2})(\operatorname{sgn}(y) + q^2 y^{-2})}\, f(\lambda, q^{-1}x, \mu, q^{-1}y)\end{aligned}$$

for all $\lambda, \mu \in \mathbb{T}$ and $x, y \in I_q$. Here $\mathcal{F}(\mathbb{T} \times I_q \times \mathbb{T} \times I_q)$ denotes functions on the space $\mathbb{T} \times I_q \times \mathbb{T} \times I_q$. A straightforward calculation reveals that if $f \in E \odot E$, then $\Delta_0(\gamma_0^\dagger \gamma_0)\, f = L(f)$. From this, it is a standard exercise to check that $f \in D(\Delta_0(\gamma_0^\dagger \gamma_0)^*)$ and $\Delta_0(\gamma_0^\dagger \gamma_0)^* f = L(f)$ if $f \in L^2(\mathbb{T} \times I_q \times \mathbb{T} \times I_q)$ and $L(f) \in L^2(\mathbb{T} \times I_q \times \mathbb{T} \times I_q)$. Here we identify functions and elements of L^2-spaces.

Exercise 5.9. Prove the statements in this paragraph. Show also that $D(\Delta_0(\gamma_0^\dagger \gamma_0)^*)$ consists precisely of such elements f.

If $\theta \in q^{\mathbb{Z}}$, we define $\ell'_\theta = \{\, (\lambda, x, \mu, y) \in \mathbb{T} \times I_q \times \mathbb{T} \times I_q \mid y = \theta x \,\}$. We consider $L^2(\ell'_\theta)$ naturally embedded in $L^2(\mathbb{T} \times I_q \times \mathbb{T} \times I_q)$. It follows easily from the above discussion that $\Delta_0(\gamma_0^\dagger \gamma_0)^*$ leaves $L^2(\ell'_\theta)$ invariant. Thus, if T is a self-adjoint extension of $\Delta_0(\gamma_0^\dagger \gamma_0)$, the obvious inclusion $T \subseteq \Delta_0(\gamma_0^\dagger \gamma_0)^*$ implies that T also leaves $L^2(\ell'_\theta)$ invariant.

Therefore every self-adjoint extension T of $\Delta_0(\gamma_0^\dagger \gamma_0)$ is obtained by choosing a self-adjoint extension T_θ of the restriction of $\Delta_0(\gamma_0^\dagger \gamma_0)$ to $L^2(\ell'_\theta)$ for every $\theta \in q^{\mathbb{Z}}$ and setting $T = \oplus_{\theta \in q^{\mathbb{Z}}} T_\theta$. So fix $\theta \in q^{\mathbb{Z}}$. Define $J_\theta = \{\, z \in I_{q^2} \mid \kappa(\theta)\, z \in I_{q^2} \,\}$ which is a q^2-interval around 0. On J_θ we define a measure ν_θ such that $\nu_\theta(\{x\}) = |x|$ for all $x \in J_\theta$.

Now define the unitary transformation $U_\theta\colon L^2(\mathbb{T} \times \mathbb{T} \times J_\theta) \to L^2(\ell'_\theta)$ such that $U_\theta(f) = g$ where $f \in L^2(\mathbb{T} \times \mathbb{T} \times J_\theta)$ and $g \in L^2(\ell'_\theta)$ are such that

$$g(\lambda, z, \mu, \theta z) = (\lambda\bar{\mu})^{\chi(z)}\, (-\operatorname{sgn}(\theta z))^{\chi(z)}\, |z|\, f(\lambda, \mu, \kappa(z))$$

for almost all $\lambda, \mu \in \mathbb{T}$ and all $z \in I_q$ such that $\theta z \in I_q$.

Define the linear operator $L_\theta\colon \mathcal{F}(J_\theta)\to\mathcal{F}(J_\theta)$ such that

$$
\begin{aligned}
(L_\theta f)(x) = \frac{1}{\theta^2 x^2}\Big(&-\sqrt{(1+x)(1+\kappa(\theta)\,x)}\,f(q^2x) - q^2\sqrt{(1+q^{-2}x)(1+q^{-2}\kappa(\theta)\,x)}\,f(q^{-2}x)\\
&+[(1+\kappa(\theta)\,x)+q^2(1+q^{-2}x)]\,f(x)\Big)
\end{aligned}
\tag{5.6}
$$

for all $f\in\mathcal{F}(J_\theta)$ and $x\in J_\theta$.

Then an easy calculation shows that $U_\theta^*\,(\Delta_0(\gamma_0^\dagger\gamma_0)|_{E\odot E})\,U_\theta = 1\odot(L_\theta|_{\mathcal{K}(J_\theta)})$, where $\mathcal{K}(J_\theta)$ is the space of compactly supported functions on J_θ. So our problem is reduced to finding self-adjoint extensions of $L_\theta|_{\mathcal{K}(I_q)}$. This operator $L_\theta|_{\mathcal{K}(I_q)}$ is a second order q-difference operator for which eigenfunctions in terms of q-hypergeometric functions are known. We can use a standard reasoning to get hold of the self-adjoint extensions of $L_\theta|_{\mathcal{K}(I_q)}$. Let $\beta\in\mathbb{T}$. Then we define a linear operator $L_\theta^\beta\colon D(L_\theta^\beta)\subseteq L^2(J_\theta,\nu_\theta)\to L^2(J_\theta,\nu_\theta)$ such that

$$
\begin{aligned}
D(L_\theta^\beta) = \{\, f\in L^2(J_\theta,\nu_\theta)\;|L_\theta(f)\in L^2(J_\theta,\nu_\theta), f(0+)=\beta\,f(0-)&\\
\text{and } (D_qf)(0+)=\beta\,(D_qf)(0-)\,\}&
\end{aligned}
$$

and L_θ^β is the restriction of L_θ to $D(L_\theta^\beta)$. Here, D_q denotes the Jackson derivative, that is, $(D_qf)(x) = (f(qx)-f(x))/(q-1)x$ for $x\in J_\theta$. Also, $f(0+)=\beta\,f(0-)$ is an abbreviated form of saying that the limits $\lim_{x\uparrow 0} f(x)$ and $\lim_{x\downarrow 0} f(x)$ exist and $\lim_{x\downarrow 0} f(x) = \beta\,\lim_{x\uparrow 0} f(x)$.

Then L_θ^β is a self-adjoint extension of $L_\theta|_{\mathcal{K}(I_q)}$. If $\beta,\beta'\in\mathbb{T}$ and $\beta\neq\beta'$, then $L_\theta^\beta\neq L_\theta^{\beta'}$. At this point it is not clear which of these extensions has to be chosen.

Exercise 5.10. Check the statements of the above paragraphs.

So in order to find the special functions attached to the multiplicative unitary we need to study the solutions to (5.6), and this leads to special functions of basic hypergeometric type. The eigenfunctions can all be described in terms of the ${}_1\varphi_1$-series (see [15]). We state the results as follows, and give the link to special functions later. For $a,b,z\in\mathbb{C}$, we define

$$
\Psi\left(\begin{matrix}a\\ b\end{matrix}\,;q,\,z\right) = \sum_{n=0}^{\infty}\frac{(a;q)_n\,(b\,q^n;q)_\infty}{(q\,;q)_n}\,(-1)^n\,q^{\frac12 n(n-1)}\,z^n \;=(b;q)_\infty\;{}_1\varphi_1\left(\begin{matrix}a\\ b\end{matrix}\,;q,z\right). \tag{5.7}
$$

This is an entire function in a, b and z. We use the normalisation constant

$$
c_q = \frac{1}{\sqrt{2}\,q\,(q^2,-q^2;q^2)_\infty}
$$

Definition 5.11. If $p \in I_q$, we define the function $a_p \colon I_q \times I_q \to \mathbb{R}$ such that a_p is supported on the set $\{\, (x,y) \in I_q \times I_q \mid \mathrm{sgn}(xy) = \mathrm{sgn}(p) \,\}$ and is given by

$$\begin{aligned} a_p(x,y) \;&= c_q\, s(x,y)\, (-1)^{\chi(p)}\, (-\mathrm{sgn}(y))^{\chi(x)}\, |y|\; \nu(py/x)\;\; \sqrt{\frac{(-\kappa(p), -\kappa(y); q^2)_\infty}{(-\kappa(x); q^2)_\infty}} \\ &\qquad \times \Psi\left(\begin{matrix} -q^2/\kappa(y) \\ q^2\kappa(x/y) \end{matrix} \; ; q^2,\, q^2\kappa(x/p) \right) \end{aligned}$$

for all $(x,y) \in I_q \times I_q$ satisfying $\mathrm{sgn}(xy) = \mathrm{sgn}(p)$.

Actually, the sign $s(x,y)$ in Definition 5.11 is chosen so that it corresponds to self-adjoint extensions $L_\theta^{\mathrm{sgn}(\theta)}$.

The extra vital information that we need is contained in the following proposition. For $\theta \in q^{\mathbb{Z}}$ we define $\ell_\theta = \{\, (x,y) \in I_q \times I_q \mid y = \theta x \,\}$.

Proposition 5.12. *Consider $\theta \in -q^{\mathbb{Z}} \cup q^{\mathbb{Z}}$. Then the family $\{\, a_p|_{\ell_\theta} \mid p \in I_q$ so that $\mathrm{sgn}(p) = \mathrm{sgn}(\theta) \,\}$ is an orthonormal basis for $l^2(\ell_\theta)$. In particular,*

$$\sum_{x \in I_q \text{ so that } \theta x \in I_q} a_p(x, \theta x)\, a_r(x, \theta x) = \delta_{p,r}, \qquad p, r \in I_q.$$

This result implies also a dual result, stemming from the following simple duality principle. Consider a set I and suppose that $l^2(I)$ has an orthonormal basis $(e_j)_{j \in J}$. For every $i \in I$, we define a function f_i on J by $f_i(j) = e_j(i)$. Then $(f_i)_{i \in I}$ is an orthonormal basis for $l^2(J)$. If we apply this principle to the line ℓ_θ, Proposition 5.12 implies the next one.

Proposition 5.13. *Consider $\theta \in -q^{\mathbb{Z}} \cup q^{\mathbb{Z}}$ and define $J = q^{\mathbb{Z}} \subset I_q$ if $\theta > 0$ and $J = -q^{\mathbb{N}} \subset I_q$ if $\theta < 0$. For every $(x,y) \in \ell_\theta$ we define the function $e_{(x,y)} : J \to \mathbb{R}$ such that $e_{(x,y)}(p) = a_p(x,y)$ for all $p \in J$. Then the family $\{\, e_{(x,y)} \mid (x,y) \in \ell_\theta \,\}$ forms an orthonormal basis for $l^2(J)$. In particular,*

$$\sum_{p \in J} a_p(x, \theta x)\, a_p(y, \theta y) = \delta_{x,y}, \qquad x, y \in I_q.$$

Finally, these functions satisfy symmetry relations

$$\begin{aligned} a_p(x,y) &= (-1)^{\chi(yp)} \mathrm{sgn}(x)^{\chi(x)} \left| \frac{y}{p} \right| a_y(x,p); \\ a_p(x,y) &= \mathrm{sgn}(p)^{\chi(p)} \mathrm{sgn}(x)^{\chi(x)} \mathrm{sgn}(y)^{\chi(y)} a_p(y,x); \\ a_p(x,y) &= (-1)^{\chi(xp)} \mathrm{sgn}(y)^{\chi(y)} \left| \frac{x}{p} \right| a_x(p,y), \end{aligned} \tag{5.8}$$

which follow from transformation formulas for ${}_1\varphi_1$-series that can be obtained from limiting cases of Heine's transformation [15].

Originally, the multiplicative unitary has been defined using the functions $a_p(x,y)$ of Definition 5.11. We now first introduce the Haar weight and its GNS-representation.

5.3 The GNS-construction for the Haar weight

We will not work in the Hilbert space H but in the GNS-space K, which we indicate how to construct.

Proposition 5.14. *The von Neumann algebra M defined in Definition 5.5 equals $L^\infty(\mathbb{T}) \otimes B\big(L^2(I_q)\big)$*

The proof of Proposition 5.14 follows using the spectral theorem and showing that multiplication by $f \in L^\infty(\mathbb{T})$ and all rank-one operators on $L^2(I_q)$ are contained in M, see [19, Lemma 2.4].

We define the following operators in M, by Proposition 5.14,

$$\Phi(m,p,t) \colon \zeta^r \otimes \delta_x \mapsto \delta_{xt}\, \zeta^{m+r} \otimes \delta_p, \qquad m, r \in \mathbb{Z},\ p, t, x \in I_q.$$

A straightforward calculation gives

$$\Phi(m_1,p_1,t_1)\,\Phi(m_2,p_2,t_2) = \delta_{p_2,t_1}\,\Phi(m_1+m_2,p_1,t_2), \qquad \Phi(m,p,t)^* = \Phi(-m,t,p)$$

In particular the finite linear span of the operators $\Phi(m,p,t)$ form a σ-weakly dense $*$-subalgebra in M.

We now construct the nsf weight φ, for which we later show that it is actually the left Haar weight. We construct of the nsf weight by writing down its GNS-construction. Define $\mathrm{Tr} = \mathrm{Tr}_{L^\infty(\mathbb{T})} \otimes \mathrm{Tr}_{B(L^2(I_q))}$ on M, where $\mathrm{Tr}_{L^\infty(\mathbb{T})}$ and $\mathrm{Tr}_{B(L^2(I_q))}$ are the canonical traces on $L^\infty(\mathbb{T})$, i.e. $\mathrm{Tr}_{L^\infty(\mathbb{T})}(f) = \int_{\mathbb{T}} f(\zeta)\, d\zeta$ with normalization $\mathrm{Tr}_{L^\infty(\mathbb{T})}(1) = 1$, and on $B(L^2(I_q))$, normalized by $\mathrm{Tr}_{B(L^2(I_q))}(P) = 1$ for any rank one orthogonal projection. Note that Tr is a tracial weight on M so in particular its modular group is trivial. For Tr we have the following GNS-construction:

- a Hilbert space $\mathcal{K} = H \otimes L^2(I_q) = L^2(\mathbb{T}) \otimes L^2(I_q) \otimes L^2(I_q)$ equipped with the orthonormal basis $\{f_{mpt} \mid m \in \mathbb{Z},\ p, t \in I_q\}$;
- a unital $*$-homomorphism $\pi \colon M \to B(\mathcal{K})$, $\pi(a) = a \otimes \iota_{L^2(I_q)}$ for $a \in M$;
- $\Lambda_{\mathrm{Tr}} \colon \mathcal{N}_{\mathrm{Tr}} \to \mathcal{K}$, $a \mapsto \sum_{p \in I_q} (a \otimes \iota_{L^2(I_q)})) f_{0,p,p}$.

Exercise 5.15. Check that the above construction is the GNS-construction for Tr.

We define the left invariant nsf weight φ formally as $\varphi(x) = \mathrm{Tr}(|\gamma|\, x\, |\gamma|)$ with the operator $|\gamma|$ affiliated to M. We proceed by defining the set D as the set of elements of $x \in M$ such that $x|\gamma|$ extends to a bounded operator on H, denoted by $\overline{x|\gamma|}$, and such that $\overline{x|\gamma|} \in \mathcal{N}_{\mathrm{Tr}}$, and for $x \in D$ we put $\Lambda(x) = \Lambda_{\mathrm{Tr}}(\overline{x|\gamma|})$. The set D is then a core for the operator Λ which is closable for the σ-strong-$*$–norm topology.

Definition 5.16. The nsf weight φ on M is defined by its GNS-construction $(\mathcal{K}, \pi, \Lambda)$.

So in particular, $\varphi(b^*a) = \langle \Lambda(a), \Lambda(b) \rangle_{\mathcal{K}}$ for all $a, b \in \mathcal{N}_\varphi$. Note the remarkable fact that for both the quantum group analogues of $SU(2)$ and $SU(1,1)$ the Haar weight are both of the form $\mathrm{Tr}_{\gamma^*\gamma}$, see Section 3.4.

Remark 5.17. Note that in particular we can use π to identify $M \subset B(H)$ with its image $\pi(M) \subset B(\mathcal{K})$. From now on we use this identification, and we work with M realized as von Neumann algebra in $B(\mathcal{K})$.

Lemma 5.18. *From the general theory of nsf weights, see Section 2, we know that φ comes with a modular automorphism group σ, a modular conjugation J and modular operator ∇. We have*

- $\sigma_t(x) = |\gamma|^{2it} x |\gamma|^{-2it}$ *for all* $x \in M$, $t \in \mathbb{R}$*;*
- $\Phi(m,p,t) \in \mathcal{N}_\varphi$ *and* $\Lambda(\Phi(m,p,t)) = |t|^{-1} f_{mpt}$*;*
- $\Phi(m,p,t) \in \mathcal{M}_\varphi$ *and* $\varphi(\Phi(m,p,t)) = |t|^{-2} \delta_{m,0} \delta_{p,t}$*;*
- $\Phi(m,p,t)$ *is analytic for* σ *and* $\sigma_z(\Phi(m,p,t)) = |p^{-1}t|^{2iz} \Phi(m,p,t)$ *for all* $z \in \mathbb{C}$*;*
- $J\, f_{mpt} = f_{-m,t,p}$*;*
- f_{mpt} *in the domain of* ∇ *and* $\nabla f_{mpt} = |p^{-1}t|^2\, f_{mpt}$.

Proof. The first statement immediately follows from general theory: Connes' cocycle derivative, see [28]. Observe that $\Phi(m,p,t)^* \Phi(m,p,t) = \Phi(0,t,t)$ is in $\mathcal{N}_{\mathrm{Tr}}$ and that $\Phi(m,p,t)|\gamma$ is a bounded operator, then the second statement follows from a calculation. The third statement follows by observing that $\Phi(m,p,t) = \Phi(0,p,p)\Phi(m,p,t)$, so that

$$\varphi\big(\Phi(m,p,t)\big) \;=\; \langle \Lambda(\Phi(m,p,t)), \Lambda(\Phi(0,p,p)) \rangle \;=\; |pt|^{-1} \langle f_{mpt}, f_{0pp} \rangle$$

which gives the result.

A calculation shows that

$$|\gamma|^{2is} \Phi(m,p,t) \;=\; |p^{-1}t|^{2is} \Phi(m,p,t) |\gamma|^{2is}$$

and this implies $\sigma_z(\Phi(m,p,t)) = |p^{-1}t|^{2iz} \Phi(m,p,t)$ for all $z \in \mathbb{C}$ by the first result.

Consider the antilinear map S as in Section 2, then

$$S f_{m,p,t} = |t| J\, \Lambda\big(\Phi(m,p,t)\big) \;=\; |t| \Lambda\big(\Phi(m,p,t)^*\big) \;=\; |t| \Lambda\big(\Phi(-m,t,p)\big) \;=\; \frac{|t|}{|p|}\, f_{-m,t,p}.$$

The last two statements follow from the polar decomposition of S. $\square$

5.4 The multiplicative unitary and the comultiplication

The multiplicative unitary $W \in B(\mathcal{K} \otimes \mathcal{K})$ has a useful description in terms of the functions $a_p(\cdot, \cdot)$, as defined in Definition 5.11,

$$W^*(f_{m_1,p_1,t_1} \otimes f_{m_2,p_2,t_2}) = \sum_{\substack{y,z \in I_q \\ \mathrm{sgn}(p_2t_2)yzq^{m_2}/p_1 \in I_q}} \left|\frac{t_2}{y}\right| a_{t_2}(p_1, y) a_{p_2}(z, \mathrm{sgn}(p_2t_2)yzq^{m_2}/p_1) \times f_{m_1+m_2-\chi(p_1p_2/t_2z),z,t_1} \otimes f_{\chi(p_1p_2/t_2z),\mathrm{sgn}(p_2t_2)yzq^{m_2}/p_1,y}. \tag{5.9}$$

For convenience we state the corresponding result for W as well, which follows directly from (5.9):

$$W(f_{m_1,p_1,t_1} \otimes f_{m_2,p_2,t_2}) = \sum_{\substack{r,s \in I_q \\ \mathrm{sgn}(rp_2t_2)sp_1q^{m_2} \in I_q}} \left|\frac{s}{t_2}\right| a_s(\mathrm{sgn}(rp_2t_2)sp_1q^{m_2}, t_2)\, a_r(p_1, p_2) \times f_{m_1-\chi(sp_2/t_2),\mathrm{sgn}(rp_2t_2)sp_1q^{m_2},t_1} \otimes f_{m_2+\chi(sp_2/t_2),r,s}. \tag{5.10}$$

Proposition 5.19. *$W \in B(\mathcal{K} \otimes \mathcal{K})$ is unitary operator.*

Proof. This follows by a straightforward check using Propositions 5.12 and 5.13. □

Now we define the comultiplication $\Delta(x) = W^*(1 \otimes x)W$ for $x \in M$.

Theorem 5.20. *Δ is coassociative, i.e. $(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta$.*

The proof of Theorem 5.20 is intense and laborious, and we will not repeat it here. The idea is to check it for generators of M, which has certain complications involving domains as α and γ are unbounded. We refer to [19] for details.

Theorem 5.21. *(M, Δ) is a locally compact quantum group with left and right Haar weight φ and multiplicative unitary W.*

Proof. According to Definition 3.1 and Theorem 5.20 it suffices to check that φ is a left Haar weight and that the right Haar weight can be taken equal to φ. This requires some von Neumann algebra techniques, for which we refer to [19]. □

6 Cocycle twisting

The aim of this section is to present a technique to obtain new quantum groups by 'twisting' an existing quantum group using a cocycle. It was shown by Fima and Vainerman [14] that for special cases of the cocycle one can indeed obtain an operator algebraic quantum group satisfying all the conditions of the Kustermans-Vaes definition. The hard part is to show existence of the Haar weights. For general (unitary) cocycles this result was obtained by De Commer [8]. In fact De Commer obtains an even more general result that led to an alternative approach of defining/finding $SU_q(1,1)$.

6.1 Cocycle twisitng

Let $\mathbb{G} = (\mathsf{M}, \Delta)$ be a locally compact quantum group. A cocycle is a unitary $\Omega \in \mathsf{M} \otimes \mathsf{M}$ that satisfies the cocycle relation

$$(\Omega \otimes 1)(\Delta \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \Delta)(\Omega).$$

Some texts will speak about *unitary* cocycles, but we shall simply make that part of the definition here. There is an easy way of obtaining a cocycle. Namely, let $X \in \mathsf{M}$ be unitary. Then

$$(X \otimes X)\Delta(X^*) \tag{6.1}$$

is always a cocycle (it is an easy exercise to verify the cocycle relation). The cocycles of the form (6.1) are called coboundaries.

To a cocycle $\Omega \in \mathsf{M} \otimes \mathsf{M}$ we may associate a new quantum group in the following way. The underlying von Neumann algebra will again be M but then with comultiplication

$$\Delta_\Omega(x) = \Omega\Delta(x)\Omega^*.$$

It follows from the coycle identity that the comultiplication is coassociative. Indeed,

$$\begin{aligned}
&(\Delta_\Omega \otimes \iota) \circ \Delta_\Omega(x) \\
=&(\Omega \otimes 1)(\Delta \otimes \iota)(\Omega\Delta(x)\Omega^*)(\Omega^* \otimes 1) \\
=&(\Omega \otimes 1)(\Delta \otimes \iota)(\Omega) \quad \times \quad (\Delta \otimes 1)\Delta(x) \quad \times \quad (\Delta \otimes \iota)(\Omega^*)(\Omega^* \otimes 1) \\
=&(1 \otimes \Omega)(\iota \otimes \Delta)(\Omega\Delta(x)\Omega^*)(1 \otimes \Omega^*) \\
=&(\iota \otimes \Delta_\Omega) \circ \Delta_\Omega(x).
\end{aligned}$$

Note that if Ω is a coboundary, say $\Omega = (X \otimes X)\Delta(X^*)$ with $X \in \mathsf{M}$ unitary, then the twisted pair $(\mathsf{M}, \Delta_\Omega)$ is isomorphic to (M, φ). Indeed the map $x \mapsto XxX^*$ gives this isomorphism. What is not so trivial is that in general $(\mathsf{M}, \Delta_\Omega)$ admits Haar weights which turn it into a locally compact quantum group.

Theorem 6.1 (See [8])**.** *There exist Haar weights $\varphi_\Omega, \psi_\Omega$ such that the 4-tuple $\mathbb{G}_\Omega = (\mathsf{M}, \Delta_\Omega, \varphi_\Omega, \psi_\Omega)$ is a locally compact quantum group.*

We will give the main ingredients of the proof of this theorem in this section, which are more general Galois co-objects. Every result here is proved by De Commer in [8], see also his thesis [7] which contains a couple of additional results (partly unpublished). Before going into details let us also mention the following.

Theorem 6.2 (See [9]). *Let $\mathbb{G}$ be a compact quantum group. The twisted quantum group $\mathbb{G}_\Omega$ is not necessarily compact.*

Let us give the idea behind Theorem 6.2, at least to some extend. Firstly let us recall something on infinite tensor products of Von Neumann algebras, see [29, Section XIV.1]. Let $(\mathsf{M}_k, \varphi_k)$ be von Neumann algebras equipped with normal, faithful state φ_k indexed by $k \in \mathbb{N}$. Then we may construct an infinite tensor product $\otimes_k(\mathsf{M}_k, \varphi_k)$ as follows. Let $(\mathcal{H}_k, \xi_k)$ be its GNS space with cyclic vector ξ_k. We have a canonical embedding $\otimes_{k=0}^n \mathcal{H}_k \to \otimes_{k=0}^{n+1}\mathcal{H}_k$ by means of the mapping $(\otimes_{k=0}^n \eta_k) \mapsto (\otimes_{k=0}^n \eta_k) \otimes \xi_{n+1}$. This turns

$$\otimes_{k=0}^n \mathcal{H}_k \to \otimes_{k=0}^{n+1}\mathcal{H}_k,$$

into an inductive system of Hilbert spaces of which we may take the direct limit which we denote by $\otimes_{k=0}^\infty \mathcal{H}_k$. The embedding of a finite tensor $\eta_0 \otimes \ldots \otimes \eta_n \in \otimes_{k=0}^n \mathcal{H}_k$ in $\otimes_{k=0}^\infty \mathcal{H}_k$ is given by $\eta_0 \otimes \ldots \otimes \eta_n \otimes \xi_{n+1} \otimes \xi_{n+2} \otimes \ldots$. This way the finite tensors $\otimes_{k=0}^n \mathsf{M}_k$ act on $\otimes_{k=0}^\infty \mathcal{H}_k$ and the double commutant of $\cup_n \otimes_{k=0}^n \mathsf{M}_k$ is by definition $\otimes_{k=0}^\infty(\mathsf{M}_k, \varphi_k)$. We emphasize that the construction *does* depend on the choice of the states. For example, take $(\mathsf{M}_k, \varphi_k) = (M_2(\mathbb{C}), \mathrm{Tr}(\,\cdot\, A))$ with A the matrix given by

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}, \qquad 0 < \alpha \leq \frac{1}{2}.$$

Then $\otimes_{k=0}^\infty(\mathsf{M}_k, \varphi_k)$ is the hyperfinite III_λ-factor with $\lambda = \frac{\alpha}{1-\alpha}$ in case $0 < \alpha < \frac{1}{2}$. And $\otimes_{k=0}^\infty(\mathsf{M}_k, \varphi_k)$ is the hyperfinite II_1-factor in case $\alpha = \frac{1}{2}$, see [29] (it is also summarized in [6]).

The idea behind Theorem 6.2 is – very roughly – to take for the quantum group $\mathbb{G}$ an infinite tensor product of $SU_{q_k}(2)$ for parameters q_k that satisfy $\sum_k q_k^2 < \infty$. The tensor product is taken with respect to the Haar states. Recall that the von Neumann algebra of $SU_{q_k}(2)$ is given by $B(\ell^2(\mathbb{N})) \otimes L^\infty(\mathbb{T})$. Define the unitary,

$$w_k = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \otimes 1 \in B(\ell^2(\mathbb{N})) \otimes L^\infty(\mathbb{T}).$$

In [9] it is shown that $\Omega := \otimes_{k=0}^\infty (w_k \otimes w_k)\Delta(w_k^*)$ is a well-defined element of the von Neumann algebra of $(\otimes_{k=0}^\infty SU_{q_k}(2)) \otimes (\otimes_{k=0}^\infty SU_{q_k}(2))$ that satisfies the cocycle relation. Then twisting $\otimes_{k=0}^\infty SU_{q_k}(2)$ with Ω turns out to yield a non-compact quantum group. Note that whereas the tensor legs $(w_k \otimes w_k)\Delta(w_k^*)$ are coboundaries, the cocycle Ω is not a coboundary (as then the twist would be compact again).

6.2 Twisting by Galois coobjects

In this section we review the construction from [8]. The input is a *Galois coobject* the output is a new quantum group, of which cocycle twists are a special case. Therefor fix a locally compact quantum group (M, φ).

Definition 6.3. A coaction of (M, φ) on a von Neumann algebra N is a normal $*$-homomorphism $\alpha : \mathsf{N} \to \mathsf{M} \otimes \mathsf{N}$ satisfying the relation $(\Delta \otimes \iota) \circ \alpha = (\iota \otimes \alpha) \circ \alpha$. We define the fixed point algebra as

$$\mathsf{N}^\alpha := \{x \in \mathsf{N} \mid \alpha(x) = 1 \otimes x\}.$$

The coaction is called *ergodic* if $\mathsf{N}^\alpha = \mathbb{C}1$. A coaction is called *integrable* if $(\varphi \otimes \iota) \circ \alpha : \mathsf{N}^+ \to \mathsf{N}^+$ is a semi-finite operator valued weight.

Remark 6.4. What we defined as coaction is sometimes called a left coaction, where a right coaction would mean that α maps $\mathsf{N} \to \mathsf{N} \otimes \mathsf{M}$ and the rest of the definition changes accordingly.

From this point we shall fix a coaction $\alpha : \mathsf{N} \to \mathsf{M} \otimes \mathsf{N}$ that is both integrable and ergodic. Note that we have $(\varphi \otimes \iota) \circ \alpha : \mathsf{N} \to \mathsf{N}^\alpha \simeq \mathbb{C}$ so that $(\varphi \otimes \iota) \circ \alpha$ is a normal semi-finite faithful weight (and not just an operator valued weight). We assume that N is equipped with this weight from now on. A particular case of this situation is where $\alpha = \Delta$, the comultiplication of M. The constructions below then 'trivialize' meaning that the twisted quantum group we shall obtain equals (M, φ) it self. Let us now carry out the constructions of [8]. Firstly we recall the definition of the crossed product, see [34].

Definition 6.5. The crossed product von Neumann algebra $\mathsf{N} \rtimes_\alpha \mathsf{M}$ is defined as the von Neumann algebra algebra acting on $L^2(\mathsf{N}) \otimes L^2(\mathsf{M})$ that is generated by $1 \otimes \widehat{\mathsf{M}}'$ and $\alpha(\mathsf{N})$.

To the action α we may associate the following operator that resembles the multiplicative unitary:

$$G : L^2(\mathsf{N}) \otimes L^2(\mathsf{N}) \to L^2(\mathsf{N}) \otimes L^2(\mathsf{M}) : (\Lambda_\mathsf{N} \otimes \Lambda_\mathsf{N})(x \otimes y) \mapsto (\Lambda_\mathsf{N} \otimes \Lambda_\mathsf{M})(\alpha(x)(y \otimes 1)). \quad (6.2)$$

We also set $\widetilde{G} = \Sigma G : L^2(\mathsf{N}) \otimes L^2(\mathsf{N}) \to L^2(\mathsf{M}) \otimes L^2(\mathsf{N})$, where $\Sigma : L^2(\mathsf{N}) \otimes L^2(\mathsf{M}) \to L^2(\mathsf{M}) \otimes L^2(\mathsf{N})$ is the flip map. It is not hard to check that G is an isometry (the proof is the same as for the definition of W^*) and hence in particular (6.2) is a well-defined and bounded map. If $\alpha = \Delta$ then in fact $G = W^*$. In general G is not a unitary map and therefore we introduce the following definition at this point.

Definition 6.6. We call an integrable, ergodic coaction α a *Galois coaction* if the map (6.2) is unitary.

Remark 6.7. There is an alternative – perhaps better – way of defining a Galois coaction. Assume that $(\mathsf{M},\Delta) = (L^\infty(\mathsf{G}),\Delta_{\mathsf{G}})$ is a classical group and G acts on a measure space (X,μ), call this action $(g,x)\mapsto \beta_g(x)$. Given $g\in\mathsf{G}$ we let $\beta_g^*\in B(L^2(X))$ be the pull back of this map. Let $\alpha : L^\infty(X,\mu)\to L^\infty(\mathsf{G})\otimes L^\infty(X)\simeq L^\infty(\mathsf{G}\times X) : f\mapsto f\circ\beta$. We shall assume that α is ergodic and integrable. Then we may define the *Galois homomorphism*, see [34, Theorem 5.3], which is the normal $*$-homorphism given by,

$$\rho : L^\infty(X,\mu)\rtimes_\alpha \mathsf{G}\to B(L^2(X,\mu)),$$

which is determined by

$$\begin{aligned}\alpha(x) &\mapsto x,\\ \int_{\mathsf{G}} f(s)\rho_s ds &\mapsto \int_{\mathsf{G}} f(s)\left(\frac{d\mu\circ\beta_s}{d\mu}\right)^{\frac{1}{2}}\beta_s^* ds.\end{aligned}$$

Because α is ergodic the map ρ is in fact surjective. It turns out that ρ *is an isomorphism if and only if the operator G introduced above is unitary*, see [8, Theorem 2.1]. These constructions can be generalized to arbitrary locally compact quantum groups and [8, Theorem 2.1] holds in fact in full generality. We did not introduce these general constructions here as this would mean that we had to introduce several new objects first (such as the unitary implementation of a coaction from [34]).

We now set

$$\widehat{\mathsf{O}} := \overline{\left\{(\iota\otimes\omega)(\widetilde{G}) \mid \omega\in B(L^2(\mathsf{N}))\right\}}^{\sigma\text{-wk}} \subseteq B(L^2(\mathsf{N}),L^2(\mathsf{M})),$$

and $\widehat{\mathsf{N}} := \widehat{\mathsf{O}}^* \subseteq B(L^2(\mathsf{M}),L^2(\mathsf{N}))$. Finally put $\widehat{\mathsf{P}}$ to be the closed linear span of $\widehat{\mathsf{O}}\widehat{\mathsf{N}}$. Define

$$\begin{aligned}\Delta_{\widehat{\mathsf{N}}} : \widehat{\mathsf{N}}\to\widehat{\mathsf{N}}\otimes\widehat{\mathsf{N}} :& x\mapsto \widehat{G}^*(1\otimes x)W,\\ \Delta_{\widehat{\mathsf{O}}} : \widehat{\mathsf{O}}\to\widehat{\mathsf{O}}\otimes\widehat{\mathsf{O}} :& x\mapsto (\Delta_{\widehat{\mathsf{N}}}(x^*))^*,\\ \Delta_{\widehat{\mathsf{P}}} : \widehat{\mathsf{P}}\to\widehat{\mathsf{P}}\otimes\widehat{\mathsf{P}} :& x\mapsto \widehat{G}^*(1\otimes x)\widehat{G}.\end{aligned}$$

One of the main results of [8] is that $(\widehat{P},\Delta_{\widehat{P}})$ admits left and right Haar weights that complete it to a locally compact quantum group. Moreover

$$\begin{pmatrix}\widehat{P} & \widehat{O}\\ \widehat{N} & \widehat{M}\end{pmatrix}$$

together with the pointwise application of

$$\begin{pmatrix}\Delta_{\widehat{P}} & \Delta_{\widehat{O}}\\ \Delta_{\widehat{N}} & \Delta_{\widehat{M}}\end{pmatrix} : \begin{pmatrix}\widehat{P} & \widehat{O}\\ \widehat{N} & \widehat{M}\end{pmatrix} \to \begin{pmatrix}\widehat{P}\otimes\widehat{P} & \widehat{O}\otimes\widehat{O}\\ \widehat{N}\otimes\widehat{N} & \widehat{M}\otimes\widehat{M}\end{pmatrix}$$

determines a locally compact quantum groupoid with base space $M_2(\mathbb{C})$.

Remark 6.8. This is just a very brief account of what happens in [8]. Note that in [8] the spaces $\widehat{N}, \widehat{O}, \widehat{P}$ are introduced in a different way. They are the intertwining algebras between suitable representations of the von Neumann algebra N on $L^2(\mathsf{N})$ and $L^2(\mathsf{M})$. The representation of N on $L^2(\mathsf{M})$ requires again the (fully general) Galois isomorphism from Remark 6.7. As we have chosen not to include this Galois map in these notes we have given the alternative definition above.

Theorem 6.9. *Let $\Omega \in \widehat{M} \otimes \widehat{M}$ be a cocycle. Then there is a Galois coobject N of (M, Δ) such that the twisted quantum group $\mathsf{P}(= \widehat{\widehat{\mathsf{P}}})$ constructed above equals the cocycle twist $(\mathsf{M}, \Delta_\Omega)$.*

Proof. We give a proof sketch, i.e. we say how to construct the Galois coobject. It relies on crossed product duality. We let

$$\mathsf{N} := \mathbb{C} \rtimes_\Omega \widehat{\mathsf{M}} = \{(\omega \otimes \iota)(\widehat{W}\Omega^*) \mid \omega \in \widehat{\mathsf{M}}_*\},$$

be the twisted crossed product of the trivial action of $(\widehat{\mathsf{M}}, \widehat{\Delta})$. By crossed product duality [32] there exists a dual coaction $\alpha : \mathsf{N} \to \mathsf{N} \otimes \mathsf{M}$ of (M, Δ) on N that is determined by

$$\alpha((\omega \otimes \iota)(\widehat{W}\Omega^*)) = (\omega \otimes \iota \otimes \iota)(\widehat{W}_{13}\widehat{W}_{12}\Omega_{12}^*),$$

with $\omega \in \widehat{\mathsf{M}}_*$. This turns N into a Galois coaction for (M, Δ). We refer to [8] for details. $\square$

6.3 Concluding remarks about $SU_q(1,1)$

In [10] it is shown that there exists a Galois coobject for $\widehat{SU_q(2)}$ such that the twisted quantum group equals $SU_q(1,1)$. The coobject turns out to be a Podlés sphere. Note that we really mean the dual of $SU_q(2)$ and not $SU_q(2)$ itself. Such Galois coobjects can be found through projective corepresentations of $SU_q(2)$, see [8]. Using crossed product duality it is in fact shown that these projective corepresentations of a quantum group are in 1–1 correspondence with Galois coobjects of the dual quantum group.

The advantage of this approach is that the coassociativity of the comultiplication of the twisted group is automatic: the problem reduces/changes from *constructing* a quantum group to *identifying* a quantum group. That is, one really has to show that the outcoming quantum group is a suitable form of $SU_q(1,1)$. This is in general not so trivial (though in some examples it is known by now) as from the twisting procedure it is not so clear which quantum group one *a priori* obtains. The hope/aim/guess is of course that these methods can be used to find higher rank versions of these quantum groups.

Finally we need to comment that the presentation in this chapter is highly simplified. The full theory can be found in [7] which includes the universal and reduced C*-algebraic theory as well as a more "global" picture of monoidal/morita equivalences.

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