# On the endpoints of De Leeuw restriction theorems

Martijn Caspers

*Dedicated to Ben de Pagter's 65th birthday.*

**Abstract.** We prove a De Leeuw restriction theorem for Fourier multipliers on certain quasi-normed spaces. The proof is based on methods that were recently used in order to resolve problems on perturbations of commutators.

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## 1. Introduction

In 1965 Karel de Leeuw proved three fundamental theorems on $L_p$-multipliers [Lee65]: his restriction theorem, lattice approximation theorem and compactification theorem. The strongest De Leeuw theorem is the compactification theorem, which shows that boundedness of an $L_p$-Fourier multiplier does not depend on the topology. More precisely, let $m \in L_\infty(\mathbb{R}^n)$ be continuous and let $m_d \in \ell_\infty(\mathbb{R}^n_{\rm disc})$ be equal to $m$ but then on $\mathbb{R}^n$ with the discrete topology. Then, the Fourier multipliers $T_m : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$ and $T_{m_d} : L_p(\mathbb{R}^n_{\rm Bohr}) \to L_p(\mathbb{R}^n_{\rm Bohr})$ have the same norm. Here $\mathbb{R}^n_{\rm Bohr}$ is the Bohr compactification of $\mathbb{R}^n$ which can be viewed as the Pontrjagin dual group of $\mathbb{R}^n_{\rm disc}$. In order to show this, De Leeuw proved that for the discrete subgroup $\mathbb{Z}^n$ of $\mathbb{R}^n$ and $m \in L_\infty(\mathbb{R}^n)$, the symbol $m_{\mathbb{Z}^n} = m|_{\mathbb{Z}^n}$ gives rise to a Fourier multiplier with norms related by,

$$\| T_{m|_{\mathbb{Z}^n}} : L_p(\mathbb{T}^n) \to L_p(\mathbb{T}^n) \| \leq \| T_m : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n) \|, \qquad 1 \leq p < \infty. \quad (1.1)$$

The inequality (1.1) is known as De Leeuw's restriction theorem. Shortly after [Lee65] De Leeuw theorems have been obtained for arbitrary locally compact abelian groups by Saeki [Sae70]. These theorems are widely applied in harmonic analysis and the analysis of singular integrals.

Recent developments in non-commutative integration motivated an extension of De Leeuw theorems in various contexts (see e.g. [CPPR15], [CPSZ18]). One of these motivations comes from the theory of perturbations of commutators. A central question in this theory (going back at least to Krein [Kre64]) asks the following. Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz. Is there an absolute constant $C_{abs}$ such that for every $x \in B(H)$ and $A \in B(H)$ self-adjoint we have a commutator estimate,

$$\|[f(A), x]\| \leq C_{abs} \|[A, x]\|. \tag{1.2}$$

The question has a long history and is proven to be false [Far67], [Far68], [Far68], [Kat73], [Dav88], unless further differentiability conditions are imposed on $f$ [BiSo67], [Pel85] or of the uniform is replaced by the Schatten $\mathcal{S}_p$-norm [PoSu11], [CMPS14]. Important relations (in fact equivalences) between commutator estimates and non-commutative Lipschitz functions were obtained in [DDPS97], [PoSu08] and the problem was recasted into the language of double operator integrals [dPWS02].

Nazarov and Peller [NaPe09] were the first ones to obtain a weak $(1, 1)$ estimate for commutators (this would be the optimal solution to estimates of the form (1.2)). They showed that if $[A, x]$ is in $\mathcal{S}_1$ and has rank 1, then $[f(A), x] \in L_{1,\infty}$ and moreover we have the estimate

$$\|[f(A), x]\|_{1,\infty} \leq C_{abs} \|[A, x]\|_1.$$

The question whether the rank 1 condition could be removed remained open for quite some time. The question was resolved in [CPSZ18] (see also [CPSZ15] for the absolute value map). The proof is based on two important ingredients: (1) Parcet's semi-commutative Calderón-Zygmund theorem [Par09], [Cad17], (2) a De Leeuw theorem for $L_{1,\infty}$-spaces which was implicitly proved in [CPSZ18].

The aim of this text is to prove this De Leeuw theorem more explicitly and in a more general context of symmetric spaces (with conditions), see [DDP89]. More precisely, we show that for $\mathbb{Z}^n$ as a discrete subgroup of $\mathbb{R}^n$ and $\| \, \|_\star$ a suitable norm on both $L_\infty(\mathbb{R}^n)$ and $L_\infty(\mathbb{T}^n)$ (to be made precise) we get for $m \in L_\infty(\mathbb{R}^n)$ smooth that,

$$\|T_m : L_1(\mathbb{T}^n) \to L_\star(\mathbb{T}^n)\| \preceq \|T_m : L_1(\mathbb{R}^n) \to L_\star(\mathbb{R}^n)\|.$$

We also prove the analogous statement in the completely bounded setting. Natural examples of such norms $\| \, \|_\star$ are weak $L_1$-norms and $M_{1,\infty}$-norms. We state some open questions in this direction in Section 4.

## 2. Preliminaries and notation

### 2.1. General notation

For von Neumann algebra theory we refer to [Tak02]. We write $\prec$ for an inequality that holds up to some constant and $\prec_n$ for an inequality that holds up to a constant only depending on the dimension $n$. Let $B_r \in \mathbb{R}^n$ be the ball with radius $r$. For a function $f : \mathbb{R}^n \to \mathbb{C}$ and $s \in \mathbb{R}^n$ we set $f_s(t) = f(t - s)$. The circle $\mathbb{T}$ is identified with the unit circle in $\mathbb{C}$ and is equipped with the Haar measure with total mass $2\pi$. We set harmonic functions $e_\gamma(z) = z^\gamma, \gamma \in \mathbb{Z}, z \in \mathbb{T}$. We say that a function $\widehat{f} = \sum_{\gamma \in F} c_\gamma e_\gamma, c_\gamma \in \mathbb{C}$ on $\mathbb{T}$ has finite frequency support if $F$ is finite.

### 2.2. Symmetric spaces

Let $\mathcal{N}$ be a semi-finite von Neumann algebra with faithful, normal tracial weight $\tau : \mathcal{N}^+ \to [0, \infty]$. Let $T(\mathcal{N})$ be the space of closed, densely defined operators that are affiliated with $\mathcal{N}$. A closed densely defined operator $x$ is in $T(\mathcal{N})$ if for the polar decomposition $x = u|x|$ we have that $u \in \mathcal{N}$ and all spectral projections $E_{[0,\lambda]}(|x|), \lambda > 0$ are in $\mathcal{N}$. Let $T_\tau(\mathcal{N})$ be the subspace of $\tau$-measurable operators, namely all $x \in \mathcal{N}$ for which there exists a $\tau$-finite projection $p \in \mathcal{N}$ such that $xp$ is bounded. $T_\tau(\mathcal{N})$ forms a $*$-algebra with respect to the strong sum and strong product (closures of sums and products).

For $x \in T_\tau(\mathcal{N})$ we define the decreasing rearrangement function,

$$\mu_t(x) = \inf \{ \|xp\| \mid p \in \mathcal{N} \text{ projection with } \tau(1-p) \leq t \} .$$

We have $\mu_t(x) = \mu_t(|x|)$. In this paper we consider quasi-norms $\| \, \|_\star$ where $\star$ is to be specified. Our prime examples are,

$$L_{1,\infty}(\mathcal{N}) = \{ x \in T_\tau(\mathcal{N}) \mid \|x\|_{1,\infty} < \infty \},$$

where

$$\|x\|_{1,\infty} = \sup_{t>0} t\mu_t(x), \qquad x \in T_\tau(\mathcal{N}).$$

And further,

$$M_{1,\infty}(\mathcal{N}) = \{ x \in T_\tau(\mathcal{N}) \mid \|x\|_{M_{1,\infty}} < \infty \},$$

where

$$\|x\|_{M_{1,\infty}} = \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(x) ds.$$

We have relations,

$$\|x\|_{M_{1,\infty}} \leq \|x\|_{1,\infty} \leq \|x\|_1, \qquad x \in T_\tau(\mathcal{N}).$$

In general, we take $\| \, \|_\star : T_\tau(\mathcal{N}) \to [0, \infty]$ and set

$$L_\star(\mathcal{N}) = \{ x \in T_\tau(\mathcal{N}) \mid \|x\|_\star < \infty \},$$

which is assumed to be a vector space. Moreover, we restrict $\| \, \|_\star$ to $L_\star(\mathcal{N})$ and assume it has the following properties.

**Assumptions on $\| \, \|_\star$.**

1. $\| \, \|_\star$ is a (non-degenerate) quasi-norm. In particular, there is a constant $K \geq 1$ such that we have the quasi-triangle inequality,

$$\|x + y\|_\star \leq K(\|x\|_\star + \|y\|_\star), \qquad \forall x, y \in L_\star(\mathcal{N}).$$

2. $\| \|_\star$ satisfies the following dilation invariance property. There exists a constant $K > 0$ such that,
$$K^{-1}\lambda \|x\|_\star \leq \|y\|_\star \leq K\lambda\|x\|_\star.$$
for any $\lambda > 0$ and for any $x, y \in L_\star(\mathcal{N})$ such that $\mu_t(x) = \mu_{\lambda t}(y)$.
3. $L_\star(\mathcal{N})$ is complete with respect to $\| \|_\star$.

Property (2) implies that (up to a constant) the quasi-norm $\| \|_\star$ is rearrangement invariant. Below we need to make a stronger assumption on the norm to relate norms of different spaces.

It is clear that the $L_{1,\infty}$-quasi-norm satisfies these properties (see also [Ran05, Lemma 1.4]). Further, so does the $M_{1,\infty}$-quasi norm by the following lemma.

**Lemma 2.1.** $\| \|_{M_{1,\infty}}$ *satisfies properties* (1), (2) *and* (3) *above.*

*Proof.* By [CGRS12, Theorem 2.1] (see also [CRSS07, Theorem 4.5]) the $M_{1,\infty}$-quasi norm on $T_\tau(\mathcal{N})$ is equivalent to the norm,
$$\|x\|_{\mathcal{Z}_1} = \sup_{p>1}(p-1)\|x\|_p, \qquad x \in \mathcal{N}.$$
Suppose that $x$ and $y$ are elements of $\mathcal{N}$ and suppose there exists $\lambda > 0$ such that for the decreasing rearrangements for every $t > 0$ we have $\mu_t(x) = \mu_{\lambda t}(y)$. Then because the $\| \|_p$-norm has the property that $\|x\|_p = \lambda^{1/p}\|y\|_p$ we find that $\|x\|_{\mathcal{Z}_1} = \|y\|_{\mathcal{Z}_1}$. By equivalence of $\mathcal{Z}_1$- and $M_{1,\infty}$-norms this shows that there exists an absolute constant $K > 0$ such that
$$K^{-1}\lambda\|x\|_{M_{1,\infty}} \leq \|y\|_{M_{1,\infty}} \leq K\lambda\|x\|_{M_{1,\infty}}.$$
Completeness of $\mathcal{Z}_1$ (and hence $M_{1,\infty}$) follows from completeness of $L_p$-spaces and the quasi-norm property may be derived from [Ran05, Lemma 1.4]. □

*Remark* 2.2. Suppose that for all $x \in \mathcal{N}$ we have that $\|x\|_p \leq \|x\|_\star$ for some $1 \leq p \leq \infty$. Then $p = 1$ because otherwise (2) would be violated. This shows that the spaces to which our De Leeuw theorems apply can be viewed as end-point spaces at $p = 1$.

### 2.3. Fourier multipliers

Let $\mathsf{G}$ be a locally compact abelian group. We shall only be concerned with $\mathsf{G} = \mathbb{R}^n$ and $\mathsf{G} = \mathbb{T}^n$. Let $\widehat{\mathsf{G}}$ be the Pontrjagin dual group of characters. So $\widehat{\mathbb{R}^n} = \mathbb{R}^n$ and $\widehat{\mathbb{Z}^n} = \mathbb{T}^n$. Further, let $\mathcal{F}$ be the Fourier transform $L_2(\mathsf{G}) \to L_2(\widehat{\mathsf{G}})$. Consider a function $m \in L_\infty(\mathsf{G})$ and set $T_m : L_2(\mathsf{G}) \to L_2(\mathsf{G})$ by $T_m = \mathcal{F}^{-1} \circ m \circ \mathcal{F}$ where we view $m$ as a multiplication operator. Concretely, $T_m = m(\nabla)$ where $\nabla = -i(\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n})$ is the gradient operator on either $\mathsf{G} = \mathbb{R}^n$ or $\mathsf{G} = \mathbb{T}^n$.

If the operator $T_m$ extends to a bounded map $L_p(\mathsf{G}) \to L_p(\mathsf{G})$ for some $1 \leq p < \infty$ then we call $m$ an $L_p$-Fourier multiplier or briefly an $L_p$-multiplier. We have the translational behaviour of an $L_p$-multiplier $m \in L_\infty(\mathbb{R}^n)$:
$$T_m(fe_s) = T_{m_s}(f)e_s, \qquad s \in \mathbb{R}^n, f \in L_p(\mathbb{R}^n), \tag{2.1}$$

with $m_s(t) = m(t-s)$. Similarly, we will say that $m$ is an $L_1 - L_\star$ multiplier if for every $f \in L_1 \cap L_2$ we have that $T_m(f) \in L_\star$ and further there exists a constant $K > 0$ such that $\|T_m(f)\|_\star \leq K\|f\|_1$. The infimum over such $K > 0$ is then the norm $\|T_m : L_1 \to L_\star\|$.

*Remark* 2.3. Multipliers $L_1 \to L_{1,\infty}$ are extensively studied in the context of Calderón-Zygmund theory [Gra04]. The (weaker) multipliers $L_1 \to M_{1,\infty}$ naturally occur in problems finding the best constants of certain commutative and non-commutative estimates. See [CMPS14, Corollary 5.6] for such an application.

## 3. De Leeuw restriction theorems at the endpoints

### 3.1. Symmetric quasi-norms

Let $\mathcal{M}$ be a semi-finite von Neumann algebra. In this section we presume that $\| \, \|_\star$ is a quasi-norm on either $L_\star(\mathcal{M} \otimes L_\infty(\mathbb{R}^n))$ and $L_\star(\mathcal{M} \otimes L_\infty(\mathbb{T}^n))$ satisfying the properties (1) and (2). We will not distinguish in notation to view $\| \, \|_\star$ on either $L_\star(\mathcal{M} \otimes L_\infty(\mathbb{R}^n))$ or $L_\star(\mathcal{M} \otimes L_\infty(\mathbb{T}^n))$. However, we will impose the following natural condition relating quasi-norms of different spaces.

**Assumption:** If $x \in L_\star(\mathcal{M} \otimes L_\infty(\mathbb{R}^n))$ and $y \in L_\star(\mathcal{M} \otimes L_\infty(\mathbb{T}^n))$ have decreasing rearrangements such that $\mu_t(x) \leq \mu_t(y)$ then $\|x\|_\star \leq \|y\|_\star$. In particular, the quasi-norms on the spaces $L_\star(\mathcal{M} \otimes L_\infty(\mathbb{R}^n))$ and $L_\star(\mathcal{M} \otimes L_\infty(\mathbb{T}^n))$ themselves are rearrangement invariant (i.e. only depend on the decreasing rearrangement).

For functions $f_i : \mathbb{R}^{n_i} \to \mathbb{C}$ we write $f_1 \otimes f_2 : \mathbb{R}^{n_1+n_2} \to \mathbb{C}$ for $(f_1 \otimes f_2)(s_1, s_2) = f_1(s_1)f_2(s_2)$. Let $G_k^{(n)} : \mathbb{R}^n \to \mathbb{C}$ be the $L_1$-normalized Gaussian function set by

$$G_k(\xi) = \frac{1}{k\sqrt{\pi}} \exp(-|\xi|^2/k^2), \qquad \xi \in \mathbb{R}.$$

and then $G_k^{(n)} = G_k^{\otimes n}$. For $f : \mathbb{T}^n \to \mathcal{M}$ set the periodization $\mathrm{per}(f) : \mathbb{R}^n \to \mathcal{M}$ by $f(x) = f(x \bmod 2\pi)$ where we identify $\mathbb{T}^n$ with the block interval $[0, 2\pi)^n$ and the mod $2\pi$ is taken coordinate-wise. Then set,

$$\Phi_k(f)(t) = \mathrm{per}(f)(t)G_k^{(n)}(t), \qquad k \in \mathbb{N}_{\geq 1}, t \in \mathbb{R}^n.$$

We say that an element $x \in \mathcal{M} \otimes L_\infty(\mathbb{T}^n)$ has finite frequency support if $x = \sum_{\gamma \in F} x_\gamma \otimes e_\gamma$ with $F$ finite.

**Lemma 3.1.** *For $x \in L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^n))$ we have that $\Phi_k(x) \in L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{R}^n))$. Moreover, $\|x\|_{1,\infty} \prec_n \|\Phi_k(x)\|_{1,\infty}$.*

*Proof.* As $\Phi_k(|x|) = |\Phi_k(x)|$ we may assume without loss of generality that $x \geq 0$ by considering $|x|$ instead. Set constants $c_n = \pi^{n/2}e^{-1}$. We estimate,

$$c_n k^{-n} \sum_{l \in \mathbb{Z}^n, |l| \leq k-1} \chi_{2\pi l + [0,2\pi)^n} \leq G_k^{(n)},$$

so that,

$$c_n k^{-n} \sum_{l\in\mathbb{Z}^n, |l|\leq k-1} \chi_{2\pi l + [0,2\pi)^n} \mathrm{per}(x) \leq G_k^{(n)} \mathrm{per}(x).$$

Taking $\| \, \|_\star$-quasi-norms we find,

$$c_n k^{-n} \| \sum_{l\in\mathbb{Z}^n, |l|\leq k-1} \chi_{2\pi l + [0,2\pi)^n} \mathrm{per}(x) \|_\star \leq \| G_k^{(n)} \mathrm{per}(x) \|_\star = \| \Phi_k(x) \|_\star. \qquad (3.1)$$

Set,

$$\lambda_n = \#\{ l \in \mathbb{Z}^n \mid |l| \leq k-1 \}.$$

Note that $\lambda_n \approx (k-1)^n$. Then,

$$\mu_t(c_n k^{-n} \sum_{l\in\mathbb{Z}^n, |l|\leq k-1} \chi_{2\pi l + [0,2\pi)^n} \mathrm{per}(x)) = c_n k^{-n} \mu_{\lambda_n^{-1} t}(x).$$

Therefore, by our assumptions on $\| \, \|_\star$ the left hand side of (3.1) is up to a constant equal to,

$$c_n k^{-n} \lambda_n \|x\|_\star \approx c_n k^{-n} (k-1)^n \|x\|_\star.$$

As $k^{-n}(k-1)^n \to 1$ if $k \to \infty$ this concludes that for all $k$ we have $c_n \|x\|_{1,\infty} \prec \|\Phi_k(x)\|_{1,\infty}$. This concludes the lemma. □

**Lemma 3.2.** *For $x \in L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^n))$ we have that $\Phi_k(x) \in L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^n))$. Moreover, $\lim_k \|\Phi_k(x)\|_1 = \|x\|_1$.*

*Proof.* Without loss of generality we may assume that $x$ is positive by replacing it with $|x|$. For $l \in \mathbb{Z}^n$ we let again $A_l = 2\pi l + [0, 2\pi)^n \subseteq \mathbb{R}^n$. We set the functions $G_k^+$ and $G_k^-$ on $\mathbb{Z}^n$ by,

$$G_k^+(l) = \max_{s\in A_l} G_k^{(n)}(s), \qquad G_k^-(l) = \min_{s\in A_l} G_k^{(n)}(s).$$

We estimate,

$$\sum_{l\in\mathbb{Z}^n} \mathrm{per}(x) \chi_{A_l} G_k^-(l) \leq \Phi_k(x) \leq \sum_{l\in\mathbb{Z}^n} \mathrm{per}(x) \chi_{A_l} G_k^+(l). \qquad (3.2)$$

And so the same estimates hold after taking the $L_1$-norm. We further get,

$$\| \sum_{l\in\mathbb{Z}^n} \mathrm{per}(x) \chi_{A_l} G_k^\pm(l) \|_1 = \| \bigoplus_{l\in\mathbb{Z}^n} G_k^\pm(l) x \|_{L_1(\oplus_{l\in\mathbb{Z}^n} \mathcal{M}\otimes L_\infty(\mathbb{T}^n))} = \|x\|_1 \sum_{l\in\mathbb{Z}^n} G_k^\pm(l).$$

Then as we have more and more refined approximations,

$$\sum_{l\in\mathbb{Z}^n} G_k^\pm(l) = \frac{1}{k^n} \sum_{l\in\mathbb{Z}^n} \max_{s\in A_l} G_1^\pm\left(\frac{s}{k}\right),$$

has limit $\|G_1^{(n)}\|_1 = 1$ for both $\pm$ either $+$ or $-$. We conclude from (3.2) that

$$\|x\|_1 \leq \lim_k \|\Phi_k(x)\|_1 \leq \|x\|_1.$$

□

We define the Fourier algebra (see also [Eym64]) as,

$$\mathcal{A}(\mathbb{R}^n) = \mathcal{F}(L_1(\widehat{\mathbb{R}^n})).$$

So $\mathcal{A}(\mathbb{R}^n)$ consists of all functions in $C_0(\mathbb{R}^n)$ whose (distributional) Fourier transform lies in $L_1(\mathbb{R}^n)$. Every $m \in \mathcal{A}(\mathbb{R}^n)$ is in particular an $L_1 \to L_1$ Fourier multiplier and $T_m(f) = \widehat{m} * f$ with $\widehat{m} = \mathcal{F}(m)$. Moreover, such multipliers are completely bounded and therefore for any (semifinite) von Neumann algebra $\mathcal{M}$ we obtain a bounded map $(\mathrm{id}_{\mathcal{M}} \otimes T_m) : L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^n)) \to L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^n))$, see also [CaHa85] for these results as well as far reaching generalizations. For $m \in C_0(\mathbb{R}^n)$ we write $m_d \in c_0(\mathbb{Z}^n)$ for its discretization $m_d = m|_{\mathbb{Z}^n}$ (restriction to $\mathbb{Z}^n$).

**Proposition 3.3.** *Let $m \in \mathcal{A}(\mathbb{R}^n)$ and let $x \in L_\infty(\mathbb{T}^n, \mathcal{M}) \simeq \mathcal{M} \otimes L_\infty(\mathbb{T}^n)$ have finite frequency support. We have that,*

$$\| \Phi_k((id_{\mathcal{M}} \otimes T_{m_d})(x)) - (id_{\mathcal{M}} \otimes T_m)(\Phi_k(x)) \|_1 \to 0.$$

*Proof.* By the triangle inequality for $\| \cdot \|_1$ and the translation behaviour (2.1) of Fourier multipliers we may assume without loss of generality that $x = x_0 \otimes e_0, x_0 \in \mathcal{M}$. In fact, this shows that we may assume that $\mathcal{M} = \mathbb{C}$ and $x = e_0$. Suppose that $\widehat{m} = \mathcal{F}(m)$ is positive and $\|\widehat{m}\|_1 = 1$. Each multiplier in $\mathcal{A}(\mathbb{R}^n)$ may be written as a linear combination of four positive multipliers in $\mathcal{A}(\mathbb{R}^n)$. It follows that $m(0) = \|\widehat{m}\|_1 = 1$. So it remains to prove that,

$$\|G_k^{(n)} - T_m(G_k^{(n)})\|_1 \to 0.$$

Let $\varepsilon > 0$ and let $B_r \subseteq \mathbb{R}^n$ be a ball such that $\|\widehat{m}\chi_{B_r} - \widehat{m}\|_1 \leq \varepsilon$. Set $\widehat{m}_r = \widehat{m}\chi_{B_r}$. Then,

$$\|(\widehat{m} - \widehat{m}_r) * G_k^{(n)}\|_1 \leq \|\widehat{m} - \widehat{m}_r\|_1 \|G_k^{(n)}\|_1 \leq \varepsilon.$$

Further, we have

$$T_{m_r}(G_k^{(n)}) = \widehat{m}_r * G_k^{(n)},$$

and

$$\min_{|s| \leq r} G_k^{(n)}(t+s) \leq (\widehat{m}_r * G_k^{(n)})(t) \leq \max_{|s| \leq r} G_k^{(n)}(t+s).$$

We therefore get, with $G_k'$ the gradient of $G_k$,

$$G_k^{(n)}(t) - r \max_{|s| \leq r} |G_k^{(n)'}(t+s)| \leq (\widehat{m}_r * G_k^{(n)})(t) \leq G_k^{(n)}(t) + r \max_{|s| \leq r} |G_k^{(n)'}(t+s)|.$$

We find,

$$|G_k^{(n)} - \widehat{m}_r * G_k^{(n)}| \leq r \max_{|s| \leq r} |G_k^{(n)'}(\,\cdot\, + s)|.$$

Further, the right hand side of this inequality converges to 0 in the $\| \cdot \|_1$-norm as $k \to \infty$. In all we conclude that for $\varepsilon > 0$ for $k$ large we have,

$$\begin{aligned} \|G_k^{(n)} - T_m(G_k^{(n)})\|_1 \leq & \|G_k^{(n)} - T_{m_r}(G_k^{(n)})\|_1 + \|(T_m - T_{m_r})(G_k^{(n)})\|_1 \\ \leq & \|G_k^{(n)} - \widehat{m}_r * G_k^{(n)}\|_1 + \|\widehat{m} - \widehat{m}_r\|_1 \leq 2\varepsilon. \end{aligned}$$

□

Now we arrive at the following De Leeuw restriction theorem.

**Theorem 3.4.** *Let $m \in \mathcal{A}(\mathbb{R}^n)$ and let $m_d = m|_{\mathbb{Z}^n} \in c_0(\mathbb{Z}^n)$ be its restriction. If $id_{\mathcal{M}} \otimes T_m$ is bounded as a multiplier $L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^n)) \to L_\star(\mathcal{M} \otimes L_\infty(\mathbb{R}^n))$. Then also $id_{\mathcal{M}} \otimes T_{m_d}$ is bounded as a multiplier $L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^n)) \to L_\star(\mathcal{M} \otimes L_\infty(\mathbb{T}^n))$. Further,*

$$\begin{aligned} &\| id_{\mathcal{M}} \otimes T_{m_d} : L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^n)) \to L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^n)) \| \\ &\quad \prec \| id_{\mathcal{M}} \otimes T_m : L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^n)) \to L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{R}^n)) \|. \end{aligned}$$

*Proof.* Let $x \in L_1(\mathcal{M} \otimes \mathbb{T}^n)$ with finite frequency support. We get by Lemma 3.1, Proposition 3.3 and Lemma 3.2 respectively that,

$$\begin{aligned} \| (\mathrm{id}_{\mathcal{M}} \otimes T_{m_d})(x) \|_{1,\infty} \prec & \limsup_k \| \Phi_k((\mathrm{id}_{\mathcal{M}} \otimes T_{m_d})(x)) \|_{1,\infty} \\ \prec & \limsup_k \| (\mathrm{id}_{\mathcal{M}} \otimes T_m)(\Phi_k(x)) \|_{1,\infty} \\ & \quad + \quad \| \Phi_k((\mathrm{id}_{\mathcal{M}} \otimes T_{m_d})(x)) - (\mathrm{id}_{\mathcal{M}} \otimes T_m)(\Phi_k(x)) \|_{1,\infty} \\ \leq & \limsup_k \| (\mathrm{id}_{\mathcal{M}} \otimes T_m)(\Phi_k(x)) \|_{1,\infty} \\ & \quad + \quad \| \Phi_k((\mathrm{id}_{\mathcal{M}} \otimes T_{m_d})(x)) - (\mathrm{id}_{\mathcal{M}} \otimes T_m)(\Phi_k(x)) \|_{1} \\ = & \limsup_k \| (\mathrm{id}_{\mathcal{M}} \otimes T_m)(\Phi_k(x)) \|_{1,\infty} \\ \leq & \| (\mathrm{id} \otimes T_m) : L_1 \to L_{1,\infty} \| \limsup_k \| \Phi_k(x) \|_1 \\ = & \| T_m : L_1 \to L_{1,\infty} \| \| x \|_1. \end{aligned} \tag{3.3}$$

This concludes our proof from the following density argument. The elements in $L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^n))$ with finite frequency support are dense in $L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^n))$ (this follows directly from Fejér's theorem [CSZ18, Lemma A.2] for example). Then let $x \in L_1(\mathcal{M} \otimes \mathbb{T}^n)$ and let $x_n \in L_1(\mathcal{M} \otimes \mathbb{T}^n)$ be such that $x_n \to x$ and $x_n$ having finite frequency support. Then by (3.3) we see that $(\mathrm{id}_{\mathcal{M}} \otimes T_{m_d})(x_n)$ is Cauchy and hence converges within $L_\star(\mathcal{M} \otimes L_\infty(\mathbb{T}^n))$ which is assumed to be complete. Say that the limit is $T(x)$. If $x$ is also in $L_2(\mathcal{M} \otimes \mathbb{T}^n)$ then also the approximating sequence $x_n$ can be taken in $L_2(\mathcal{M} \otimes \mathbb{T}^n)$ ([CSZ18, Remark A.1]). Then $T(x) = (\mathrm{id}_{\mathcal{M}} \otimes T_m)(x)$. So the multiplier is an extension of the $L_2$-multiplier. □

We extend the result to smooth multipliers.

**Theorem 3.5.** *Let $m \in L_\infty(\mathbb{R}^n \backslash \{0\})$ be smooth on $\mathbb{R}^n \backslash \{0\}$ and let $m_d = m|_{\mathbb{Z}^n} \in \ell^\infty(\mathbb{Z}^n)$ be its restriction. If $id_{\mathcal{M}} \otimes T_m$ is bounded as a multiplier $L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^n)) \to L_\star(\mathcal{M} \otimes L_\infty(\mathbb{R}^n))$. Then also $id_{\mathcal{M}} \otimes T_{m_d}$ is bounded as a multiplier $L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^n)) \to L_\star(\mathcal{M} \otimes L_\infty(\mathbb{T}^n))$. Moreover,*

$$\begin{aligned} &\| id_{\mathcal{M}} \otimes T_{m_d} : L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^n)) \to L_\star(\mathcal{M} \otimes L_\infty(\mathbb{T}^n)) \| \\ &\quad \prec \| id_{\mathcal{M}} \otimes T_m : L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^n)) \to L_\star(\mathcal{M} \otimes L_\infty(\mathbb{R}^n)) \|. \end{aligned}$$

*Proof.* Let $\varphi_k$ be Schwartz functions such that $\varphi_k(0) = 0$, such that for all $s \in B_k \backslash B_{\frac{1}{k}}$ we have $\varphi_k(s) = 1$ and finally such that $\|\widehat{\varphi}_k\|_1$ is uniformly bounded in $k$, say by a constant $A$. Such functions exist as was justified in [CPSZ18, Footnote 5]. We have that,

$$\begin{aligned} \|T_{m\varphi_k} : L_1 \to L_\star\| \leq & \|T_{\varphi_k} : L_1 \to L_1\| \|T_m : L_1 \to L_\star\| \\ \leq & \|\widehat{\varphi}_k\| \|T_m : L_1 \to L_\star\|. \end{aligned}$$

By construction $m\varphi_k$ is Schwartz. Let $x \in L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^n))$ with finite frequency support. Let $\mathcal{E}_0$ be the conditional expectation of $L_\infty(\mathbb{T}^n)$ onto $\mathbb{C}1_{\mathbb{T}^n}$. It extends to a complete contraction of $L_1(\mathbb{T}^n)$ onto $\mathbb{C}1_{\mathbb{T}}$. Set $\mathcal{E}_0(x) = x_0$ and $x_1 = x - x_0$. So,

$$\|(\mathrm{id}_{\mathcal{M}} \otimes T_{m_d})(x)\|_\star \leq \|(\mathrm{id}_{\mathcal{M}} \otimes T_{m_d})(x_0)\|_\star + \|(\mathrm{id}_{\mathcal{M}} \otimes T_{m_d})(x_1)\|_\star.$$

We have that,

$$\|(\mathrm{id}_{\mathcal{M}} \otimes T_{m_d})(x_0)\|_\star \prec |m_d(0)| \|x_0\|_1 \leq |m_d(0)| \|x\|_1.$$

Further, we get for $k$ large such that the frequency support of $x_1$ lies in $B_k$ that

$$\begin{aligned} \|(\mathrm{id}_{\mathcal{M}} \otimes T_{m_d})(x_1)\|_\star = & \|(\mathrm{id}_{\mathcal{M}} \otimes T_{m_d\varphi_k})(x_1)\|_\star \\ \prec & \|(\mathrm{id}_{\mathcal{M}} \otimes T_{m\varphi_k}) : L_1 \to L_\star\| \|x_1\|_1 \\ \prec & \|\widehat{\varphi}_k\| \|(\mathrm{id}_{\mathcal{M}} \otimes T_m) : L_1 \to L_\star\| \|x_1\|_1 \\ \leq & A \|(\mathrm{id}_{\mathcal{M}} \otimes T_m) : L_1 \to L_\star\| \|x\|_1. \end{aligned}$$

We conclude by density of the functions with finite frequency support in $L_1(\mathbb{R}^n)$, just as in the proof of Theorem 3.4. □

We single out the commutative case.

**Corollary 3.6.** *Let $m \in L_\infty(\mathbb{R}^n \backslash \{0\})$ be smooth on $\mathbb{R}^n \backslash \{0\}$ and let $m_d = m|_{\mathbb{Z}^n} \in \ell^\infty(\mathbb{Z}^n)$ be its restriction. If $T_m$ is $L_1 \to L_\star$-bounded then $T_{m_d}$ is $L_1 \to L_\star$-bounded. Moreover,*

$$\|T_{m_d} : L_1(\mathbb{T}^n) \to L_\star(\mathbb{T}^n)\| \prec \|T_m : L_1(\mathbb{R}^n) \to L_\star(\mathbb{R}^n)\|.$$

## 4. Open questions

We conclude this paper with a couple of questions which we believe are interesting.

*Question.* Does a De Leeuw theorem hold for general multipliers $m \in L_\infty(\mathbb{R}^n)$ of weak type $(1, 1)$? That is, can one drop additional (smoothness) assumptions as in Theorems 3.4 and 3.5.

*Question.* The classical de Leeuw theorem, the constant in Theorem 3.4 that is incorporated in the symbol $\prec$ is in fact 1, i.e. one gets a true inequality $\leq$. We do not know if this is true in the weak type $(1, 1)$ case.

*Question.* Naturally the question arises if the lattice approximation and the compactification theorem of De Leeuw hold at the endpoints.

*Question.* In [CPPR15] a non-commutative De Leeuw restriction theorem was proved for multipliers $T_m$ acting on $L_p(\widehat{\mathsf{G}})$. Here $\mathsf{G}$ is a locally compact group and $L_p(\widehat{\mathsf{G}})$ is the non-commutative $L_p$-space of its group von Neumann algebra. If $\Gamma$ is a discrete subgroup of $\mathsf{G}$ and $\mathsf{G}$ has small almost invariant neighbourhoods with respect to $\Gamma$ (see [CPPR15] for the precise definition) then a De Leeuw theorem holds for $L_p$-spaces. The weak $(1,1)$ restriction theorem for any class of multipliers on such non-commutative spaces is open. For example, is there a De Leeuw restriction theorem for weak $(1,1)$ type multipliers for the Heisenberg group?

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Martijn Caspers
TU Delft, EWI/DIAM, P.O.Box 5031, 2600 GA Delft, The Netherlands
e-mail: `m.p.t.caspers@tudelft.nl`