Gradient forms and strong solidity of free quantum groups

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The area of quantum groups is concerned with non-commutative versions of groups in the following sense. The celebrated Gelfand-Naimark theorem states that a topological compact Hausdorff space X can fully be understood in terms of the commutative C*-algebra of continuous functions $X \to \mathbb{C}$. This gives a categorical contragredient duality between compact Hausdorff spaces and commutative C*-algebras. Therefore C*-algebras are often considered as non-commutative spaces, eventhough there might not be a single point visible in such C*-algebras.

In the same spirit a compact quantum group is the non-commutative geometric analogue of a compact group (or even locally compact groups are part of the theory). A suitable framework of C*-algebraic compact quantum groups was found by Woronowicz in the late 1980's. In this framework all examples that were known at that time could be incorporated. Most notably $SU_q(2)$, the quantum $SU(2)$ group is a compact quantum group.

Two important classes of quantum groups were found by Wang and van Daele [14]. They arise as libertations of the group of orthogonal matrices, or unitary matrices. Their quantum versions are well-studied nowadays and are called the *free orthogonal quantum groups* and the *free unitary quantum groups*. As a C*-algebra they are generated by the operators $u_{i,j}, 1 \leq i,j \leq N$ such that the matrix $u = (u_{i,j})_{1 \leq i,j \leq N}$ is unitary and such that $\overline{u} = u$, where the bar denotes the entrywise adjoint. A comultiplication (the analogue of group multiplication) is given by $\Delta(u_{i,j}) = \sum_k u_{i,k} \otimes u_{k,j}$. We refer to this quantum group as O_N^+. In fact more general Q-deformations of this quantum group can be defined, but we do not give details here. Through a canonical GNS-construction the C*-algebra of O_N^+ generates a von Neumann algebra (a non-commutative measure space) which we call $L_\infty(O_N^+)$ (the reader should keep in mind that these are thus non-commutative algebras).

Ever since its introduction the interest in O_N^+ has been quite large, especially because of its remarkable analogues with the free group factors. Many results for its C*-algebra and von Neumann algebra have been obtained, in particular in the past 5 years. There are factoriality results, approximation properties, Cartan algebras, results on the Baum-Connes conjecture, the Connes embedding problem, etc. Though that $L_\infty(O_N^+)$ shares many properties with the free group factors, by recent results of Brannan-Vergnioux [3] we know that they are non-isomorphic. In particular, this renders the investigation of the above properties non-trivial.

In [9] and [10] it was proved that $L_\infty(O_N^+)$ is strongly solid. This means that if $A \subseteq L_\infty(O_N^+)$ is an amenable diffuse von Neumann subalgebra then the normalizing algebra generated by all unitaries $u \in L_\infty(O_N^+)$ such that $uAu^* = A$ is amenable again. Amenability is a very strong property for von Neumann algebras: Connes [8] showed that amenable von Neumann algebras can be approximated by matrix algebras in a strong sense and based on this can be classified. Much later Ozawa and Popa [11] introduced this notion of strong solidity to find a new proof that free group factors do not have Cartan subalgebras. Strong solidity then became a standard tool in the theory and many techniques have been introduced to show that specific classes of von Neumann algebras possess this property.

The strong solidity results by Fima-Vergnioux [9] and Isono [10] hold for the tracial versions of $L_\infty(O_N^+)$. But as mentionned above these algebras have Q-deformations that renders them into non-tracial algebras; in fact in many cases we know that they are non-amenable type III factors [15]. The strong solidity question there remained open.

What we show in this talk is that also the type III deformations of our algebras have the property of strong solidity. This is our main result.

Let us comment here on some ingredients of the proof. All approaches to strong solidity are based on a two step strategy (as in [11]): (1) one needs a notion of 'weak compactness' of actions of the (stable) normalizing algebra and (2) one needs to construct a 'deformation' of the algebra. Point (1) of the proof we can overcome through techniques recently introduced in [1]. For (2) we introduce a new deformation based on earlier results by Peterson [13] and again Ozawa-Popa [12]. The deformation is based on the construction of a derivation from results of Cipriani-Sauvageot [6] in combination with very recent results on the Haagerup property of $L_\infty(O_N^+)$ [2], [5], [4].

The result completes the strong solidity question for O_N^+. In fact we show that our results also apply to some other quantum groups. This involves U_N^+, the free unitary quantum groups, as well as some free products of these algebras.

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