Hecke von Neumann algebras, operator spaces and absence of Cartan subalgebras

Martijn Caspers

Hecke algebras arise as q-deformations of Coxeter groups. As a $*$-algebra they are generated by self-adjoint operators T_s with s in some generating set S, that satisfy a relation $(T_sT_t)^{m(s,t)} = 1$ as well as the Hecke relation

$$(T_s - q)(T_s + 1) = 0,$$

for a deformation parameter $q > 0$. Hecke algebras were used by Jones in his knot invariants and play important roles in representation theory. Taking a GNS-representation, these algebras generate a so-called Hecke von Neumann algebra, say $\mathcal{M}_q$. These were studied earlier on by Dymara [3] and later by Davis–Dymara–Januszkiewicz–Okun [4]. In the right-angled case it was proved by Garncarek [5] that $\mathcal{M}_q$ is a factor for a certain range of $q \in [\rho, \rho^{-1}]$ and otherwise it is a sum of a factor and $\mathbb{C}$. Outside the right-angled case this question is still open. We take Garncarek's result as a starting point and assume that $\mathcal{M}_q$ is the Hecke von Neumann algebra of a right-angled Coxeter system.

The aim of this talk is to explain how operator spaces can be used to study the von Neumann algebra $\mathcal{M}_q$. In particular we are able to show that these algebras are non-injective, have the completely bounded approximation property (CBAP) and are strongly solid algebras. For example recall that a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ is injective if it is the image of a conditional expectation $B(\mathcal{H}) \to \mathcal{M}$. It follows from Connes' characterization of injectivity that an injective II_1-factor $\mathcal{M}$ with trace τ has the property that for all $x_i \in \mathcal{M}$ we have

$$\| \sum_i x_i \otimes \overline{x_i} \|_{\mathcal{M} \otimes \mathcal{M}^{op}} \geq \tau(\sum_i x_i^* x_i).$$

We can violate this inequality for $\mathcal{M}_q$ as it is possible to identify Σ_d, the linear span of T_w with w a word with letters in S of length d, into a finite sum of Haagerup tensor products of row and column Hilbert spaces,

$$\mathcal{H}_c \otimes_h \mathcal{H}_c \otimes_h \mathbb{C} \otimes_h \mathcal{H}_r \otimes_h \mathcal{H}_r.$$

Details can be found in [2]. Such techniques have been found in different contexts before in the study of free Araki-Woods factors or q-Gaussian algebras, see for instance [7].

Using explicit Stinespring decompositions for radial multipliers and word length cut-downs we show that $\mathcal{M}_q$ has the CBAP. Using the Gromov boundary action of the Coxeter group generated by S we are able to show that $\mathcal{M}_q$ is strongly solid.

Recall that a Cartan subalgebra of a von Neumann algebra is a maximal abelian subalgebra whose normalizer generates the von Neumann algebra itself. Cartan subalgebras typically arise in crossed products of free ergodic probability measure preserving actions of discrete groups on a probability measure space. Their study is central in classification programs for von Neumann algebras (for instance by

Popa–Vaes). On the other hand the group von Neumann algebra of a free group does not have a Cartan subalgebra as was shown by Voiculescu; moreover it is strongly solid as was shown by Ozawa–Popa. Later on other examples of von Neumann algebras with no Cartan subalgebra have been found, see for instance [1] or [6]. The conclusion of our results above is that also $\mathcal{M}_q$ does not have a Cartan subalgebra.

Acknowledgements. I thank the organizers of this workshop. Fruitful discussions, especially with T. Crisp, M. Kennedy and G. Pisier, helped me to put some of the material above into a wider context and may lead to follow-up projects.

References

[1] S. Avsec, *Strong solidity of the q-gaussian algebras for all* $1 < q < 1$, arXiv: 1110.4918.
[2] M. Caspers, *Absence of Cartan subalgebras for Hecke von Neumann algebras*, arXiv: 1601.00593.
[3] J. Dymara, *Thin buildings*, Geom. Topol. **10** (2006), 667–694.
[4] M. W. Davis, J. Dymara, T. Januszkiewicz, B. Okun, *Weighted L^2-cohomology of Coxeter groups*, Geom. Topol. **11** (2007), 47–138.
[5] L. Garncarek, *Factoriality of Hecke-von Neumann algebras of right-angled Coxeter groups*, J. Funct. Anal. **270** (2016), 1202–1219.
[6] Y. Isono, *Examples of factors which have no Cartan subalgebras*, Trans. Amer. Math. Soc. **367** (2015), 7917–7937.
[7] A. Nou, *Non injectivity of the q-deformed von Neumann algebra*, Math. Ann. **330** (2004), no. 1, 17–38.