NONCOMMUTATIVE DE LEEUW THEOREMS

MARTIJN CASPERS

Based on joint work with Javier Parcet, Mathilde Perrin, Eric Ricard.

1. Classical De Leeuw theorems

In 1965 Karel de Leeuw [4] proved fundamental theorems about Fourier multipliers acting on L^p-spaces. These theorems play a major role in commutative and noncommutative harmonic analysis and have many applications to for example partial differential equations. In order to state De Leeuw's main results recall the following. Let $m : \mathbb{R}^n \to \mathbb{C}$ be measurable and consider the linear mapping T_m that is determined by

$$\widehat{T_m f} = m\widehat{f},$$

where $f \mapsto \widehat{f}$ is the Fourier transform.

(1) **Restriction.** Let H be a subgroup of $\mathbb{R}^n$. Suppose that T_m acts boundedly on $L^p(\mathbb{R}^n)$ then the mapping

$$T_{m|_H} : \int_H \widehat{f}(h)\chi_h d\mu(h) \mapsto \int_H m(h)\widehat{f}(h)\chi_h d\mu(h)$$

extends to a $L^p(\widehat{H})$-bounded multiplier for any subgroup $H \subseteq \mathbb{R}^n$ where the χ_h's stand for the characters on the dual group $\widehat{H}$ and μ is the Haar measure.

(2) **Compactification.** Let $\mathbb{R}^n_{\text{Bohr}}$ be the Pontryagin dual of $\mathbb{R}^n_{\text{disc}}$ equipped with the discrete topology. Given $m : \mathbb{R}^n \to \mathbb{C}$ bounded and continuous, the $L^p(\mathbb{R}^n)$-boundedness of T_m is equivalent to the boundedness in $L^p(\mathbb{R}^n_{\text{Bohr}})$ of the multiplier with the same symbol,

$$T_m : \sum_{\mathbb{R}^n_{\text{disc}}} \widehat{f}(\xi)\chi_\xi \mapsto \sum_{\mathbb{R}^n_{\text{disc}}} m(\xi)\widehat{f}(\xi)\chi_\xi.$$

The proof of the compactification theorem proceeds through the restriction theorem. In fact the restriction theorem easily follows from the compactification theorem which therefore is the stronger statement. We also take into consideration periodization and lattice approximation (i.e. Igari's theorem [3]). After De Leeuw's fundamental paper [4] these theorems were soon generalized to nonabelian groups by Saeki [8].

2. Noncommutative De Leeuw theorems

The development of noncommutative integration theory (especially in the second half of the 20th century) naturally raises the question if there are noncommutative De Leeuw theorems. Noncommutative means that $\mathbb{R}^n$ can be replaced by an arbitrary locally compact group. In this case the Fourier multipliers T_m act on its Pontryagin dual, which only exists as a so-called quantum group, whose underying space is the group von Neumann algebra. Very recently a prolific series of papers was devoted to this topic, see [2], [5], [6], [7] and references given there.

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In [1] we show to what extent De Leeuw theorems can be generalized. Let G be a locally compact group and let $m : G \to \mathbb{C}$ which we shall always assume to be bounded and continuous. To avoid technicalities in our exposition here, we assume that G is unimodular. Let

$$\mathcal{L}(G) = \{\lambda(f) \mid f \in L^1(G)\}'',$$

be the group von Neumann algebra generated by the left regular representation λ. Let φ be the Plancherel weight on $\mathcal{L}(G)$ which is given by $\varphi(x^*x) = \|f\|_{L^2(G)}$ in case there exists f such that $xg = f * g, g \in L^2(G)$. Otherwise $\varphi(x^*x) = \infty$. As G is unimodular φ is tracial. We may construct noncommutative $L^p(\mathcal{L}(G))$ as the completion of the space $\{x \mid \|x\|_p := \tau(|x|^p)^{1/p} < \infty\}$ with respect to the $\| \, \|_p$ norm. It follows that $C_c(G)^{*2}$ (second convolution power) spans a dense subset of $L^p(\mathcal{L}(G))$ and that we may set

$$T_m : L^p(\mathcal{L}(G)) \to L^p(\mathcal{L}(G)) : \lambda(f) \mapsto \lambda(mf), \qquad m \in C_c(G)^{*2}.$$

We call m an *L^p-Fourier multiplier* in case T_m extends boundedly.

In [1] we prove the following theorems which involve two assumptions. We say that G has small almost invariant neighbourhoods with respect to a subgroup Γ if for every finite subset $F \subseteq \Gamma$ there exists a net of open sets $U_i \to \{e\}$ of G such that for all $s \in F$ we have $\mathrm{measure}(U_i \cap sU_is^{-1})/\mathrm{measure}(U_i) \to 1$. We say that G is approximable by discrete subgroups if there exists a net Γ_i of subgroups of G with fundamental domains shrinking to the identity. Our main results include:

(1) **Restriction.** Let $1 \leq p \leq \infty$. Let $\Gamma \subseteq G$ be a discrete subgroup and suppose that G has small almost invariant neighbourhoods with respect to Γ. Then,

$$\|T_{m|_H} : L^p(\mathcal{L}(H)) \to L^p(\mathcal{L}(H))\| \leq \|T_m : L^p(\mathcal{L}(G)) \to L^p(\mathcal{L}(G))\|,$$

(2) **Compactification.** Let $1 \leq p \leq \infty$. Suppose that G is approximable by discrete subgroups and that G_{disc} with the discrete topology is amenable. Then,

$$\|T_m : L^p(\mathcal{L}(G_{\mathrm{disc}})) \to L^p(\mathcal{L}(G_{\mathrm{disc}}))\| = \|T_m : L^p(\mathcal{L}(G)) \to L^p(\mathcal{L}(G))\|,$$

These theorems recover the classical De Leeuw theorems. The proof strategy is (in a suitable sense) the same as De Leeuw's. However the techniques are totally different and involve an intricate analysis of ucp maps on von Neumann algebras.

3. Final comments

We conclude with three remarks. Firstly in [1] we also prove noncommutative periodization and lattice approximation results as in the classical case. Secondly the unimodularity condition on G can be removed in which case proper noncommutative L^p-spaces of a group von Neumann algebra were defined by Haagerup and Connes-Hilsum. Finally, the above theorems also hold in the operator space setting, meaning that bounds can be replaced by complete bounds. In fact in the operator space setting one gets additional result using the technique of transference to Schur multipliers (found in [6] for discrete groups and generalized in [2] to arbitrary groups).

References

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