Formalizing Arrow’s Theorem in Logics of Dependence and Independence

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Logics for Social Behaviour
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Joint work with Eric Pacuit
Outline

1. Arrow’s Theorem
2. Logics of Dependence and Independence
3. Formalizing Arrow’s Theorem in Independence Logic
Arrow’s Theorem

- A (finite) set of **alternatives**: \( A = \{ a, b, c, d, \ldots \} \)
- A finite set of **voters**: \( V = \{ v_1, \ldots, v_n \} \)
- A **ranking** \( R \subseteq A \times A \) is a transitive and complete relation on \( A \).
- A **linear ranking** \( R \) is a ranking that is a linear relation.

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<tr>
<th>year</th>
<th>voter 1</th>
<th>voter 2</th>
<th>voter 3</th>
<th>group decision</th>
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- **Preference aggregation function** \( F : \mathcal{P}(A)^n \rightarrow \mathcal{P}(A) \), where \( \mathcal{P}(A) = O(A) \) or \( L(A) \)
Arrow’s Theorem

Theorem (Arrow)

If \(|A| \geq 3\) and \(|V| = n\), then any preference aggregation function \(F : \mathcal{P}(A)^n \rightarrow \mathcal{P}(A)\) satisfying Independence of Irrelevant Alternatives and Unanimity is a dictatorship.

- Functionality of Preference Aggregation Rule (dependence)
- Independence of Irrelevant Alternatives (dependence)
- Unanimity (dependence)
- Dictatorship (dependence)
- Universal Domain (independence)
  - All Rankings
  - Independence
Logics of Dependence and Independence
First Order Quantifiers:
\[ \forall x_1 \exists y_1 \forall x_2 \exists y_2 \phi \]

Henkin Quantifiers (Henkin, 1961):
\[ \left( \forall x_1 \exists y_1 \forall x_2 \exists y_2 \right) \phi \]

Independence Friendly Logic (Hintikka, Sandu, 1989):
\[ \forall x_1 \exists y_1 \forall x_2 \exists y_2 / \{ x_1 \} \phi \]

Dependence Logic (Väänänen 2007):
\[ \forall x_1 \exists y_1 \forall x_2 \exists y_2 \left( = (x_2, y_2) \land \phi \right) \]

Independence Logic (Grädel, Väänänen 2013):
\[ \forall x_1 \exists y_1 \forall x_2 \exists y_2 \left( x_1 \perp y_2 \land \phi \right) \]

The value of \( y_2 \) is completely determined by the value of \( x_2 \).

The value of \( x_1 \) is completely independent of the value of \( y_2 \).
Dependence Logic:

\[ \phi ::= \alpha \mid \neg \alpha \mid \phi \land \phi \mid \phi \lor \phi \mid \exists x \phi \mid \forall x \phi \mid = (\vec{x}, y) \]

Independence Logic:

\[ \phi ::= \alpha \mid \neg \alpha \mid \phi \land \phi \mid \phi \lor \phi \mid \exists x \phi \mid \forall x \phi \mid \vec{x} \perp \vec{y} \]

- The ranking of voter \( v_i \) is *completely independent of* the rankings of the other voters \( v_j \) \((j \neq i)\).

\[ v_i \perp \langle v_j \rangle_{j \neq i} \]

- The group decision \( u \) is *completely determined by* the rankings of the voters \( v_1, \ldots, v_n \).

\[ = (v_1, \ldots, v_n, u) \]
Team semantics (Hodges 1997)

The group decision $u$ is completely determined by the rankings of the voters $v_1, v_2, v_3$.

\[ M \models S = (v_1, v_2, v_3, u) \]

A team $S$

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{year} & \text{voter 1} & \text{voter 2} & \text{voter 3} & \text{group decision} \\
\hline
s_1 & 2000 & abc & cab & acb & abc \\
\hline
s_2 & 2001 & abc & cab & acb & abc \\
\hline
s_3 & 2002 & cab & bca & acb & cba \\
\hline
s_4 & 2003 & cab & bca & acb & cba \\
\hline
\end{array}
\]

$M \models S = (\vec{x}, y)$ iff for any $s, s' \in S$,

\[ s(\vec{x}) = s'(\vec{x}) \implies s(y) = s'(y). \]

- $\langle R_i, F \rangle \mapsto s_{R_i, F}; v_1 \mapsto R_1, \ldots, v_n \mapsto R_n, u \mapsto F(R)$.
- $\\{ \langle R_i, F \rangle \mid i \in I \} \mapsto \text{team } S_F = \{ s_{R_i, F} \mid i \in I \}$
- Functionality of Preference Aggregation Rule

$\theta_F := (v_1, \ldots, v_n, u)$
The ranking of voter $v_1$ is completely independent of the rankings of the other voters $v_2$ and $v_3$. 

$v_1 \perp \langle v_2, v_3 \rangle$

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- $M \models_S \vec{x} \perp \vec{y}$ iff for all $s, s' \in S$, there exists $s'' \in S$ s.t. $s''(\vec{x}) = s(\vec{x})$ and $s''(\vec{y}) = s'(\vec{y})$.

**Independence:** For any profiles $(R_1, \ldots, R_n), (R'_1, \ldots, R'_n) \in \text{dom}(F)$ and any voter $x_i \in V$, there is a profile $(R''_1, \ldots, R''_n) \in \text{dom}(F)$ such that $R''_i = R_i$ and $R''_j = R'_j$ for all $j \neq i$. 

$$\theta_I := \bigwedge \{ v_i \perp \langle v_j \rangle_{j \neq i} : 1 \leq i \leq n \}$$
Fix

- a finite set \( A = \{a, b, c, \ldots \} \) of alternatives containing at least three elements
- and a set \( V = \{v_1, \ldots, v_n\} \) of voters.

Let \( v_1, \ldots, v_n, u \) be distinguished variables.

The signature \( \mathcal{L}_A \) consists of
- unary predicate symbols \( R_{ab}(x) \) for each pair \((a, b) \in A \times A\).
- Strict preference \( P_{ab}(x) := R_{ab}(x) \land \neg R_{ba}(x) \)

An intended first-order model \( M \):
- \( \text{dom}(M) = L(A) \) the set of all linear rankings of \( A = \{a, b, c\} \)
- \( R_{ab}^M = \{abc, cab, acb\} \), i.e., \( R_{ab}^M(R') \) iff \( aR'b \) for all \( R' \in L(A) \)

Fact: There is a set \( \Gamma_{DM} \) of first-order \( \mathcal{L}_A \)-formulas such that \( M \) is a model of \( \Gamma_{DM} \) iff \( M \) is (isomorphic to) the intended model.
(Flatness) For every formula $\phi$ of first-order logic,

$$M \models_S \phi \iff \forall s \in S : M \models_s \phi.$$ 

- $M \models_S \alpha$ iff for all $s \in S$, $M \models_s \alpha$;
- $M \models_S \neg \alpha$ iff for all $s \in S$, $M \not\models_s \alpha$;

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$M \models_{S_1} R_{ab}(v_1)$ \quad $M \not\models_{S_1 \cup S_2} R_{ab}(v_1)$

$M \models_{S_2} \neg R_{ab}(v_1)$ \quad $M \not\models_{S_1 \cup S_2} \neg R_{ab}(v_1)$
For any first-order formulas $\phi$ and $\psi$,

- $M \models_S \phi \rightarrow \psi$ iff for all $s \in S$, if $M \models \{s\} \phi$ then $M \models \{s\} \psi$.

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- $P_{ab}(v_2) \rightarrow P_{ab}(u)$
- Voter $v_i$ is a dictator: $\theta_d(v_i) := \bigwedge_{a,b \in A}(P_{ab}(v_i) \rightarrow P_{ab}(u))$
- There is a dictator: $\theta_D := \bigvee_{i=1}^{n} \theta_d(v_i)$, where $\bigvee$ is the intuitionistic disjunction.
- $M \models_S \phi \lor \psi$ iff $M \models_S \phi$ or $M \models_S \psi$. 

12/22
Theorem (Väänänen, Grädel, Kontinen, Galliani)

- Over sentences, both dependence logic ($D$) and independence logic ($I$) have the same expressive power as $\Sigma_1^1$. In particular, on finite structures, both logics capture NP.
- $D$-formulas characterize all $\Sigma_1^1$ downward monotone properties.
- $I$-formulas characterize all $\Sigma_1^1$ properties.
- $D < I$

$$\phi ::= \alpha \mid \neg \alpha \mid \bot \mid = (x_1, \ldots, x_k, y) \mid x_1 \ldots x_k \perp y_1 \ldots y_m \mid \phi \land \phi \mid \phi \lor \phi \mid \forall x \phi \mid \exists x \phi \mid x_1 \ldots x_k \subseteq y_1 \ldots y_k$$

Every value of $\bar{x}$ is also a value of $\bar{y}$.

Theorem (Galliani, Hella 2013)

Inclusion logic ($\text{FO} + \bar{x} \subseteq \bar{y}$) has the same expressive power as positive greatest fixed point logic. In particular, on ordered finite structures, inclusion logic captures PTIME.
Every value of the variable $x$ is also a ranking of voter $v_1$ in the set of profiles.

$$x \subseteq v_1$$

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* $M \models S \vec{x} \subseteq \vec{y}$ iff for all $s \in S$, there exists $s' \in S$ such that $s(\vec{x}) = s'(\vec{y})$.

$$\theta_{AR} := \bigwedge \{ \forall x (x \subseteq v_i): 1 \leq i \leq n \}$$

(Recall: $\text{dom}(M) = L(A)$.)

**All rankings:** For any voter $v_i$ and any ranking $R$, there is a profile $(R_1, \ldots, R_n) \in \text{dom}(F)$ such that $R_i = R$. 
Independence logic

- $\phi ::= \alpha \mid \neg \alpha \mid \bot \mid (\vec{x}, y) \mid \vec{x} \perp \vec{y} \mid \vec{x} \subseteq \vec{y} \mid \phi \land \phi \mid \phi \lor \phi \mid \forall x \phi \mid \exists x \phi$

- Independence logic $\equiv \Sigma^1_1$
Definition: $M \models_S \sim \phi$ iff $M \not\models_S \phi$ whenever $S \neq \emptyset$.

Definition: $\phi$ is negatable iff $\sim \phi$ is definable in the logic.

Theorem

There is a natural deduction system such that for any set $\Gamma \cup \{\phi\}$ of formulas of independence logic with $\phi$ negatable,

$$\Gamma \vdash \phi \iff \Gamma \models \phi.$$

Example:

- First-order formulas are negatable.
- $\theta_D$ is negatable.
- $\bot$ is negatable. Hence $\Gamma \vdash \bot$ (i.e., $\Gamma$ is inconsistent) can be derived.
Formalizing Arrow’s Theorem in Independence Logic
Theorem (Arrow)

If $|A| \geq 3$ and $|V| = n$, then any preference aggregation function $F : \mathcal{P}(A)^n \to \mathcal{P}(A)$ satisfying Independence of Irrelevant Alternatives and Unanimity is a dictatorship.

\[ \Gamma_{Arrow} \models \theta_D \quad \text{or} \quad \Gamma_{Arrow}, \sim \theta_D \models \bot \]

\[ \implies \Gamma_{Arrow} \not\models \theta_D \quad \text{or} \quad \Gamma_{Arrow}, \sim \theta_D \not\models \bot \]

(by Completeness Theorem)

- Functionality of Preference Aggregation Rule [$\theta_F := = (v_1, \ldots, v_n, u)$]
- Independence of Irrelevant Alternatives
- Unanimity
- Dictatorship [$\theta_D := \bigwedge_{i=1}^{n} \left( \bigwedge_{a,b \in A} (P_{ab}(v_i) \rightarrow P_{ab}(u)) \right)$]
- Universal Domain
  - All Ranking [$\theta_{AR} := \bigwedge \{ \forall x (x \subseteq v_i) : 1 \leq i \leq n \}$]
  - Independence [$\theta_I := \bigwedge \{ v_i \perp \langle v_j \rangle_{j \neq i} : 1 \leq i \leq n \}$]
Independence of Irrelevant Alternatives (IIA)

The social decision on the relative preference between two alternatives $a, b$ depends only on the individual preferences between these alternatives. It is independent of their rankings with respect to other alternatives.

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$\theta_{IIA} := \bigwedge \{ (R_{ab}(v_1), R_{ba}(v_1), \ldots, R_{ab}(v_n), R_{ba}(v_n), R_{ab}(u)) | a, b \in A \}$

$M \models S = (\phi_1, \ldots, \phi_k, \psi)$ iff for all $s, s' \in S$, if $s \sim_{\{\phi_1, \ldots, \phi_k\}} s'$, then $s \sim_{\{\psi\}} s'$, where $s \sim_{\Gamma} s'$ is defined as

for all $\gamma \in \Gamma : M \models \{s\} \gamma \iff M \models \{s'\} \gamma$. 
Unanimity

If every voter prefers \( a \) to \( b \), then so should the group decision.

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\[
\theta_U := \bigwedge \left\{ \left( P_{ab}(v_1) \land \cdots \land P_{ab}(v_n) \right) \rightarrow P_{ab}(u) \mid a, b \in A \right\}
\]
**Theorem (Arrow)**

If $|A| \geq 3$ and $|V| = n$, then any preference aggregation function $F : \mathbb{P}(A)^n \rightarrow \mathbb{P}(A)$ satisfying **Independence of Irrelevant Alternatives** and **Unanimity** is a dictatorship.

**Theorem (Arrow’s Theorem)**

- $\Gamma_{Arrow} \models \theta_D$, where $\Gamma_{Arrow} = \Gamma_{DM} \cup \{\theta_F, \theta_{IIA}, \theta_U, \theta_{AR}, \theta_I\}$.
- $\Gamma_{Arrow} \vdash \theta_D$