

# An integrated benchmark model for Counterparty Credit Risk

T.P.S. Heijnders





# An integrated benchmark model for Counterparty Credit Risk

by

T.P.S. Heijnders

to obtain the degree of Master of Science  
at the Delft University of Technology  
to be defended publicly on Jan 18, 2023

Date: December 22, 2022  
Studentnumber: 4362659  
Project duration: March 7, 2022 – January 18, 2023  
Thesis committee: Dr. F. Fang, TU Delft, supervisor  
Prof. Dr. ir. C. Vuik, TU Delft  
Prof. Dr. A. Papapantoleon, TU Delft  
Dr. X. Shen, FFQuant, supervisor

An electronic version of this thesis is available at <http://repository.tudelft.nl/>.





# Preface

*T.P.S. Heijnders  
Delft, December 22, 2022*

This thesis has been submitted for the degree Master of Science in Applied Mathematics at Delft University of Technology. The research has been conducted at FF Quant under the supervision of Dr. F. Fang. The responsible professor of this thesis is Prof.dr.ir. C. Vuik.

I would like to thank Dr. F. Fang for the set up of this project and the guidance throughout. She helped me to become a better researcher, writer and gave me some wise lessons along the way. Furthermore, I would like to thank Dr. X. Shen from FF Quant for the help throughout this period. Also, I would like to thank Prof.dr.ir. C. Vuik and Prof. Dr. A. Papapantoleon for taking the time to read this report and to take place in my thesis committee. Furthermore, I want to thank Gijs Mast with whom I could always discuss difficult topics and who could help me with coding from time to time.

Lastly, I want to thank all my friends and family for giving me support and encouragement throughout these sometimes stressfull times. A special thanks to my girlfriend Lot who supported me during the most stressfull moments.



# Abstract

The EAD metric is widely used in the calculations for the capital requirements concerning Counterparty Credit Risk (CCR). In this thesis we compare several methods for calculating this EAD. Basel III gives us two methods, the Standardized Approach for CCR (SA-CCR) and the Internal Model Method (IMM). Furthermore, we introduce an integrated benchmark model, whereby we estimate the EAD by integrating the Wrong Way Risk (WWR) directly, while in SA-CCR and IMM this WWR is captured by an alpha factor of 1.4. In this benchmark model, we derive the formula for backing out the probability of default using CDSs and then, via a Gaussian copula, we include a correlation between the exposures and default probability to model the WWR. The ultimate goal of this thesis is to find out how conservative the SA-CCR is compared to the IMM and the integrated benchmark model, and if the alpha factor of 1.4 is a reasonable value to account for WWR.

The test portfolios consist of interest rate swaps, cross currency swaps and FX forwards, which are the most liquid product types in the market. To value these products we model the interest rate using the Hull-White model and the exchange rate using a GBM, following industry standard.

Testing results suggest that the SA-CCR is at least a factor of 1.5 more conservative than the IMM in the presence of collateral, even with stressed parameters. The level of conservatism is even higher because no diversification is allowed between different asset classes by SA-CCR. Furthermore, we observe that using the parameters backed out from calibration and a default correlation of about 30 to 40%, our integrated benchmark model based on copula returns more or less comparable EADs of IMM times the alpha factor of 1.4. This indicates that this value of 1.4 is a reasonable value to cover WWR in the IMM framework.

**Key words.** counterparty credit risk, wrong way risk, exposure at default, collateral, credit default swap, Hull-White model, geometric Brownian motion, Monte Carlo simulation, Gaussian copula model





# List of Abbreviations

<b>CCR</b>	Counterparty Credit Risk
<b>CDF</b>	Cumulative Density Function
<b>CDS</b>	Credit Default Swap
<b>EAD</b>	Exposure at Default
<b>EE</b>	Expected Exposure
<b>FX</b>	Foreign Exchange
<b>GBM</b>	Geometric Brownian Motion
<b>IMM</b>	Internal Model Method
<b>IRS</b>	Interest Rate Swap
<b>LGD</b>	Loss Given Default
<b>MPOR</b>	Margin Period of Risk
<b>MTA</b>	Minimum Transfer Amount
<b>MtM</b>	Mark-to-Market
<b>NICA</b>	Net Independent Collateral Amount
<b>OTC</b>	Over-The-Counter
<b>PD</b>	Probability of Default
<b>PDF</b>	Probability Density Function
<b>PFE</b>	Potential Future Exposure
<b>SA-CCR</b>	standardized approach for CCR
<b>TH</b>	Threshold
<b>WWR</b>	Wrong Way Risk
<b>XCS</b>	Cross-Currency Swap



# List of Figures

1.1	Image of Exposure, future Exposure, Expected Exposure (EE) and Potential Future Exposure (PFE) from [17]. . . . .	2
4.1	Calibration of normal distribution on the log increments of the historical exchange rate data. . . . .	30
4.2	Normal distribution with parameters from table 4.2 fitted on the log-increments of the GBM modeled with the same parameters. . . . .	30
4.3	EE with and without collateral. Here we use 10000 paths, $252 \cdot T$ steps, and parameters 4.1. For the exposure with collateral we have set $MPOR = 10, TH = 40, MTA = 10$ . . . . .	31
4.4	Expected exposure with and without collateral. Here we use 10000 paths, $252 \cdot T$ steps, and parameters 4.1, 4.2, 4.3. For the exposure with collateral we have set $MPOR = 10, TH = 60, MTA = 10$ . . . . .	32
4.5	Expected exposure with and without collateral. Here we use 10000 paths, $252 \cdot T$ steps, and parameters 4.1, 4.2, 4.3. For the exposure with collateral we have set $MPOR = 10, TH = 40, MTA = 10$ . . . . .	32
4.6	Expected exposure of the three same products as in figures 4.3, 4.4 and 4.5 using the stressed parameters observed in table 4.4. . . . .	33
5.1	Results for constant hazard rate and piece-wise constant hazard rate. . . . .	38
5.2	Conditional expected exposure using the Copula model with different values of $\rho$ . We have taken a spread equal to 400 and IRS parameters from the calibration described in section 4.6. . . . .	41
5.3	Conditional expected exposures using the Copula model for different spread values with corresponding probability of default curve and expected loss. We have taken $\rho$ equal to 0.3 and IRS parameters from the calibration described in section 4.6 . . . . .	42
5.4	Comparison between the effect of $\rho$ on the Copula model using a low spread of 10 on the left and a high spread of 1000 on the right. . . . .	42
6.1	Comparison of Internal Model Method and Copula method for portfolio's consisting of 10 same type of products using normal parameters. . . . .	46
6.2	Comparison of Internal Model Method and Copula method for portfolio's consisting of 30 and 90 products using normal parameters. . . . .	47



# List of Tables

4.1	The parameters for the domestic and foreign short-rate. . . . .	29
4.2	Results of the calibration of the GBM. . . . .	30
4.3	Results of the calibration of the correlations. . . . .	31
4.4	Parameters calibrated by stressed data. . . . .	33
5.1	Default spreads linked to ratings according to [21]. . . . .	36
6.1	Parameters for SA-CCR. . . . .	43
6.2	Comparison EAD from SA-CCR and EAD IMM for single trades. . . . .	44
6.3	Comparison SA-CCR, IMM and Copula model using normal parameters. . . . .	45
6.4	Comparison SA-CCR, IMM and Copula model using stressed parameters. . . . .	45



# Contents

<b>List of Abbreviations</b>	<b>vii</b>
<b>List of Figures</b>	<b>ix</b>
<b>List of Tables</b>	<b>xi</b>
<b>1 Introduction</b>	<b>1</b>
<b>2 Mathematical Framework</b>	<b>5</b>
2.1 Probability theory . . . . .	5
2.2 Copula. . . . .	6
2.3 Stochastic processes . . . . .	7
2.4 Changing measures. . . . .	8
2.5 GBM . . . . .	8
2.6 Dependent Brownian motions . . . . .	9
<b>3 Counterparty Credit Risk (CCR) and CCR Measures</b>	<b>11</b>
3.1 Financial Risk . . . . .	11
3.2 Exposure . . . . .	12
3.3 Wrong Way risk . . . . .	12
3.4 Mitigation of CCR . . . . .	12
3.4.1 Netting set . . . . .	12
3.4.2 Collateral . . . . .	13
3.4.3 Margin . . . . .	13
3.5 EAD under Standardized Approach. . . . .	15
3.5.1 RC. . . . .	15
3.5.2 PFE add-on . . . . .	15
3.5.3 Effective Notional . . . . .	17
3.6 EAD under the Internal Model Method . . . . .	18
<b>4 Simulation Framework without WWR</b>	<b>19</b>
4.1 Various interest rate definitions . . . . .	19
4.1.1 Forward interest rate . . . . .	20
4.1.2 IRS swap. . . . .	21
4.2 Foreign exchange derivatives . . . . .	22
4.3 Interest rate models . . . . .	22
4.3.1 Hull-White model . . . . .	23
4.3.2 G1++ model . . . . .	24
4.4 Monte Carlo simulation . . . . .	24
4.5 Simulation algorithms . . . . .	25
4.6 Calibration . . . . .	26
4.6.1 Calibration of the short rate model . . . . .	27
4.6.2 Calibration of the GBM. . . . .	29
4.6.3 Calibration of correlation. . . . .	30
4.7 Results . . . . .	31

4.8	Sensitivity analysis and stressed parameters.. . . . .	32
<b>5</b>	<b>Simulation Framework with WWR</b>	<b>35</b>
5.1	Extracting Probability of default from CDS spreads . . . . .	35
5.1.1	CDS. . . . .	35
5.1.2	Extracting the hazard rate . . . . .	36
5.2	Copula model . . . . .	39
5.2.1	Expected exposure . . . . .	40
5.2.2	Sensitivity analysis Copula model . . . . .	40
<b>6</b>	<b>Numerical results</b>	<b>43</b>
6.1	Comparing SA-CCR with IMM . . . . .	43
6.2	Comparison on testing portfolios . . . . .	44
<b>7</b>	<b>Conclusions and Discussion</b>	<b>49</b>
<b>A</b>	<b>Proofs</b>	<b>51</b>
A.1	Change of measure exchange rate . . . . .	51
A.2	Change of measure foreign to domestic . . . . .	52
A.3	Copula model exact method . . . . .	53
<b>B</b>	<b>Portfolio's</b>	<b>55</b>
B.1	Portfolio with 30 derivatives . . . . .	56
B.2	Portfolio with 90 derivatives . . . . .	57
	<b>Bibliography</b>	<b>59</b>



# 1

## Introduction

Before the credit crisis of 2008, some institutions and large banks were considered as “too big to fail” and therefore risk-free. However, by 2008 five of the largest investment banks in the United States became insolvent. Bear Stearns and Merrill Lynch had been sold for very low prices while Morgan Stanley and Goldman Sachs had to give up their private bank status to be bailed out by the U.S. government [17]. On top of it all, the fourth largest investment bank in the United States, Lehman Brothers, had to file for bankruptcy. This event swept over the world like a large shock wave. In the Netherlands alone, the government had to take drastic measures to save their largest banks [11]. It was the beginning of a deep recession around the world starting in 2008.

Because of this crisis, trillions of dollars were lost from the financial market mainly because of mispricing mortgage risk but also because of the bad identification of Counterparty Credit Risk (CCR). Here we see CCR as the risk that a counterparty goes in to default and can no longer pay their obligations. The crisis made clear that, regardless whether a counterparty is a large investment bank or a Triple-A entity, it can never be considered risk-free. Because of this, CCR has become a major subject for the global financial markets. It was always present in the financial markets, but with the introduction of Basel III new rules were set in place to strengthen bank capital for CCR.

Basel III is the successor of Basel II introduced by the Basel Committee on Banking Supervision (BCBS) and was issued in December 2010 [26]. It was mostly an effort to control the causes of the crisis and it therefore aims to strengthen the capital requirements of banks. The risk-based capital requirements for CCR in Basel III cover two important characteristics of CCR: the risk of counterparty defaults and the volatility in the creditworthiness of the counterparty [20]. The risk of counterparty defaults was already covered in Basel I and Basel II but was strengthened in Basel III. Moreover, the Basel III reforms introduced a new capital charge for the risk of loss due to the decrease of the creditworthiness of the counterparty. This potential mark-to-market loss is known as CVA risk. It captures changes in counterparty credit spreads and other market risk factors. CVA risk was a major source of unexpected losses for banks during the credit crisis.

In this thesis we will focus on the risk for counterparty default in the capital requirements of CCR, which we will refer to as Regulatory Capital. Regulatory Capital for CCR is based on the “loan-equivalent approach”, which means that each Over-The-Counter (OTC) derivative or alike can be viewed as a loan from credit risk perspective. Hence Regulatory Capital for CCR

is calculated by the product of Loss Given Default (LGD), Exposure at Default (EAD) and Probability of Default (PD). Since LGD and PD information can be retrieved from the market or from the credit risk models, the calculation of Regulatory Capital for CCR lies in the quantification of EAD, where the EAD of a trade is based on a metric called exposure.

The metric exposure is widely used in measuring CCR. When a bank has an outstanding contract with a counterparty, we define the exposure of this contract as the positive part of the Mark-to-Market (MtM) price of this contract at time of default. We define the MtM price as the present value of the financial contract. The reason is that, when the counterparty goes into default the bank has a claim of the positive MtM price, but when the MtM price is negative we are in debt to the counterparty and from the valuation perspective our positions will be unchanged in case of a default [17]. Exposure is therefore defined as the greater of the MtM price and zero. The bank is mostly interested in the exposure at some future time because the difference between CCR and the credit risk in loans is mainly in the additional uncertainty in the EAD of the former, compared to the latter. This could be referred to as future exposure and can be observed in figure 1.1 as the grey area. We define the PFE as the 97.5% percentile of the future exposure. This can be seen as the 25 worst cases of the future exposure among 1000 scenarios and is illustrated in figure 1.1. We then define the EE as the average of all the possible scenarios of the exposure at a given time point in the future.

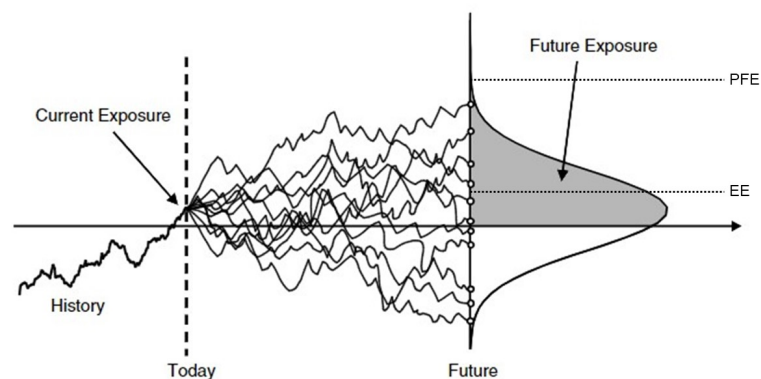


Figure 1.1: Image of Exposure, future Exposure, EE and PFE from [17].

Banks measure and manage CCR internally using mainly the PFE and EE metrics. Here PFE is mainly used internally for setting limits on the credit quality of the counterparties [10]. EE is used combined with other quantities for the calculation of EAD and the capital requirements due to counterparty risk. As said before, estimating the EAD is critical for calculating the Regulatory Capital for CCR. We distinguish two methods for calculating this EAD, a non-modelled and a modelled approach. The modelled approach is named the Internal Model Method (IMM), as defined in Basel regulations, a bank must undergo some strict model validation and approval procedures to have the permission to use this approach by the regulator. The non-modelled approach is named the standardized approach for CCR (SA-CCR) in the reformed Basel III regulations, which replaces two old non-internal models approaches, the Current Exposure Method (CEM) and the Standardised Method (SM).

In both methods the calculation of EAD requires a factor alpha of 1.4 to cover the so-called Wrong Way Risk (WWR) in CCR quantifications. WWR is the risk arising due to the unfavourable dependence between exposure and counterparty credit quality. In this thesis we

will model the SA-CCR, IMM and an integrated benchmark model inspired by the approach taken in [23]. In this integrated benchmark model we integrate the probability of default directly in the EAD calculations to take into account the WWR correlation between the exposures and probability of default of a counterparty, instead of the alpha factor used by the regulator. We name this integrated benchmark model the Copula model since it uses a Gaussian copula to include a correlation between a market factor and a credit worthiness indicator.

In this thesis, we will answer the question if the Copula model can be a good benchmark for the IMM, i.e., if the alpha factor of 1.4 is a reasonable value to account for WWR. Furthermore, we answer the question how conservative the SA-CCR is compared with the IMM and Copula model. We continue this thesis by introducing a mathematical framework with definitions and formulas we need further on. After that we will dive in somewhat deeper in CCR and the mitigation of it in chapter 3, where we will also analyze the SA-CCR and IMM as described in Basel III. In chapter 4 we will set up the algorithms and structures to value the financial products which we will use in the simulation framework without WRR. In chapter 5 we will first use the pricer of a Credit Default Swap to construct a probability of default. We will then use this probability within the Copula model where we integrate the WWR. In the last chapter we will compare numerical results and draw our conclusions.



# 2

## Mathematical Framework

In this chapter, we discuss some known definitions and theorems which we will need later on in the thesis.

### 2.1. Probability theory

**Definition 2.1.1.** For a random variable  $X$  the Cumulative Density Function (CDF) is given by:

$$F_x(x) = \mathbb{P}[X \leq x], \quad (2.1)$$

which is also often known as just the probability distribution function. The Probability Density Function (PDF) for a continuous random variable is given by:

$$f_x(x) = \frac{d}{dx} F_x(x). \quad (2.2)$$

**Definition 2.1.2.** For a continuous random variable  $X$  the expectation is given by:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx. \quad (2.3)$$

The variance of  $X$  is then given by:

$$\text{Var}[X] = \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 f_X(x) dx. \quad (2.4)$$

**Definition 2.1.3.** We say that  $X$  is normally distributed with expectation  $\mu$  and variance  $\sigma^2$ ,  $X \sim \mathcal{N}(\mu, \sigma^2)$ , when the CDF is given by:

$$F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x \exp\left(\frac{-(z - \mu)^2}{2\sigma^2}\right) dz. \quad (2.5)$$

Consequently, the PDF is given by:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x - \mu)^2}{2\sigma^2}\right). \quad (2.6)$$

**Definition 2.1.4.** For random variables  $X$  and  $Y$ , the joint cumulative distribution is given by:

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y). \quad (2.7)$$

If both variables are continuous the joint probability density function is given by:

$$f_{x,y}(x, y) = \frac{d^2}{dx dy} F_{X,Y}(x, y). \quad (2.8)$$

We say that  $X$  and  $Y$  follow a **bivariate normal distribution** with correlation coefficient  $\rho$  and parameters  $\mu_X, \sigma_X^2, \sigma_Y^2, \mu_Y$  when their joint probability density function is given by:

$$f_{x,y}(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right\}. \quad (2.9)$$

## 2.2. Copula

When we have variables which can not be fitted to a bivariate or multivariate normal distribution we can use a copula model. These are useful when choosing joint distributions to model random variables whose marginal distributions are known [24], where the marginal distribution of one variable is the probability distribution independent of the other variables.

**Definition 2.2.1** (Copula). A joint probability function that results in both marginal distributions being uniformly distributed on  $(0, 1)$  is called a copula. That is, the joint distribution function  $C(x, y)$  is a copula if  $C(0, 0) = 0$  and for  $0 \leq x, y \leq 1$ :

$$C(x, 1) = x, \quad C(1, y) = y. \quad (2.10)$$

Say we want to find a joint distribution function for random variables  $X$  and  $Y$ , whose marginal functions are known continuous probability distribution functions:  $\mathbb{P}(X \leq x) = F(x)$  and  $\mathbb{P}(Y \leq y) = G(y)$ . Now we want to find the joint probability function:  $H(x, y) = \mathbb{P}(X \leq x, Y \leq y)$ . We take  $C(F(X), G(Y))$  as the joint probability function of  $G(Y)$  and  $F(X)$ , this is a copula since both  $G(Y)$  and  $F(X)$  are uniform. Since both  $F$  and  $G$  are increasing we know that  $X \leq x$  if and only if  $F(x) \leq F(x)$ . The same holds for  $Y$  and  $G(Y)$ . Using this we can find the joint probability function of  $X$  and  $Y$ ,  $H(x, y)$ :

$$\begin{aligned} H(x, y) &= \mathbb{P}(X \leq x, Y \leq y) \\ &= \mathbb{P}(F(X) \leq F(x), G(Y) \leq G(y)) \\ &= C(F(x), G(y)). \end{aligned} \quad (2.11)$$

This copula approach of describing or modeling the joint distribution function for  $X$  and  $Y$  is first deciding on their marginal distributions  $F$  and  $G$ , and then choosing an appropriate copula to model the joint distribution of  $F(X)$  and  $G(Y)$ . An appropriate copula to model this joint distribution is one that takes into account the dependency between  $F(X)$  and  $G(Y)$ .

A popular copula in the financial industry is the **Gaussian copula**. Let  $X$  and  $Y$  be standard normal variables whose joint probability function is given by (2.9) using  $\mu = 0$  and  $\sigma = 1$ . Let  $\Phi$  be the standard normal distribution, then the joint distribution of  $\Phi(X)$  and  $\Phi(Y)$  is called the Gaussian copula and is given by:

$$C(x, y) = \mathbb{P}(\Phi(X) \leq x, \Phi(Y) \leq y) = \mathbb{P}(X \leq \Phi^{-1}(x), Y \leq \Phi^{-1}(y)) \quad (2.12)$$

$$= \int_{-\infty}^{\Phi^{-1}(x)} \int_{-\infty}^{\Phi^{-1}(y)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{s^2 - 2\rho st + t^2}{2(1-\rho^2)}\right) ds dt. \quad (2.13)$$

### 2.3. Stochastic processes

Throughout this paper we want to model different market scenario's or "paths". These scenario's are driven by risk factors which are typically modeled by stochastic processes. We usually calibrate the model parameters for all risk factors from their historical scenario's, based on which we simulate scenarios of risk factors into the future. A stochastic process is a collection of random variables indexed by a time variable  $t$ , denoted by  $(X_t)_{t \geq 0}$ . In this section the definitions and theorems are based on [14], [27] and [18]. We consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and we define the following:

**Definition 2.3.1** (Filtration). A filtration on probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is a collection of  $\sigma$ -algebras  $(\mathcal{F}_t)_{0 \leq t \leq \infty}$  of  $\mathcal{F}$  and is an increasing sequence  $\mathcal{F}_s \subset \mathcal{F}_t, \forall s \leq t \leq \infty$ .

**Definition 2.3.2** (Adapted). A process  $(X_t)_{t \geq 0}$  is adapted if for every  $t \geq 0, X_t$  is  $\mathcal{F}_t$  - measurable.

**Definition 2.3.3** (Martingale). An adapted real-valued process  $(X_t)_{t \geq 0}$  is called a martingale if  $X_t$  is integrable for every  $t \geq 0$  and satisfies the martingale property for every  $0 \leq s < t$ :

$$\mathbb{E}(X_t | \mathcal{F}_s) = X_s. \quad (2.14)$$

The process is called a submartingale if for every  $0 \leq s < t$ :

$$\mathbb{E}(X_t | \mathcal{F}_s) \geq X_s. \quad (2.15)$$

**Definition 2.3.4** (Semimartingale). A semimartingale  $S = (S_t)_{t \geq 0}$  is a adapted process of the form:

$$S_t = S_0 + M_t + A_t \quad (t \geq 0), \quad (2.16)$$

for a finite and  $\mathcal{F}_0$  measurable  $S_0$ , a local martingale  $(M_t)_{t \geq 0}$  with  $M_0 = 0$  and a finite variation process  $(A_t)_{t \geq 0}$  with  $A_0 = 0$ .

**Definition 2.3.5.** A real valued process  $\{W(t), t \geq 0\}$  or  $(W_t)_{t \geq 0}$  is a Brownian motion if:

- $W(0) = 0$
- For all  $0 \leq s < t : W(t) - W(s) \sim \mathcal{N}(0, t - s)$ .
- for  $0 \leq t_0 < t_1 < \dots < t_n, W(t_i) - W(t_{i-1})$  with  $i = 1, \dots, n$  are independent.
- $W(t)$  is almost surely continuous.

**Theorem 2.3.6.** The Brownian motion  $W(t)$  is a martingale.

**Definition 2.3.7.** For a simple process  $\theta(t) = \sum_i v_i \mathbb{1}_{(t_{i-1}, t_i]}(t)$  with stopping times  $t_i$  and  $v \in \mathcal{F}_{t_{i-1}}$  the integral of simple functions with respect to the Brownian motion is defined as:

$$I_m(t) = \int_0^t \theta_m(s) dW(s) = \sum_{i \leq n} v_i (W(t_i \wedge t) - W(t_{i-1} \wedge t)). \quad (2.17)$$

**Definition 2.3.8** (Ito's integral). For any  $\theta(s) \in \mathbb{L}^2$ , we can define the stochastic integral or Ito's integral as:

$$\int_0^t \theta(s) dW(s) = \lim_{m \rightarrow \infty} \int_0^t I_m(t), \text{ in } \mathbb{L}^2, \quad (2.18)$$

with  $\theta_n(s)$  a sequence of simple processes such that:

$$\mathbb{E}_{n \rightarrow \infty} \left[ \int_0^t |\theta(s) - \theta_n(s)| ds \right] = 0. \quad (2.19)$$

**Theorem 2.3.9** (Ito's formula). *Let  $f \in C^2(\mathbb{R})$  and consider a semi martingale  $S = M + A$ . Then it holds:*

$$f(S_t) = f(S_0) + \int_0^t f'(S_u) dS_u + \int_0^t f''(S_u) d[S]_u, \quad (2.20)$$

where we use Ito's integral and  $[S]$  is the quadratic variation of  $S$ .

## 2.4. Changing measures

In this paper we will distinguish two probability measures: the real-world measure  $\mathbb{P}$  and the risk-neutral measure  $\mathbb{Q}$ . In pricing derivatives we often use the risk neutral measure. Under which the ratio of a derivative's price and a numeraire becomes a martingale, following the no-arbitrage pricing theory. The real-world measure is the measure which we can directly observe and quantify based on data series in the real world. For example, price quotes in the stock market are usually no martingales. We can switch between measures using Girsanov's theorem.

**Definition 2.4.1.** *Two probability measures  $\mathbb{P}$  and  $\mathbb{Q}$  defined on probability space  $(\Omega, \mathcal{F}, \mathbb{F})$  are equivalent if:*

$$\mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0 \text{ for all } A \in \mathcal{F}. \quad (2.21)$$

**Definition 2.4.2** (Girsanov's Theorem). *Let  $W(t)$  be a Brownian motion on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We can define a probability measure  $\mathbb{Q}$  such that the Radon-Nikodym derivative is given by:*

$$\left. \frac{\mathbb{Q}}{\mathbb{P}} \right|_{\mathcal{F}_t} = Z_t = \mathcal{E}(X)_t, \quad (2.22)$$

with  $X = (X_t)_{t \geq 0}$  a stochastic process and  $\mathcal{E}$  given by the stochastic exponential:

$$\mathcal{E}(X)_t = \exp\left(X_t - \frac{1}{2}[X]_t\right). \quad (2.23)$$

Then we know if  $W(t)$  is a Brownian motion under measure  $\mathbb{P}$ , then  $W(t)^*$  is a Brownian motion under  $\mathbb{Q}$  and given by:

$$W(t)^* = W(t) - [W(t), X]. \quad (2.24)$$

## 2.5. GBM

A process is called a Geometric Brownian Motion (GBM) process when it satisfies the following SDE:

$$dS(t) = \mu S(t) dt + \sigma S(t) dW(t), \quad (2.25)$$

with  $S(t_0) = S_0$ ,  $\mu$  the drift parameter and  $\sigma$  the volatility. Here  $W(t)$  is a Brownian motion and for the increment we have:

$$dW(t) = W(t) - W(t - dt). \quad (2.26)$$

Using Ito's formula we can show that  $S(t)$  has a log-normal distribution. This is that  $\ln(S(t))$  has a normal distribution. We take  $X(t) = \ln(S(t))$  and see:

$$dX(t) = \left( \mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dW(t), \quad (2.27)$$



with  $X(t_0) = \ln(S_0)$ . Since the Brownian motion increment  $dW(t)$  are normally distributed with mean 0 and variance  $dt$ , we see that the increments of  $X(t)$  are normally distributed with mean  $(\mu - 1/2\sigma)dt$  and variance  $\sigma^2 dt$ .

## 2.6. Dependent Brownian motions

We want to be able to add a dependence structure between two independent Brownian motions  $\tilde{W}_1$  and  $\tilde{W}_2$ . We use [22] and we first establish what it means for two Brownian motions to be independent, we find:

**Definition 2.6.1.** *When two Brownian motions are correlated we have:*

$$\mathbb{E}[W_i(t) \cdot W_j(t)] = \rho_{i,j}t \text{ for } i \neq j \quad (2.28)$$

$$\mathbb{E}[W_i(t) \cdot W_j(t)] = t \text{ for } j = i \quad (2.29)$$

Two Brownian motions are said to be independent when:

$$\mathbb{E}[\tilde{W}_i(t) \cdot \tilde{W}_j(t)] = 0 \text{ for } i \neq j \quad (2.30)$$

$$\mathbb{E}[\tilde{W}_i(t) \cdot \tilde{W}_j(t)] = t \text{ for } j = i \quad (2.31)$$

And for the increments we have something similar, replacing  $t$  with  $dt$ .

Using a vector construction for our two independent Brownian motions,  $\tilde{\mathbf{W}} = [\tilde{W}_1, \tilde{W}_2]^T$ , we can implement a correlation using the Cholesky decomposition.

**Definition 2.6.2.** *Each symmetric positive definite matrix  $\mathbf{C}$  has a unique factorization, the so-called Cholesky decomposition, of the form:*

$$\mathbf{C} = \mathbf{L}\mathbf{L}^T, \quad (2.32)$$

where  $\mathbf{L}$  is a lower triangular matrix with positive diagonals.

Using a correlation matrix  $\mathbf{C}$ , the Cholesky decomposition gives us:

$$\mathbf{C} = \begin{bmatrix} 1 & \rho_{1,2} \\ \rho_{1,2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho_{1,2} & \sqrt{1 - \rho_{1,2}^2} \end{bmatrix} \begin{bmatrix} 1 & \rho_{1,2} \\ 0 & \sqrt{1 - \rho_{1,2}^2} \end{bmatrix} \quad (2.33)$$

Now take independent Brownian motion  $\tilde{W}_1$  and  $\tilde{W}_2$  and we correlate them using a matrix-vector multiplication  $\mathbf{L} * \tilde{\mathbf{W}}(t)$ .

$$\begin{bmatrix} 1 & 0 \\ \rho_{1,2} & \sqrt{1 - \rho_{1,2}^2} \end{bmatrix} \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \end{bmatrix} = \begin{bmatrix} \tilde{W}_1 \\ \rho_{1,2}\tilde{W}_1 + \sqrt{1 - \rho_{1,2}^2}\tilde{W}_2 \end{bmatrix}. \quad (2.34)$$

Now we define  $W_1(t) = \tilde{W}_1$  and  $W_2(t) = \rho_{1,2}\tilde{W}_1 + \sqrt{1 - \rho_{1,2}^2}\tilde{W}_2$ . Which gives us two Brownian motion correlated with  $\rho_{1,2}$ .



# 3

## Counterparty Credit Risk (CCR) and CCR Measures

In this chapter we dive into the definition of CCR and the quantification metrics for CCR in industry, which we will need throughout this thesis. For this we use the book of Jong Gregory [17]. Later on we will look into the methods to calculate EAD provided by Basel.

### 3.1. Financial Risk

In the financial world one can encounter different types of risk. An investor, bank or other financial entity is always looking for a way to identify, quantify, manage and/or mitigate various risks they face. Here risk mitigation refers to the means one can take to reduce and/or transform the concerned risk.

The topic of this thesis is related to the so-called **Counterparty Credit Risk**. This is the risk that the counterparty of one's financial contract will default prior to the expiration of the contract and will not make all the payments required by the contract [30]. Exchange traded derivatives are not affected by CCR: An exchange is a centralized counterparty and thus has the knowledge of all positions traded via them. Due to the trades overview it has, the exchange is able to better manage and thus guarantee the cash flows promised by the counterparty.

Other types of financial risk are typically categorized as follows: *Market risk* is the risk or the volatility in the movements of market prices. *Credit risk* is when a debtor of a loan position may be unable or unwilling to pay. *Liquidity risk* is the risk that a transaction cannot be executed at normal market prices due to illiquidity of the underlying asset. *Funding liquidity risk* is the possible inability to fund contractual payments or collateral requirements, potentially forcing an early liquidation of assets and creating losses. *Operational risk* arises from unexpected errors and losses incurred by people, systems, internal and external events, which includes human error, failed processes, model risk, fraud and legal risk.

CCR is a risk caused by both volatility in the market and in the default probability of the counterparty. However, it concerns the profit part of the Profit & Loss (P&L) distribution of a financial product, while market risk concerns the loss part. CCR is somewhat similar to credit risk, however, two aspects may differentiate these risks. First, the future value of the financial contract traded in the market is uncertain and could be affected by collateral agreements. For example, the value of a derivative on a potential default date will be the netted value of all future cash flows required under that contract. Second, in a derivatives transaction each counterparty

has credit risk to the other.

## 3.2. Exposure

Credit exposure defines the loss in the event of a counterparty default and is the basic metric to quantify trade-level CCR [16]. For many financial instruments, the creditor is not at risk for the full principal amount but only the replacement costs. We will often consider exposure independently of any default event if assuming the absence of wrong way risk described in section 3.3.

The MtM of a financial product defines the present value of the financial product under the MtM Accounting and is usually the sum of discounted values of all payments to take place in the future. These payments may be scheduled to occur many years in the future and may have present values that are strongly dependant on market variables.

In the event a counterparty has defaulted, an institution may closeout the relevant contracts and cease any future contractual payments. Following this, they may determine the net amount owing between them and their counterparty. When the outstanding amount is negative, the institution is still legally obligated to settle this amount. However, from a valuation perspective, the position appears essentially unchanged. When the outstanding amount is positive, the institution expect to recover some fraction of their claim. This unknown recovery value is not included in the definition of exposure. We can then define exposure for a portfolio with MtM price  $V(t)$  at a certain time  $t$  as:

$$E(t) = \max(V(t), 0). \quad (3.1)$$

As explained before in the introduction, the trading limit with respect to one counterparty is usually defined based on the metric PFE, which is the 25th largest value out of 1000 possible scenarios of the exposure we could have at a certain time in the future, i.e. corresponding to 97.5% quantile of the exposure distribution. Regulators also define regulatory capital to cover CCR, which is based on another metric named EE, which is the average of all possible scenarios of exposure values.

## 3.3. Wrong Way risk

WWR is the risk that there exists an unfavourable dependence between exposure and counterparty's credit quality. In case of a FX Forward or cross-currency product we can consider WWR as a possible linkage between the relevant foreign exchange rate and the default probability of the counterparty. In the case of interest rate products, WWR might exist due to a relationship between the relevant interest rates and the counterparty default probability: High interest rates may trigger defaults and whereas low interest rates may be indicative of a recession where defaults are more likely. One way of quantifying WWR is somehow modeling the relationship between default probability and exposure, which is a main target of this thesis, see chapter 5.

## 3.4. Mitigation of CCR

### 3.4.1. Netting set

One way to mitigate CCR is by entering into a netting agreement, such that the MtM prices of all trades within the same netting set with a counterparty can be aggregated, or in financial jargon, netted. This is also sometimes referred to as closeout netting.

A bank can also consider payment netting, where it can net cash flows occurring on the same day. This will typically relate to settlement risk.

We will mostly refer to a netting set which corresponds to a trade of set which can legally be

netted together in the event of a default. In this thesis we will consider for simplicity one netting set.

### 3.4.2. Collateral

Another way to mitigate CCR is collateralization. It works similar to loan collaterals: a borrower is required to post collateral to the lender as a security of the loan. The collateral can be in the form of an asset, cash or real estate depending on the type of loan. In our case when a bank and a counterparty exchange a product or portfolio, they could ask collateral from the other party as a security for high exposures. Just as netting this is in two ways, meaning that both parties can post collateral. The receiver of the collateral only becomes the owner of this if the counterparty goes into default. In the case of a positive MtM value the bank can ask for collateral and in case of a negative MtM value, collateral could be asked of the bank. When entering a contract with a counterparty, you can agree on several collateral posting. The amount of collateral is then defined through the following parameters:

- A Threshold (TH), which is defined as the threshold for the MtM price, above which collateral is posted.
- An independent amount, which is also named as initial margin by regulations, is an amount of extra collateral that must be posted irrespective of the exposure.
- The Minimum Transfer Amount (MTA), which is the minimum amount of collateral that can be posted.

Collateralization does bring other types of risk. Besides operational and liquidity risk we are also faced by the Margin Period of Risk (MPOR), which is the risk that collateral cannot be received immediately. During this period the MtM price might still change, causing possible under-collateralization of the position. We will talk more about this in the next section, based on [7].

### 3.4.3. Margin

We will consider the collateral in a bilateral way governed by the ISDA Master Agreement and its Credit Support Annex (CSA), which is the commonly adopted template for legal contracts for bilateral over-the-counter derivatives trading. The collateral structure is specified in the CSA as a combination of two types of margin: initial margin and variation margin. A margin is the collateral that an investor has to deposit with their broker or exchange to serve as collateral for the trade. When the trade drops in value the investor must bring the initial margin back to the initial level and the amount that need to be posted is referred to as variation margin. The variation margin is thus closely related to the MtM price. The initial margin is the amount that the investor must have to post to enter the trade.

Furthermore, we must take into account the MPOR. According to Basel III article 50.19 [1] the definition of MPOR is given by the time period from the last exchange of collateral covering a netting set of transactions with a defaulting counterparty until that counterparty is closed out and the resulting market risk is re-hedged. This we interpret as the time between a margin call is made and the moment when the collateral is received, which we denote with  $\partial t$ . According to Basel III article 53.28 [4] this is for re-margining with a periodicity of N-days at least equal to the supervisory floor, F, plus the N days minus one day. Suppose we have a re-margining period of 1 day and a supervisory floor of 10 business days, the length of the MPOR is equal to 10.

When we consider ourselves a bank  $B$  and have a counterparty  $C$  which pays collateral

$A_B(t)$  and  $A_C(t)$  respectively. The collateral at time  $t$  can then be defined by:

$$K(t) = A_C(t) - A_B(t), \quad (3.2)$$

where the collateral in possession of the bank is defined as  $K(t)$  and we assume the collateral payments are netted. In the presence of collateral, the definition of exposure is adjusted to the following:

$$E(t) = \max(V(t) - K(t), 0). \quad (3.3)$$

If we take into account the threshold payments for the bank and counterparty as  $TH_B$  and  $TH_C$  respectively, we can find an expression for the collateral of both parties. Note that  $V(t)$  is the MtM price of a portfolio in favour of the bank, we have:

$$A_C(t) = \max((V(t) - TH_C), 0), \quad (3.4)$$

$$A_B(t) = \max((-V(t) - TH_B), 0). \quad (3.5)$$

To fully align with the collateral requirements in practice, we must consider a few more things which we discussed earlier. In reality there is delay between the marginal call and collateral which is the MPOR indicated by  $\partial t$ . When we fill this in for (3.2) together with the expressions of the thresholds we get:

$$K(t) = \max((V(t - \partial t) - TH_C), 0) - \max(-(V(t - \partial t) - TH_B), 0). \quad (3.6)$$

Furthermore, since the default settlement is also not instant, we must consider the cash flows which happen close to a default. We will refer to these as trade flows since they do not necessarily have to be cash that is exchanged. A missed trade flow is a serious event under the ISDA Master Agreement, and a "failure to pay" can rapidly lead to default and trade termination. We will refer to the unpaid trade flows as  $UTF(t)$ . When we add this term to the simplified model, we have the following updated version of the exposure formula:

$$E(t) = \max(V(t) - K(t) + UTF(t), 0). \quad (3.7)$$

In another, less common, version of the classical model, the assumption is that both  $B$  and  $C$  will stop paying trade flows at the moment the MPOR commences, time  $t - \partial t$ . Then we would set the unpaid trade flows to:

$$UTF(t) = TF(t, [t - \partial t, t]), \quad (3.8)$$

where  $TF(t, [t - \partial t, t])$  is the time  $t$  net value of all the trade flows scheduled between  $t - \partial t$  and  $t$ .

In this thesis we will assume that both  $B$  and  $C$  will pay their trade flows for the entire MPOR and thus set  $UTF(t)$  to 0. Furthermore, we assume that only our counterparty pays collateral. We do still want to take in to account the minimal transfer amount where for simplification we can add this to the threshold and call this MTA. In our simulation, we will calculate the exposure with collateral as (3.2) with  $K(t)$  given by:

$$K(t) = \max(V(t - \partial t) - TH - MTA, 0). \quad (3.9)$$

### 3.5. EAD under Standardized Approach.

The Standardized Approach for CCR, ie. SA-CCR, can be found in the Basel framework, article CRE52 [5]. The SA-CCR can be used only for OTC derivatives, exchange-traded derivatives and long settlement transactions. It is determined by using the following formula:

$$EAD = \alpha * (RC + PFE), \quad (3.10)$$

where EAD is the exposure at default and is to be calculated separately for each netting set. In the formula above we have that  $\alpha = 1.4$  (designed to address WWR), RC the replacement costs and PFE the amount for potential future exposure.

The RC and PFE are calculated differently for a margined and a unmargined netting set. We say that a netting set is margined when the counterparty has to post variation margin. All other netting sets are considered as unmargined netting sets, including where only the bank needs to post variation margin.

#### 3.5.1. RC

If we consider an unmargined netting set, the RC intends to capture the loss that would occur if a counterparty would default and could not pay their obligation immediately. The PFE add-on represents a potential conservative increase in exposure over a one-year time horizon from the present date. The replacement costs are then given by:

$$RC = \max(V - C, 0), \quad (3.11)$$

where  $C$  is equal to the Net Independent Collateral Amount (NICA), which is the difference of the initial collateral amount between the bank and the counterparty. Note that in this case the RC takes in to account the loss at a single time point. The future component of only a 1-year period is then added with the PFE add-on.

For a margined netting set, the RC tries to capture the loss that would occur if a counterparty were to default at the present or at a future time, assuming that the closeout and replacement of transactions occur instantaneously. However, we must take into account the MPOR between the last exchange of collateral before default and replacement of the trades in the market. The PFE add-on represents the potential change in value of the trades during this time period. Here the RC is equal to:

$$RC = \max(V - (VM + NICA), TH + MTA - NICA, 0). \quad (3.12)$$

In this thesis we take NICA equal to 0, and say that the variation margin (VM) is equal to the value of the portfolio. We see that the recover costs are equal to the threshold plus minimum transfer amount. This is then the maximum loss which could occur. Note that the RC does not take into account the MPOR, this will be integrated in the PFE add-on.

Note that the RC is calculated at netting set level, whereas PFE add-ons are calculated for each asset class within a given netting set and then aggregated.

#### 3.5.2. PFE add-on

PFE is the amount of potential future exposure. It consists of an aggregate add-on component, which consist of add-ons calculated for each asset class and a multiplier that allows for recognition of excess collateral or negative MtM value for the transactions. Thus, the PFE is given by:

$$PFE = \text{multiplier} * \text{AddOn}^{\text{aggregate}}. \quad (3.13)$$

Since over-collateralization should reduce capital requirements for CCR, the PFE component also takes into account the risk-reducing property of collateral. This is why the Basel committee decided to apply a multiplier component that decreases when the collateral increase. This multiplier is floored at 5% so it does not reach 0. The multiplier will be activated when the current market value is negative or when the initial collateral is higher than the initial value. The multiplier is given by:

$$\text{multiplier} = \min \left\{ 1; \text{Floor} + (1 - \text{Floor}) * \exp \left( \frac{V - C}{2 * (1 - \text{Floor}) * \text{AddOn}^{\text{agg}}} \right) \right\}. \quad (3.14)$$

In our case we won't take into consideration the initial collateral so the multiplier is only activated when the initial value is negative. Note that the multiplier decreases when the value decreases and that the multiplier increases when the aggregated add-on increases.

To calculate the aggregate add-on, banks must calculate add-ons for each asset class within the netting set. The SA-CCR uses the following five asset classes:

- interest rate derivatives
- Foreign exchange derivatives
- Credit derivatives
- Equity derivatives
- Commodity derivatives

The respective add-ons for each asset class are simply aggregated using the following formula:

$$\text{AddOn}^{\text{aggregate}} = \sum_a \text{AddOn}^a, \quad (3.15)$$

where  $a$  is the asset class. When this primary risk driver is clearly identifiable, the transaction will fall into one of the asset classes described above.

Per trade in one asset class we must calculate the effective notional (EN) which is described below. Then, we need to divide the trades into hedging sets by separating them according to the corresponding currency. Then per hedging sets we can divide the trades into one of the three maturity buckets. Bucket 1 for trades with a maturity lower than one year, bucket 2 for maturities between one and five years, and bucket 3 for maturities higher than five years. We calculate the effective notional per maturity bucket by adding the effective notional on trade level. We name these  $EN_{B1}$ ,  $EN_{B2}$  and  $EN_{B3}$ .

To calculate the effective notional per hedging set  $h$  we follow the following formulas, whereby we distinguish the cases whether the bank chooses to recognise offset:

$$\text{Offset formula: } EN_h = \sqrt{EN_{B1}^2 + EN_{B2}^2 + EN_{B3}^2 + 1.4EN_{B1}EN_{B2} + 1.4EN_{B2}EN_{B3} + 0.6EN_{B1}EN_{B3}} \quad (3.16)$$

$$\text{No Offset formula: } EN_h = |EN_{B1}| + |EN_{B2}| + |EN_{B3}| \quad (3.17)$$

For foreign exchange classes we do not separate the trades into maturity buckets. The  $EN_h$  is then simply the sum of the effective national's of all the trades within the hedging set. In the foreign exchange asset class we distinguish hedging sets by the same currency pair.



We now want to derive the add-on per asset class. We do this by using the effective notional per hedging set  $EN_h$  described above, and with the Supervisory Factor (SF) per asset-class. This is a factor specific to each asset class used to convert the effective notional amount into effective EPE based on the measured volatility of the asset class. For the interest rate class this is equal to 0.5% and for the foreign exchange class 4%. The hedging set add-on is derived when we multiply  $EN_h$  with SF. The asset class add-on is calculated by taking the sum of the hedging set add-ons:

$$\text{AddOn}^a = \sum_h \text{AddOn}_h = \sum_h \text{SF} * |EN_h| \quad (3.18)$$

We need to divide the three products IRS, XCS and FX Forward to their corresponding asset class. We will say that the IRS and XCS are in the interest rate asset class since this is the most dominant risk factor. Within the IRS asset class we will divide the hedging sets where the product is in domestic currency and foreign currency. Within the XCS asset class we will divide the hedging sets in the cases where the fixed leg is paid in the domestic currency and where the fixed leg is paid in foreign currency. Within the FX asset class we have only one hedging set.

### 3.5.3. Effective Notional

The effective notional is a measure of the sensitivity of the trade to the underlying risk factors and is calculated as:

$$EN = d * M_f * \delta, \quad (3.19)$$

where  $d$  is the adjusted notional. For interest rate and credit derivatives, the adjusted notional is the product of the trade notional amount  $N$  converted to the domestic currency, and the supervisory duration  $D_s$ . This last term is given by the following formula:

$$D_s = \frac{\exp(-0.05 * S) - \exp(-0.05 * T)}{0.05}, \quad (3.20)$$

where  $S$  is the start date and  $T$  the end date of the time period referenced by the interest rate or credit derivative. Thus  $d$  is given by:

$$d = N * D_s. \quad (3.21)$$

For derivatives in the foreign exchange asset class the adjusted notional is the notional value of the foreign currency leg of the derivative contract converted to the domestic currency.

Supervisory delta  $\delta$  is also defined at the trade level and is applied to the adjusted notional amount to reflect the direction of the transaction and its non-linearity. It is defined for all derivatives as +1 for a long position and -1 for a short position.

For XCS and IRS products  $\delta = 1$  when we pay the fixed leg and receive the floating leg. Then consequently we have  $\delta = -1$  when we pay the floating leg and receive the fixed leg. For FX products  $\delta = 1$  when we pay the domestic currency and receive the foreign currency. Consequently  $\delta = -1$  when we pay the foreign currency and receive the domestic currency.

The maturity factor  $M_f$  takes in to account the time period over which the PFE is calculated. This is different for margined and unmargined netting sets. For unmargined netting set the maturity factor is calculated as the minimum of one year and the remaining maturity  $M$  of the derivative contract, floored at ten business days. This gives the following formula for the

maturity factor:

$$M_f = \sqrt{\frac{\min\{M; 1\}}{1}}. \quad (3.22)$$

For a margined netting set, the maturity factor is calculated using the MPOR. Since we have set the MPOR to 10 business days and  $M_f$  is expressed in years, the formula is given by:

$$M_f = \frac{3}{2} \sqrt{\frac{10}{252}}. \quad (3.23)$$

### 3.6. EAD under the Internal Model Method

Another way to calculate the EAD is by an Internal Model Method, i.e the IMM. Detailed model requirements of this method are described in the Basel framework article CRE53 [4]. A bank that wants to use the IMM to measure exposure or EAD can only do so when it has approval from its supervisor.

The IMM measures the CCR exposure or EAD at the level of a netting set. It must forecast the changes in market value due to changes in market variables, i.e. risk factors, such as interest and exchange rates. For margined counterparty's the model may also capture future collateral movements. The particular model that a bank should use is not specified in regulations. Instead it must meet certain requirements. When using such a model, the exposure of default is defined as follows:

$$\text{EAD} = \alpha * \text{EEPE}, \quad (3.24)$$

where  $\alpha$  is again 1.4 and EEPE is the average over time of the effective EE.

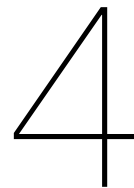
The internal model estimates EE at future time points  $t_1, t_2, \dots, t_n$ . The effective EE is the non-decreasing EE and it is given by:

$$\text{effectiveEE}(t_k) = \max(\text{effectiveEE}(t_{k-1}), \text{EE}(t_k)), \quad (3.25)$$

for  $t_k = t_1, t_2, \dots, t_n$ . The IMM requires that we calculate the EEPE over a 1 year time horizon. If all contracts expire before one year, then the we need to calculate the EEPE over the time until the last contract is expired. Using that  $dt_k = t_k - t_{k-1}$ , the EEPE is given by:

$$\text{EEPE} = \sum_{k=1}^{\min(1, M)} \text{effectiveEE}(t_k) dt_k. \quad (3.26)$$

Note that the alpha factor in the SA-CCR and in the IMM is introduced to account for WWR. Later in this thesis, as a main goal of this thesis, we will model WWR specifically and integrate it into the simulation framework we develop according to IMM requirements.



## Simulation Framework without WWR

In this chapter we set up a simulation framework for CCR according to IMM requirements, as well as the pricing functions to value the financial products which we will include in our testing portfolios. In accordance with IMM, in this simulation framework we won't take the WWR into consideration yet. Below we first describe the mathematical framework for interest rates, after which we define the products that we consider in this thesis. We then describe the algorithms to calculate the EE. Lastly, we develop the method for calibration and conduct sensitivity analysis to determine the parameters.

### 4.1. Various interest rate definitions

The concept of interest rate is quite familiar to many of us. When someone deposits money on the bank account they expect to receive some interest from it. Or when a bank loans money to a client, the client must pay interest. We provide mathematical definitions of various interest rates below so that we can model them properly in our simulation. The definitions can be found in the books [22] and [9].

**Definition 4.1.1** (Money market account). *We say  $B(t)$  is the value of a bank account at time  $t \geq 0$ . Here we assume  $B(0) = 1$  such that:*

$$B(t) = e^{\int_0^t r(s) ds}, \quad (4.1)$$

where  $r(t)$  is a function of time and is referred to as the instantaneous spot rate or the short rate.

**Definition 4.1.2** (short-rate). *The short-rate is defined as the interest rate one earns on a riskless investment over an small period of time  $\Delta t$ . We see from the previous definition that:*

$$\frac{B(t + \Delta t) - B(t)}{B(t)} \approx r(t)\Delta t. \quad (4.2)$$

Thus money on the bank account, a riskless investment, grows at each small period of time  $\Delta t$  with the return of  $r(t)$ .

Bond securities are financial products that pay a regular interest, which is called a coupon, on a predefined amount of money. The main issuers of bonds in global financial markets are central and local governments, whereas companies may issue so-called corporate bonds. A

special type of bonds are zero-coupon bonds, which are often-used numeraires for pricing interest rate derivatives.

**Definition 4.1.3.** A basic interest rate product is the zero-coupon bond,  $P(t, T)$ , which pays 1 currency unit at maturity time  $T$ , i.e.  $P(T, T) = 1$ . We are interested with its value at time  $t < T$ . The price of a zero-coupon bond at time  $t$  with maturity  $T$  is thus given by:

$$P(t, T) = \mathbb{E}^Q \left[ \frac{B(t)}{B(T)} P(T, T) | \mathcal{F}(t) \right] = \mathbb{E}^Q \left[ e^{-\int_t^T r(z) dz} | \mathcal{F}(t) \right]. \quad (4.3)$$

We can define the spot rate  $R(t, T)$  as the continuously compounded rate of return that generates the observed price of the pure discount bond. We can solve this as:

$$P(t, T) = e^{-R(t, T)(T-t)}, \quad (4.4)$$

$$R(t, T) = -\frac{\ln P(t, T)}{(T-t)}. \quad (4.5)$$

Alternative to continuous compounding we can do simple compounding.

**Definition 4.1.4.** The simply compounded interest rate at time  $t$  for maturity  $T$  is denoted by  $L(t, T)$  and is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from  $P(t, T)$  units of currency at time  $t$  when accruing occurs proportionally to the investment time.

$$L(t, T) = \frac{1 - P(t, T)}{(T-t)P(t, T)}. \quad (4.6)$$

Market LIBOR rates are simply compounded rates. It can be easily proved that the instantaneous short rate  $r(t)$  is the limit of the different rates defined above:

$$r(t) = \lim_{T \rightarrow t} R(t, T) = \lim_{T \rightarrow t} L(t, T). \quad (4.7)$$

#### 4.1.1. Forward interest rate

Forward rates are characterised by three time instants,  $t$ , expiry  $T$  and maturity  $S$  with  $t \leq T \leq S$ . Forward rates are interest rates that can be decided today for an investment in a future time period. We can derive the forward interest rate through a forward rate agreement (FRA). The contract gives its holder an interest-rate payment for the period between time  $T$  and  $S$ . At maturity  $S$  a payment based on the floating spot rate  $L(T, S)$  is exchanged for a payment based on the fixed interest rate  $K$  resetting at  $T$ . For a notional amount  $N$  the value of the FRA at time  $S$ , seen from the perspective where one pays the floating leg, is given by:

$$V_{FRA}(S) = N(S-T)(K - L(T, S)) = N \left( (S-T)K - \frac{1}{P(T, S)} + 1 \right). \quad (4.8)$$

Therefore the value for the FRA at time  $t$  is:

$$V_{FRA}(t) = P(t, S)V_S^{FRA} = P(t, S)N(S-T)(K - L(S, T)), \quad (4.9)$$

$$= N(P(t, S)(S-T)K - P(t, T) + P(t, S)). \quad (4.10)$$

There is one value for  $K$  such that the value at  $t$  is 0. This resulting rate is the forward rate.

**Definition 4.1.5.** The simply compounded forward interest rate at time  $t$  for expiry  $T > t$  and maturity  $S > t$  is denoted by  $F(t, T, S)$  and is defined by:

$$F(t, T, S) = \frac{1}{S - T} \left( \frac{P(t, T)}{P(t, S)} - 1 \right), \quad (4.11)$$

when the maturity date approaches the expiry date we can get the instantaneous forward rate by taking the limit.

**Definition 4.1.6** (Instantaneous forward interest rate). The Instantaneous forward interest rate at time  $t$  for maturity  $T > t$  is denoted by  $f(t, T)$  and given by:

$$f(t, T) = \lim_{S \rightarrow T} F(t, S, T) = -\frac{\partial \ln P(t, T)}{\partial T}. \quad (4.12)$$

#### 4.1.2. IRS swap

Swaps are financial products that enable their holders to swap two sets of interest rate payments. The most commonly known type of swap is the plain vanilla interest rate swap. We will consider the case where one party agrees to pay the fixed cash flows that are equal to the interest at a predetermined, fixed rate on a notional amount. The counterparty pays a floating interest on the same notional amount. The payments are made at predetermined future dates before a pre-defined maturity date.

We will refer to the plain vanilla interest rate swap as just the Interest Rate Swap (IRS). The IRS consists of two legs, a fixed leg with a series of fixed rate payments, at a fixed rate  $K$  at the future times  $T_{i+1}, \dots, T_m$  and the float leg consisting of a series floating rates. We take  $\tau_i$  as the time between payments in years. The payment of the fixed leg at  $T_i$  then equals  $\tau_i N K$ , with  $N$  the notional amount and  $K$  the fixed interest rate. We choose  $K$  such that value of the swap at  $t = 0$  is equal to 0.

The float leg equals  $N \tau_i F(t; T_{i-1}, T_i)$  which is the forward interest rate over the period  $[T_{i-1}, T_i]$ . Here  $T_{i-1}$  is the reset date and  $T_i$  the maturity date. This means that the interest rate over the period  $[T_{i-1}, T_i]$ , is determined at  $T_{i-1}$ <sup>1</sup>. We will assume that these dates will overlap with the payment dates. The payoff of the interest rate swap is then given by:

$$V_{IRS}(t) = N \sum_{k=i+1}^m \tau_k P(t, T_k) (F(t; T_{k-1}, T_k) - K). \quad (4.13)$$

Note that we only sum beginning at  $i + 1$ , since the start date of the swap is at  $T_i$ , the first payment is at  $T_{i+1}$ . We now only consider the price of the payer swap. Using the definition of the forward rate we see:

$$V_{IRS}(t) = N \sum_{k=i+1}^M \tau_k P(t, T_k) F(t; T_{k-1}, T_k) - NK \sum_{k=i+1}^M \tau_k P(t, T_k) \quad (4.14)$$

$$= N(P(t, T_i) - P(t, T_m)) - NK \sum_{k=i+1}^m \tau_k P(t, T_k), \quad (4.15)$$

where  $t < T_{i+1}$ , and we will use that  $P(t, T_i) = 1$  if  $t > T_i$ . Thus we will shift  $T_i$  such that  $T_i < t < T_{i+1}$

<sup>1</sup>There exist also set-in-arrear IR swaps, for which the forward rates are set at the end of each coupon period, i.e.,  $T_i$ . These are not in scope of this thesis.

## 4.2. Foreign exchange derivatives

In our testing portfolio we also have products related to currency exchanges. We will use the terms domestic and foreign interest rates to define the different interest rate per currency. For the exchange rate we will use the notation  $X(t)$ , which means  $X(t)$  unit of domestic currency equals one unit of foreign currency at time  $t$ .

A simple exchange rate product is the Foreign Exchange (FX) Forward. This is a derivative where one currency is exchanged for another currency at future date at a predetermined exchange rate. We want to exchange an amount of domestic currency  $N_d$  for an amount of foreign currency  $N_f$  at time  $T$ . Using the zero coupon bond for the domestic and foreign currency we receive the value of the forward contract at time  $t < T$ :

$$V_{FX}(t) = N_f P_f(t, T) X(t) - N_d P_d(t, T), \quad (4.16)$$

where  $P_f(t, T)$  is the zero coupon bond in the foreign currency and  $P_d(t, T)$  the zero coupon bond in the domestic currency. At the time the contract is made we want to choose  $N_d$  and  $N_f$  according to a predetermined exchange rate  $X_0$  and we have  $N_d = X_0 N_f$ . The value of an FX Forward is than fair at time  $t$  when  $X_0$  is given by the forward exchange rate.

**Definition 4.2.1.** *The forward exchange rate is given by:*

$$X_F(t, T) = X(t) \frac{P_f(t, T)}{P_d(t, T)}. \quad (4.17)$$

We will also consider a Cross-Currency Swap (XCS). This is a swap which is similar to the interest swap, only we will receive one of the legs in foreign currency. The value of a swap in which one receives the floating leg in foreign currency and pays the fixed leg in domestic currency is given in domestic currency:

$$V^{XCS}(t) = N_f X(t) \sum_{k=i+1}^M \tau_k P_f(t, T_k) F_f(t; T_{k-1}, T_k) - N_d K \sum_{k=i+1}^M \tau_k P_d(t, T_k), \quad (4.18)$$

where we also choose  $N_d$  and  $N_f$  according to a predetermined exchange rate  $X_0$ . Later in the calculations of our portfolio we allow different variates of XCS contracts such as we pay or receive the floating or fixed leg.

## 4.3. Interest rate models

To model the interest rate, a popular type of models in industry are short-rate models, in particular the Hull-White model and an adaption of the Hull-White model i.e. the G1+ model. For more details we refer to [22], [9] and [12].

An interest rate model is classified as an *affine term-structure model*, if it is a stochastic model whereby the drift and volatility are of the affine form. More precisely, when the risk free dynamic of the short rate model is given by:

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t), \quad (4.19)$$

then the model has an affine term structure if the drift and volatility are of the following form for deterministic time functions  $\beta_1$ ,  $\beta_2$ ,  $\alpha_1$  and  $\alpha_2$ :

$$\mu(t, r(t)) = \beta_1(t)r(t) + \alpha_1(t), \quad (4.20)$$

$$\sigma^2(t, r(t)) = \beta_2(t)r(t) + \alpha_2(t). \quad (4.21)$$

It is desirable for the short rate model to be of affine term structure since we can then write the zero coupon bond as an expression of the short rate. Note that for affine interest rate models, we can always write the price of a zero-coupon bond as follows:

$$P(t, T) = A(t, T) \exp(-B(t, T)r(t)), \quad (4.22)$$

where  $A$  and  $B$  can be obtained by solving the following Riccati differential equations:

$$\frac{\partial}{\partial t} B(t, T) + \beta_1(t)B(t, T) - \frac{1}{2}\beta_2(t)B(t, T)^2 + 1 = 0, \quad B(T, T) = 0 \quad (4.23)$$

$$\frac{\partial}{\partial t} [\ln A(t, T)] - \alpha_1(t)B(t, T) + \frac{1}{2}\alpha_2(t)B(t, T)^2 = 0, \quad A(T, T) = 1. \quad (4.24)$$

#### 4.3.1. Hull-White model

The Hull-White model is a single-factor, no-arbitrage yield curve model in which the short-term interest rate is driven by an extended Ornstein-Uhlenbeck mean reverting process. The dynamic of process  $r(t)$  is given by:

$$dr(t) = \lambda(\theta(t) - r(t))dt + \eta dW_r(t), \quad r(0) = r_0, \quad (4.25)$$

where  $W_r(t)$  is the Brownian Motion, parameter  $\eta$  determines the overall level of volatility and  $\lambda$  is the mean reversion rate parameter. A large value of  $\lambda$  causes short-term rate movements to dampen out rapidly, so that the long-term volatility is reduced. Here  $\theta(t)$  is a time-dependent drift term, which is used to fit the mathematical bond prices to the yield curve observed in the market, which is given by:

$$\theta(t) = \frac{1}{\lambda} \frac{\partial}{\partial t} f^r(t) + f^r(t) + \frac{\eta^2}{2\lambda^2} (1 - e^{-2\lambda t}). \quad (4.26)$$

The Hull-White process is free of arbitrage if and only if we have the following expression in (4.26):

$$f^r(t) = -\frac{\partial}{\partial t} \log P_{mkt}(t), \quad (4.27)$$

where  $P_{mkt}(t)$  is the market discount and is given by:

$$P_{mkt}(t) = e^{-r_0 t}. \quad (4.28)$$

Using Ito's lemma we get the following solution for the short rate process:

$$r(t) = e^{-\lambda t} r(s) + \lambda \int_s^t \theta(z) e^{-\lambda(t-z)} dz + \int_s^t \eta e^{-\lambda(t-z)} dW_r(z). \quad (4.29)$$

Therefore,  $r(t)$  is conditionally on  $\mathcal{F}_s$  normally distributed with mean and variance given by:

$$\mathbb{E}[r(t)|\mathcal{F}_s] = r(s)e^{-\lambda t} + \lambda \int_s^t \theta(z) e^{-\lambda(t-z)} dz, \quad (4.30)$$

$$\text{Var}[r(t)|\mathcal{F}_s] = \int_s^t \eta^2 e^{-2\lambda(t-z)} dz = \frac{\eta^2}{2\lambda} (1 - e^{-2\lambda(t-s)}). \quad (4.31)$$

From the Hull-White model we get the following results for functions  $A(t, T)$  and  $B(t, T)$  in the equation for the zero-coupon bond (4.22):

$$B(t, T) = \frac{1}{\lambda} [1 - e^{-\lambda(T-t)}] \quad (4.32)$$

$$A(t, T) = \frac{P_{mkt}(T)}{P_{mkt}(t)} \exp\left(B(t, T)f^r(t) - \frac{\eta^2}{4\lambda} (1 - e^{-2\lambda t})B(t, T)^2\right) \quad (4.33)$$

### 4.3.2. G1++ model

The G1++ model is an equivalent formulation for the Hull-White model and it is given by:

$$dx(t) = -\lambda x(t)dt + \eta dW(t), x(0) = 0, \quad (4.34)$$

$$r(t) = x(t) + \beta(t), \quad (4.35)$$

where  $\beta(t)$  is the deterministic shift having the same role as  $\theta(t)$ . Using Ito's lemma we can find the equivalence with the Hull-White model and we can find that:

$$\theta(t) = \lambda\beta(t) + \frac{\partial\beta(t)}{\partial t}. \quad (4.36)$$

Using the same method as before we can find the solution of the model which reads:

$$r(t) = x(t) + \beta(t) = x(s)e^{-\lambda(t-s)} + \int_s^t \eta e^{-\lambda(t-z)} dW_r(z) + \beta(t). \quad (4.37)$$

Therefore  $r(t)$  is again conditionally on  $\mathcal{F}_s$  normally distributed with mean and variance given by:

$$\mathbb{E}[r(t)|\mathcal{F}_s] = x(s)e^{-\lambda(t-s)} + \beta(t), \quad (4.38)$$

$$\text{Var}[r(t)|\mathcal{F}_s] = \int_s^t \eta^2 e^{-2\lambda(t-z)} dz = \frac{\eta^2}{2\lambda}(1 - e^{-2\lambda(t-s)}). \quad (4.39)$$

We see that the variance is indeed the same as the Hull-White model. We can now also derive the expression of  $\beta(t)$  such that the expectation is the same:

$$\beta(t) = f^r(t) + \frac{\eta^2}{2\lambda^2}(1 - e^{-\lambda t})^2. \quad (4.40)$$

Following the proof of [12], we can find the expression of  $P(t, T)$  using the G1++ model. Using the expression for the ZCB in (4.22), we find:

$$B(t, T) = \frac{1}{\lambda}(1 - e^{-\lambda(T-t)}), \quad (4.41)$$

$$(4.42)$$

$$A(t, T) = \frac{P_{mkt}(T)}{P_{mkt}(t)} e^{1/2(V(t, T) - V(0, T) + V(0, t))}, \quad (4.43)$$

here  $V(t, T)$  is the variance of  $\int_t^T x(s)$  conditional on  $\mathcal{F}_t$  given by:

$$V(t, T) = \frac{\eta^2}{\lambda^2} \left( T - t - 2 \frac{1 - e^{-\lambda(T-t)}}{\lambda} + \frac{1 - e^{-2\lambda(T-t)}}{2\lambda} \right). \quad (4.44)$$

## 4.4. Monte Carlo simulation

Throughout this paper we use the Hull-White model for the short rate simulations. Furthermore, to calculate EAD according to the IMM we need the EE of different products, where the EE is the expectation of the exposure. To calculate this expectation we need the *Monte Carlo simulation*. Our products depend on the risk factors interest rate and exchange rate which are based on the stochastic models we discussed in earlier subsections. To calculate EE, we need a few steps in the Monte Carlo simulation [8]:



- Simulate sample paths of the underlying variables (short rate, exchange rate).
- Evaluate the cash flows of a product on each sample path.
- Average the cash flows over the sample paths.

The Monte Carlo simulation is used to calculate the expectation of the exposures, and converges because of the law of large numbers, which states that the sample average converges almost surely to the expected value.

## 4.5. Simulation algorithms

We will now model the value of the IRS, from which we will calculate the exposure and find the EE using a Monte Carlo simulation. To model the interest rate we need the Euler discretization of the short rate process, which is given by:

$$r(t_{i+1}) = r(t_i) + \lambda(\theta(t_i) - r(t_i))\Delta t + \eta(W(t_{i+1}) - W(t_i)). \quad (4.45)$$

A pseudo code for the algorithm is described in Algorithm 1.

---

### Algorithm 1: Calculation EE for IRS

---

**Start:** We determine model parameters  $\lambda$  and  $\eta$ , choose the number of paths  $N$  and steps  $t_0, t_1, \dots, T$  for maturity  $T$ .

Generate standard normal distribution  $\mathbf{Z}$

**Generate short rate paths  $r$  and calculate exposure for every path:**

**for**  $i=t_0, \dots, T$  **do**

*Create the Brownian Motion increments  $dW_i = \sqrt{\Delta t}Z_i$ ;*

*Make short rate using (4.45);*

*Generate ZCB with (4.22).;*

*Calculate value IRS, with  $V$  in (4.15);*

**if** collateral is True **then**

        |  $E = \max(V - C, 0)$ , with  $C$  as in (3.9)

**end**

**else**

        |  $E = \max(V, 0)$

**end**

**end**

**Calculate expected exposure of the portfolio as average of  $N$  scenarios per time point as in 4.4.**

---

Beside the IRS we also want to have FX products in our portfolio, such as the XCS and FX Forward. For this we need to model the extra risk factors i.e, the exchange rate and the foreign short rate. We write the exchange rate  $X(t)$  as a GBM described in section 2.5:

$$dX(t) = \mu X(t)dt + \sigma X(t)dW_X(t)^{\mathbb{P}}, \quad (4.46)$$

with Euler discretization:

$$X(t_{i+1}) = X(t_i) \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma(W(t_{i+1}) - W(t_i))\right) \quad (4.47)$$

Whitin the GBM we use the real world measure  $\mathbb{P}$ . To have a better understanding about the selection of measures we will determine the dynamics under the risk neutral measure  $\mathbb{Q}$ ,

where we use that the process must be a martingale under the risk neutral measure. The proof is given in A.1, the dynamics in the risk-neutral measure is given by:

$$dX(t) = (r_d - r_f)X(t)dt + \sigma X(t)dW_X^{\mathbb{Q}}(t). \quad (4.48)$$

The triangle relation that we observe in (4.48) does not hold in practice. Furthermore, the martingale assumption for the risk neutral measure also does not hold in practice. We will thus use the real world measure  $\mathbb{P}$ . For the short rate dynamics we can switch to the  $\mathbb{P}$  measure when we calibrate the parameters, which we will do in the next section. The system of equations in the real world measure  $\mathbb{P}$  is then given by:

$$\begin{cases} dX(t) = \mu X(t) + \sigma X(t)dW_X^{\mathbb{P}^d} \\ dr_d(t) = \lambda_d(\theta_d(t) - r_d(t))dt + \eta_d dW_d^{\mathbb{P}^d} \\ dr_f(t) = \lambda_f(\theta_f(t) - r_f(t))dt + \eta_f dW_f^{\mathbb{P}^f} \end{cases} \quad (4.49)$$

where we want to write the foreign interest rate dynamics of  $r_f$  in the domestic real world measure. Using Girsanov's theorem we find this dynamics and it is shown below, the proof is given in A.2.

$$dr_f(t) = (\lambda_f(\theta_f(t) - r_f(t)) - \eta_f \sigma \rho_{fX})dt + \eta_f dW_f^{\mathbb{P}^d}. \quad (4.50)$$

To add FX products to our simulation engine, we need to implement a dependency between the risk factors. This dependency is driven by the correlation between the Brownian motions. To include these correlation into our simulation, we need to apply Cholesky decomposition on the following correlation matrix. Given the vector  $\mathbf{dW} = [dW_d, dW_f, dW_X]$ , the correlation matrix is given by:

$$\mathbf{C} = \mathbf{dWdW}^T = \begin{bmatrix} 1 & \rho_{df} & \rho_{dX} \\ \rho_{df} & 1 & \rho_{fX} \\ \rho_{dX} & \rho_{fX} & 1 \end{bmatrix}. \quad (4.51)$$

Similar to the 2D case described in the mathematical framework, we want to change the system with independent Brownian motions  $\tilde{\mathbf{W}} = [\tilde{W}_d, \tilde{W}_f, \tilde{W}_X]$ , to one where we have a correlation between the Brownian motions. The Cholesky decomposition gives us a lower triangular matrix  $\mathbf{L}$  such that :  $\mathbf{C} = \mathbf{LL}^T$ , then we know that the vector with correlated matrices is given by:  $\mathbf{W} = \mathbf{L}\tilde{\mathbf{W}}$ . Using the Cholesky decomposition we find the expression for matrix  $\mathbf{L}$ , which is given below:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ \rho_{df} & \sqrt{1 - \rho_{df}^2} & 0 \\ \rho_{dX} & \frac{\rho_{fX} - \rho_{df}\rho_{dX}}{\sqrt{1 - \rho_{df}^2}} & \sqrt{\frac{\rho_{dX}^2 - 2\rho_{df}\rho_{dX}\rho_{fX} + \rho_{dX}^2 + \rho_{fX}^4 - 1}{\rho_{dX}^2 - 1}} \end{bmatrix}. \quad (4.52)$$

The pseudo code for valuing EE of the FX products is given in Algorithm 2.

## 4.6. Calibration

To determine the values of the model parameters, we perform a calibration using historical data, consistent with the  $\mathbb{P}$  measure assumptions made in FX and IR models. To value this parameters as accurate as possible, we take into account as much historical data as we can.

---

**Algorithm 2:** Calculation EE for FX products

---

**Start:** We determine model parameters  $\lambda_d, \eta_d, \lambda_f, \eta_f, \mu$  and  $\sigma$ . We choose the number of paths  $N$  and steps  $t_0, t_1, \dots, T$  for maturity  $T$ .

Generate standard normal distribution matrix  $\mathbf{Z}$  and apply to the correlation matrix the Cholesky decomposition matrix to yield lower-triangle  $\mathbf{L}$ .

**Generate paths  $r_f, r_d, X$  and calculate exposure for every path:**

**for  $i=t_0, \dots, T$  do**

*Make Brownian motions correlated using  $W = L\tilde{W}$ ;*

*Create the Brownian Motion increments  $dW_i = \sqrt{\Delta t}Z_i$ ;*

*Generate dynamics  $X, r_f$  and  $r_d$  using (4.45) and (4.47).;*

*Generate ZCB with (4.22).;*

*Calculate value cross-currency products, with  $V$  in (4.16) and (4.18);*

**if collateral is True then**

        |  $E = \max(V - C, 0)$ , with  $C$  as in (3.9)

**end**

**else**

        |  $E = \max(V, 0)$

**end**

**end**

**Calculate expected exposure of the portfolio as average of  $N$  scenarios per time point as in 4.4.**

---

We will first consider the period from 2010 until 2022. However, Basel requires us to also take into account a stressed period of 3 years for the calibration which is stated in article (CRE 53.51) [4]. For this we will use the period from 2007 until 2010. We will discuss these stressed results in section 4.8.

We take the historical data from FRED economics [3]. For the short-rate parameters we will look at the spot rate observed in the market. We will use the English pound as the example of the domestic currency, whereby the interest spot rate is represented by the Sterling Overnight Index Average (SONIA) [6]. For the foreign currency example we will use the US dollar, whereby the interest spot rate is represented by the Federal Funds Effective Rate (FFER) which is similar to the Secured overnight Financing Rate (SOFR) [2]. We then can use historical data of the spot exchange rate of 1 pound to dollars to calibrate the FX rate model.

#### 4.6.1. Calibration of the short rate model

For the calibration of the short-rate model we will follow the method given by Park [13]. This method aims to minimize the difference between the volatility of the spot rate increments found analytically and the volatility of the spot rate increments found in the market. We use the sample variance to find the daily volatility. We take  $\{\Delta R_1^{market}, \dots, \Delta R_N^{market}\}$  the daily spot rate changes observed in the market. Using the fact that there are 252 business days in a year we have:

$$\sigma_{dR}^{market} = \sqrt{252} \sqrt{\frac{1}{N-1} \sum_{i=1}^N (\Delta R_i^{market} - \mathbb{E}[\Delta R^{market}])^2}, \quad (4.53)$$

where we use the sample mean to estimate the expectation:

$$\mathbb{E}[\Delta R^{market}] = \frac{1}{N} \sum_{i=1}^N \Delta R_i^{market}. \quad (4.54)$$

We then want to find the solution to the following optimization problem:

$$\min_{\eta, \lambda} (\sigma_{dR}^{market} - \sigma_{dR})^2, \quad (4.55)$$

where  $\sigma_{dR}$  is the analytical volatility of the spot rate. We use the affine term structure given in (4.22) and using the definition of the spot rate we see:

$$R(t, T) = \frac{-\ln P(t, T)}{T - t} = -\frac{\ln(A(t, T)) - B(t, T)r(t)}{T - t}. \quad (4.56)$$

with  $A(t, T)$  and  $B(t, T)$  as in (4.33) and (4.32) respectively. Using Ito's formula we see:

$$dR(t, T) = \frac{\frac{1}{2}\eta^2 B^2(t, T) - r(t)}{T - t} dt + \frac{\eta B(t, T)}{T - t} dW(t). \quad (4.57)$$

Since we can write  $dW(t)$  as  $\sqrt{dt}Z(t)$  with  $Z(t)$  the standard normal distribution, we can write the standard deviation of  $dR(t, T)$  as:

$$\sigma_{dR}(t, T) = \frac{\eta B(t, T)}{T - t} \sqrt{dt}. \quad (4.58)$$

We can set  $T = t + \tau$ , with  $\tau$  is the time fraction until the next time point. We can then set  $\tau$  to 1/252 and we get the equation:

$$\sigma_{dR}(\tau) = \frac{\eta(1 - e^{-\lambda\tau})}{\lambda\tau} \sqrt{dt}. \quad (4.59)$$

In the past couple years unusual events have happened which have affected the observed spot rates in the market, for example the Covid pandemic, it is wise to take into account more data from the past such that this unlikely event does not have a dominant effect on our calibration. We use the "moving window" technique. We minimize over several time periods, where we take a window of two years per period and move it up one year for each different time period. Starting at 2010, we have the time periods:  $P_1 = [01 - 01 - 2010; 01 - 01 - 2013]$ ,  $P_2 = [01 - 01 - 2011; 01 - 01 - 2014]$ ,  $P_3 = [01 - 01 - 2012; 01 - 01 - 2015]$ , ...,  $P_{10} = [01 - 01 - 2019; 01 - 01 - 2022]$ . Then the objective function of our minimization problem becomes:

$$\min_{\eta, \lambda} \sum_{j=1}^{10} (\sigma_{dR}^{market} - \sigma_{dR})_{P_j}^2. \quad (4.60)$$

The calibrated  $\eta$  and  $\lambda$  are shown in table 4.1. Furthermore, we have to think of the corresponding  $r_0$  for the foreign and domestic short rate, for this we will use the spot rate observed on 01-08-2022.<sup>2</sup>

<sup>2</sup>This is the date we started testing the model and used as starting date since then.

Table 4.1: The parameters for the domestic and foreign short-rate.

	$r_0$	$\lambda$	$\eta$
Domestic	0.012	0.89	0.038
Foreign	0.023	0.91	0.056

#### 4.6.2. Calibration of the GBM

We know that the exchange rate  $X(t)$  is usually modelled in the industry by a GBM and we want to calibrate the parameters based on historical data. This effectively makes the SDE of  $X(t)$  being defined in the real world measure and we have:

$$dX(t) = \mu X(t)dt + \sigma X(t)dW^{\mathbb{P}}(t). \quad (4.61)$$

From the GBM we know that  $Y(t) = \ln X(t)$  is normally distributed, where the increments of  $Y(t)$ ,  $dY(t)$ , are also normally distributed with:

$$dY(t) = \left( \mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dW^{\mathbb{P}}(t). \quad (4.62)$$

This means that the increments of  $Y$  are normally distributed with mean  $= \left( \mu - \frac{1}{2}\sigma^2 \right) dt$  and variance  $= \sigma^2 dt$ . We can then find these variables using the maximum likelihood [15].

We take the exchange rate of 1 dollar to pounds given by FRED [3] over a period from 01-01-2010 until 01-01-2022. When we have the data of the exchange rate changing over time, we want to fit the log-normal distribution  $X(t)$  on it. We take the data points of the exchange rate given by  $\{X(t_0), X(t_1), \dots, X(t_n)\}$  and take the natural logarithms, thus  $Y_i = \ln(X(t_i))$ . Since the increments are normally distributed we have that the process  $\{Y_1 - Y_0, Y_2 - Y_1, \dots, Y_n - Y_{n-1}\}$  is normally distributed. Using log likelihood we have that:

$$\text{mean} = \frac{1}{n} \sum_{i=1}^n Y_i - Y_{i-1} = \frac{1}{n}(Y_n - Y_0). \quad (4.63)$$

Then the variance is given by:

$$\text{variance} = \frac{1}{n-1} \sum_{i=1}^n (Y_i - Y_{i-1} - \text{mean})^2. \quad (4.64)$$

In this way we can fit a normal distribution on the log-increments of our data points using this mean and variance. This fit can be seen in figure 4.1.

Now from this mean and variance we can extract the parameters we need for the GBM. We know for the variables that variance  $= \sigma^2 dt$  and mean  $= \mu - \frac{1}{2}\sigma^2 dt$ , which then gives us:

$$\sigma = \sqrt{\text{variance}/dt}, \quad (4.65)$$

$$\mu = \text{mean}/dt + \frac{1}{2}\sigma^2. \quad (4.66)$$

The calibrated results can be observed in table 4.2. Furthermore we again take  $X_0$  as the exchange rate observed at 01-08-2022.

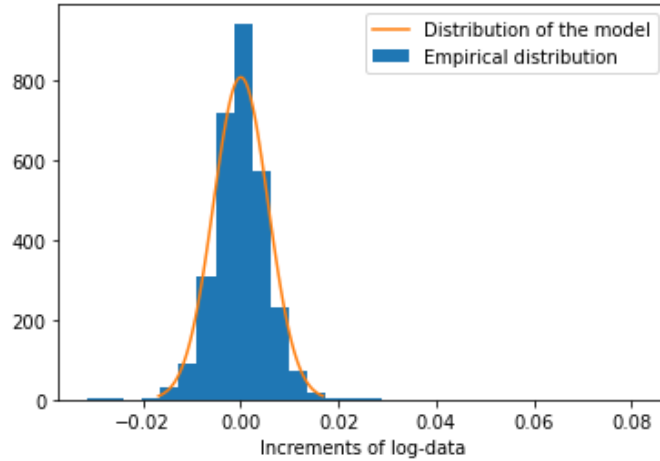


Figure 4.1: Calibration of normal distribution on the log increments of the historical exchange rate data.

Table 4.2: Results of the calibration of the GBM.

	$X_0$	$\mu$	$\sigma$
Exchange rate	0.82	0.019	0.089

If we implement the parameters as in the table in our GBM and take a look at the distribution of the log increments, we should see the model distribution match the empirical distribution. To visualize this see figure 4.2:

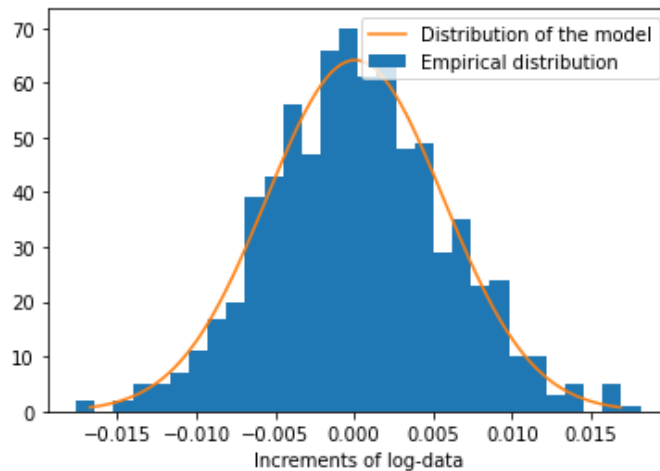


Figure 4.2: Normal distribution with parameters from table 4.2 fitted on the log-increments of the GBM modeled with the same parameters.

### 4.6.3. Calibration of correlation

Lastly, we want to calibrate the correlations between our risk factors given in (4.51). Using the expression of the spot rate (4.56), we get an expression of the short rate using the spot rate observed in the market:

$$r(t) = \frac{R(t, T) \cdot (T - t) + \ln(A_r(T, t))}{B_r(T, t)}. \quad (4.67)$$

Using the short rate and exchange rate observed in the market we can extract the informa-

tion about the random part  $dW(t_{i+1}) = W(t_{i+1}) - W(t_i)$ . We use the Euler discretization in (4.45) and (4.47), we see:

$$dW(t_{i+1})_r = \frac{r(t_{i+1}) - r(t_i) - \lambda(\theta(t_i) - r(t_i))\Delta t}{\eta} \quad (4.68)$$

$$dW(t_{i+1})_X = \frac{\ln\left(\frac{X(t_{i+1})}{X(t_i)}\right) - (\mu - \frac{1}{2}\sigma^2)\Delta t}{\sigma} \quad (4.69)$$

We can then extract the correlation directly using the build-in function in python for finding the correlation, this gives us the Pearson product-moment correlation coefficients. The results are presented in table 4.3

Table 4.3: Results of the calibration of the correlations.

	$\rho_{Xf}$	$\rho_{Xd}$	$\rho_{fd}$
Correlation	0.61	-0.22	0.5

## 4.7. Results

Now that we have found all the parameters which we need in the algorithms 2 and 1, we can then look at an example of the EE. For the IRS and XCS products we choose maturity  $T = 5$ , starting date  $T_i = 1$ , end date  $T_m = 5$ , 10 cash flows, notional  $N = 10000$  and we set  $K$  equal to  $r_0$ . The results can be observed in figure 4.3 and 4.4.

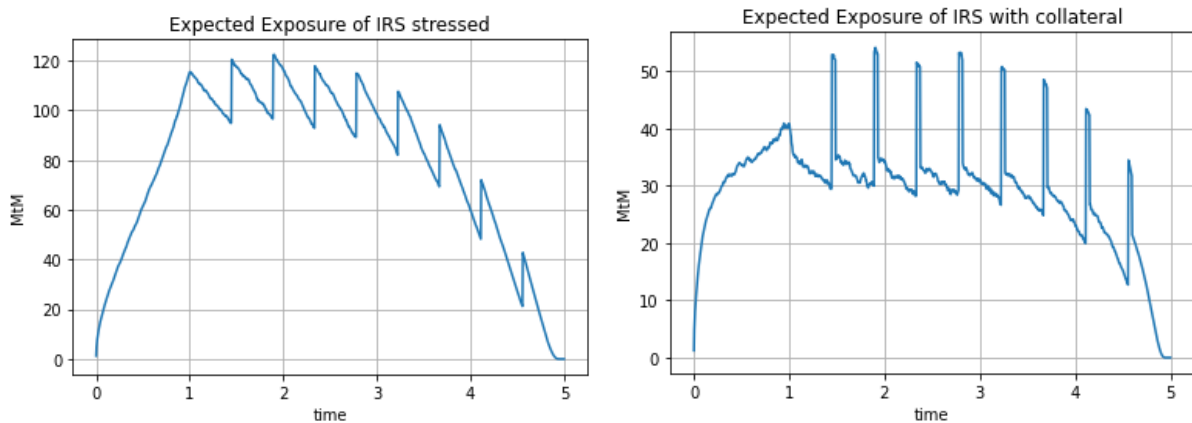


Figure 4.3: EE with and without collateral. Here we use 10000 paths,  $252 \cdot T$  steps, and parameters 4.1. For the exposure with collateral we have set  $MPOR = 10, TH = 40, MTA = 10$ .

Each time a payment is made we see that the exposure increases, when we pay the fixed leg. When we receive the fixed leg, the exposure will decrease every time a payment is made. The spikes within the collateral exposure are given because of the MPOR. When the MPOR is 0, we would see that the exposure is capped by  $VM + TH$ . However, because of the time delay in the collateral valuation, we see these spikes when the payments are made.

For the FX Forward we take a maturity  $T = 5$  and notional  $N_d = 10.000$ . The results are shown in figure 4.5.

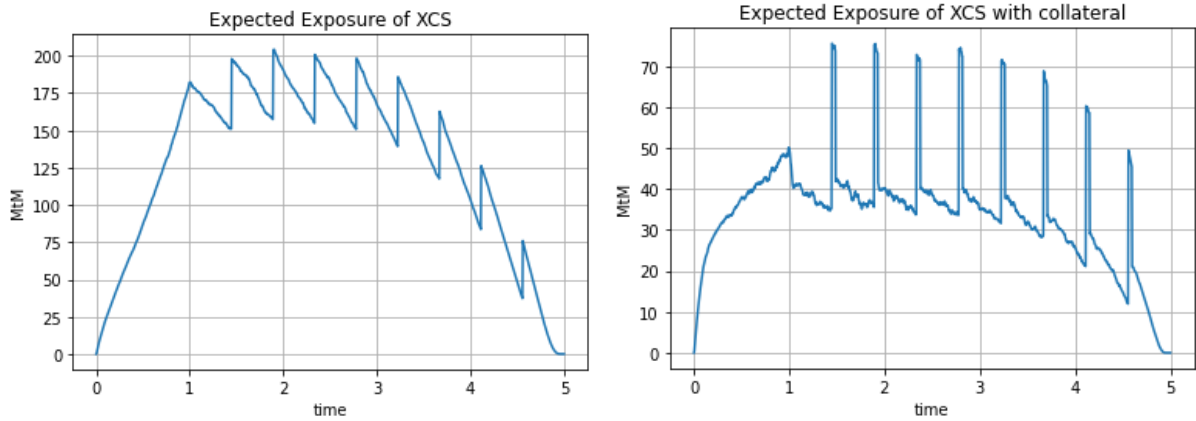


Figure 4.4: Expected exposure with and without collateral. Here we use 10000 paths,  $252 \cdot T$  steps, and parameters 4.1, 4.2, 4.3. For the exposure with collateral we have set  $MPOR = 10, TH = 60, MTA = 10$ .

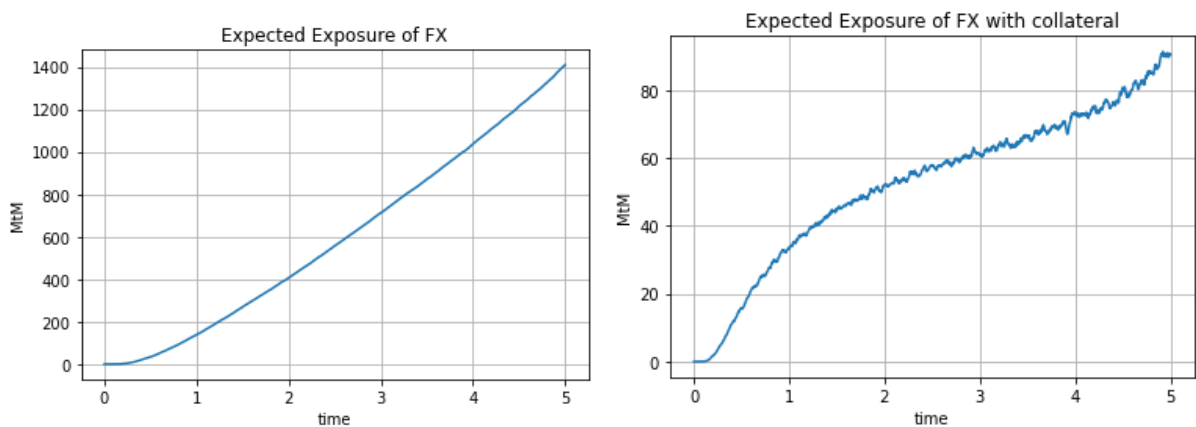


Figure 4.5: Expected exposure with and without collateral. Here we use 10000 paths,  $252 \cdot T$  steps, and parameters 4.1, 4.2, 4.3. For the exposure with collateral we have set  $MPOR = 10, TH = 40, MTA = 10$ .

## 4.8. Sensitivity analysis and stressed parameters.

To see if our implementation is correct we have performed various sensitivity analysis. We could for example see when we increase  $\eta$  in our short rate model the exposure quickly increases, also when we decrease  $\lambda$  the exposure would again increase. This is what we expect, when we increase the volatility, more extreme paths can be observed and our exposure will increase. This effect can then be limited by increasing  $\lambda$ , as stated before, a large value for  $\lambda$  will reduce the long-term volatility.

In the FX Forward we can observe significant sensitivity in the exchange rate. In the example above, we pay the domestic leg and receive the foreign leg. When the exchange rate increases, or if we increase the drift and volatility, we see an increase in the exposure of the FX Forward. The same holds for the XCS, only this difference is less prominent as in the FX Forward.

To see this effect more clearly, we will look at the stressed parameters. As said before, Basel requires us to take into account a stressed period of three years to estimate these parameters. We will use the period between 2007 and 2010 when the credit crisis took place. Using the calibration methods above, the stressed parameters can be observed in table 4.4. To see the effect of these stressed parameters we again show the expected exposure of the three



products as described above. However, this time we will see a clear increase in EE because of the stressed parameters.

Table 4.4: Parameters calibrated by stressed data.

Stressed	$\lambda$	$\eta$	$\mu_X$	$\sigma_X$
Domestic short rate	2.05	0.3		
Foreign short rate	2.1	0.37		
Exchange rate			0.075	0.13

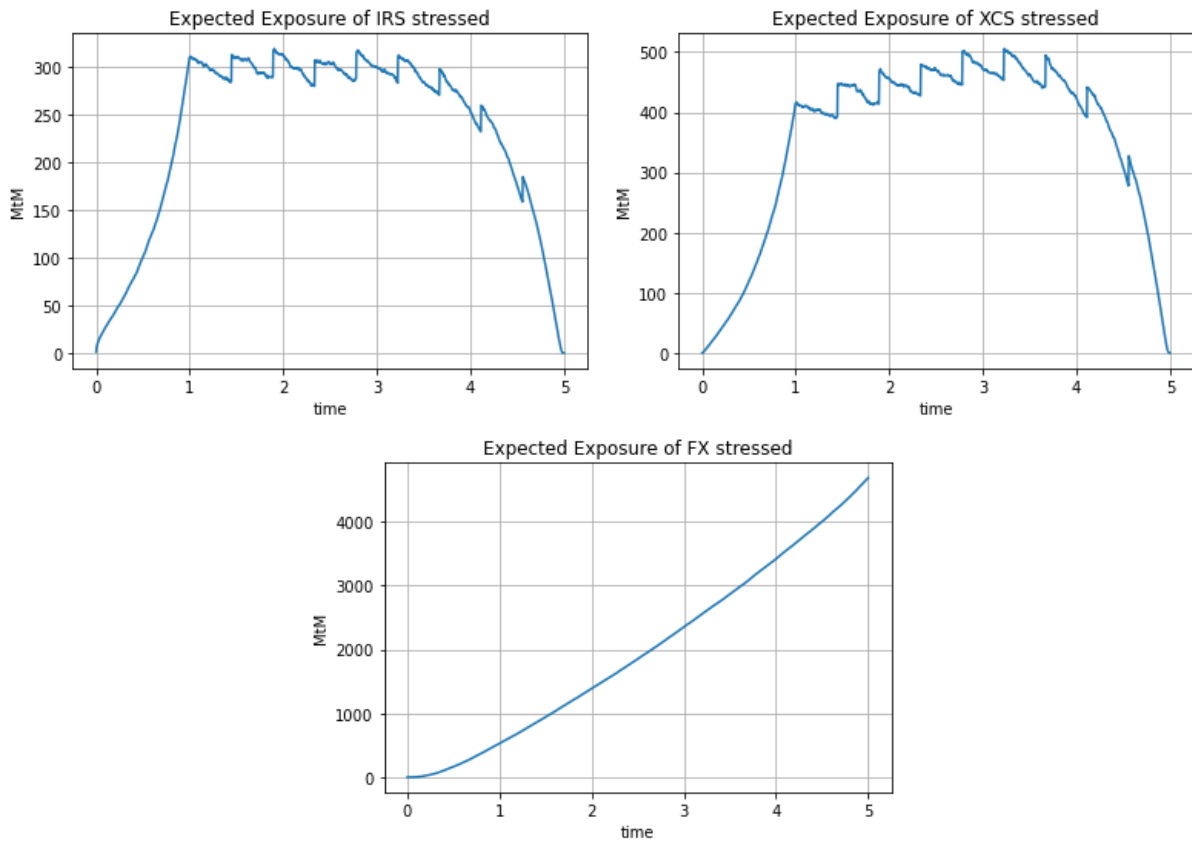


Figure 4.6: Expected exposure of the three same products as in figures 4.3, 4.4 and 4.5 using the stressed parameters observed in table 4.4.



# 5

## Simulation Framework with WWR

In this chapter we set up the framework for the EAD simulation whereby the WWR is integrated. First we describe a method to extract the probability of default using a Credit Default Swap (CDS) spread. After that we can use the probability of default in a Copula model to include the WWR in the EAD calculation.

### 5.1. Extracting Probability of default from CDS spreads

The goal of this section is to extract the probability of default of a counterparty using credit default swaps. We will use the pricer of a CDS to make an estimation about the probability of default for our counterparty. The idea is to search for the parameter values, with which the CDS pricer returns a theoretical CDS price that matches the market CDS quote. The pricing method we select for CDSs are based on the industry-standard reduced-form model, or the hazard-rate model. One of the input parameters of this model is the *hazard rate*, which corresponds to the intensity of a Poisson process which we assume the arrival of the default event to follow. With this hazard rate we can then construct the probability of default.

#### 5.1.1. CDS

A CDS is a financial instrument, which to the buyer of the swap is a protection against the default of the issuer of a bond the CDS buyer holds. The seller of the CDS is usually a bank or insurance company [28]. In the case of a default, the seller of the CDS is obligated to buy back the debt security for its face value from the buyer of the CDS in case of a physical settlement, and in return get the defaulted debt. The exact amount that the seller is obligated to pay to the buyer is calculated as a LGD multiplied by the notional amount. LGD can also be expressed as one minus the recovery rate  $R$ , with the recovery rate being prescribed in the contract if it is a fixed-recovery CDS or otherwise being decided by the market (via an auction after default). In practice, the market convention is to assume  $R$  equal to 40% for the purpose of valuing the CDSs. The notional value of a CDS refers to the face value of the underlying security. The premium, also known as the contractual spread, that is paid by the buyer to the CDS sellers is expressed in a percentage of the notional value. This percentage is then given in basis points, e.g., 10000 basis point is 1%.

Following the Big Bang Protocol, the contractual spreads are standardized these days. The CDS spreads quoted in the market are not the contractual spreads but the spread values which equalize the default legs and the related premium legs. The higher the CDS spreads, the worse the credit worthiness of the corresponding bond issuers. One can imagine that when

the credit rating of a counterparty changes, its credit spreads move up accordingly. In general ratings made by credit agencies are lagged behind of the spreads. The best known credit rating agencies are Moody's, S&P and Fitch [29]. According to a research [21], the ratings of Moody's and Fitch are linked to this default spread, which we can see in table 5.1.

Moody's rating	Fitch Rating	Default spread
Aaa	AAA	0
Aa1	AA+	25
Aa2	AA	50
Aa3	AA-	70
A1	A+	85
A2	A	100
A3	A-	115
Baa1	BBB+	150
Baa2	BBB	175
Baa3	BBB-	200
Ba1	BB+	240
Ba2	BB	275
Ba3	BB-	325
B1	B+	400
B2	B	500
B3	B-	600
Caa	CCC	700
Ca	CC	850
C	D	1000

Table 5.1: Default spreads linked to ratings according to [21].

### 5.1.2. Extracting the hazard rate

In the reduced form model, it is assumed that default is a Poisson process with hazard rate  $h(t)$ . If the default time is  $\tau$ , the probability of default over a small time period  $dt$  conditional on no default to time  $t$  is:

$$\mathbb{P}(t < \tau \leq t + dt | \tau > t) = h(t)dt. \quad (5.1)$$

The probability of surviving to at least time  $T > t$  assuming no default has occurred up to time  $t$  is given by:

$$Q(t, T) = e^{-\int_t^T h(s)ds}. \quad (5.2)$$

From this probability of survival we can then extract the probability of default within  $[t, T]$  with:

$$P(t, T) = Q(0, t) \cdot (1 - Q(t, T)). \quad (5.3)$$

To price a CDS we need to value both the protection leg and the premium leg. For the protection leg we know that at some default time  $\tau$ , we receive  $N(1 - R)$ . We assume that the recovery rate is independent of the interest rate and hazard rate. Furthermore, we assume that the interest rate is independent of the hazard rate, and that the valuation date of the CDS is at

time 0. The protection leg is now equal to:

$$V_{\text{protection leg}} = N(1 - R)\mathbb{E}[P(0, \tau)\mathbb{1}_{\tau < T}] = N(1 - R) \int_0^T P(0, s) \frac{dQ(0, s)}{ds} ds \quad (5.4)$$

$$= N(1 - R) \int_0^T P(0, s) dQ(0, s). \quad (5.5)$$

The premium leg consists of two parts: regular premium payments up to the expiry of the CDS which cease if a default occurs and a single payment of accrued premium in the event of a default. For simplicity, we won't take the last term into consideration. If there are  $M$  remaining payments with payment times  $t_1, t_2, \dots, t_M$ , period end times  $e_1, e_2, \dots, e_m$  and year fractions of  $\Delta_1, \Delta_2, \dots, \Delta_M$ , then the present value of the premium leg is:

$$V_{\text{premium leg}} = NC\mathbb{E}\left[\sum_{i=1}^M \Delta_i P(0, t_i) \mathbb{1}_{e_i < \tau}\right] = NC \sum_{i=1}^M \Delta_i P(0, t_i) Q(0, e_i). \quad (5.6)$$

We can write this as  $C \cdot V_{\text{premium}}$  with  $V_{\text{premium}} = N \sum_{i=1}^M \Delta_i P(0, t_i) Q(0, e_i)$  and  $C$  the percentage of the notional amount which need to be paid as premium, thus the spread. The present value of the CDS can now be summarized as follows:

$$V_{\text{CDS}} = V_{\text{protection leg}} - C \cdot V_{\text{premium}}. \quad (5.7)$$

We are now looking for  $C$  such that  $V_{\text{CDS}} = 0$ . This is also known as the par spread and we have:

$$C = \frac{V_{\text{protection leg}}}{V_{\text{premium}}}. \quad (5.8)$$

Using the equation above, we can for a given spread and maturity bootstrap the hazard rate. We can use common approximation for the integral and assume that any payment resulting from a default is made on the next scheduled payment date. We can then approximate the protection leg as:

$$V_{\text{protection leg}} \approx N(1 - R) \sum_{i=1}^M P(0, t_i) (Q(0, t_{i-1}) - Q(0, t_i)), \quad (5.9)$$

where  $t_i$  is a payment date. In addition, if we assume that on average default happens mid-way through the period we can approximate:

$$V_{\text{premium}} \approx N \sum_{i=1}^M \Delta_i P(0, t_i) \left[ Q(0, t_i) + \frac{Q(0, t_{i-1}) - Q(0, t_i)}{2} \right], \quad (5.10)$$

$$= N \sum_{i=1}^M \Delta_i P(0, t_i) \left[ \frac{Q(0, t_{i-1}) + Q(0, t_i)}{2} \right]. \quad (5.11)$$

Given a spread  $S$  for a CDS with maturity  $T$  and notional  $N$ , we can find the hazard rate by bootstrapping the following equation:

$$0 = N(1 - R) \sum_{i=1}^M P(0, t_i) (Q(0, t_{i-1}) - Q(0, t_i)) - S \cdot N \sum_{i=1}^M \Delta_i P(0, t_i) \left[ \frac{Q(0, t_{i-1}) + Q(0, t_i)}{2} \right]. \quad (5.12)$$

We can now distinguish two cases, one whereby the hazard rate is constant for the maturity and the other whereby the hazard rate is piece-wise constant for different maturities. If we take the hazard rate constant we see  $Q(0, t_i)$  as:

$$Q(0, t_i) = e^{-\int_0^{t_i} \lambda(s) ds} = e^{-t_i \lambda}. \quad (5.13)$$

In practise the piece-wise constant hazard rate curve is bootstrapped as follows. Suppose there are three CDSs available of maturity 1, 3 and 5 years for a bond, we then assume three different hazard rates for the maturities in years from 0 to 1, 1 to 3 and 3 to 5. Given respectively by  $h_{0,1}$ ,  $h_{1,3}$  and  $h_{3,5}$  with corresponding spreads of 100, 200 and 400 basis points. We can then bootstrap these hazard rates in order, from the short tenors to long tenors. Take the valuation of the 5-year CDS as an example: if the hazard rate curve is 0.01 for year 0 to 1 and 0.02 for year 1 to 3 then we insert 0.01 and 0.02 into the pricing formula for the valuation of the cash flows to occur before and on 1 year and 3 years, respectively, and then back out the hazard rate level for year 3 to 5. Suppose in this example, we obtain 0.075 for the hazard rate between 3 and 5 year from bootstrapping.

This we will compare with the case whereby we have a constant hazard rate and a spread equal to 400 basis points from 0 to 5 years. Furthermore we take  $N = 10.000$ ,  $\Delta_i = 0.25$  and  $R = 0$ . We can see the impact of having a term structure in the hazard rate in the figure below.

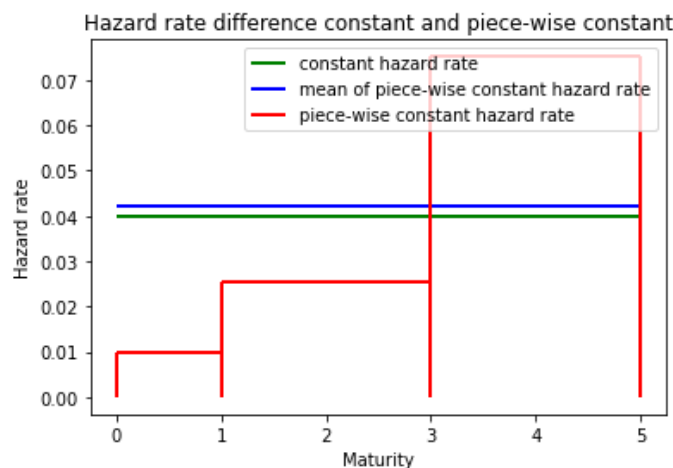


Figure 5.1: Results for constant hazard rate and piece-wise constant hazard rate.

From the piece-wise constant hazard rates we can subtract a constant hazard rate by taking the weighted mean over the piece-wise constant hazard rates:  $h_{mean} = \frac{h_{0,1} + h_{1,3} * 2 + h_{3,5} * 2}{5}$ . We can observe in figure 5.1 that this has a similar value as the constant hazard rate. However, this could be a coincidence because of the choice for the spreads in the piece-wise constant hazard rate. For computational time purposes we will from now on assume that the hazard rate is constant.

Using a constant hazard rate we can find a simple relation between the hazard rate and the par spread. We can prove that, if assuming a flat hazard rate curve, the hazard rate is equal to  $S/(1 - R)$ , with  $S$  being the par spread. If we assume that the premium leg were paid

continuously, we get from equation (5.8) that for spread  $S$ :

$$S = (1 - R) \frac{\int_0^T P(0, s) dQ(0, s)}{\int_0^T P(0, s) Q(0, s) ds} \quad (5.14)$$

$$= (1 - R) h \frac{\int_0^T P(0, s) e^{-hs} ds}{\int_0^T P(0, s) e^{-hs} ds} = (1 - R) h \quad (5.15)$$

## 5.2. Copula model

In this section we introduce the Copula model to integrate the WRR in the calculation for EAD. Especially, we want to remove the factor  $\alpha$  in the IMM by implementing a correlation between the probability of default and the exposures in the simulation engine of IMM. There are different ways to do this, one way would be to directly correlate the probability of default and the risk factors that drive the exposures. We can, for example, model the CDS spread of a counterparty as a GBM variable and then add a correlation structure between the risk factors and the Brownian motion in the GBM model. Another method to link the hazard rate and the portfolio value is described in [19]. Both methods still require a risk measure to directly measure the default loss, which involves both the exposure and probability of default. In this thesis we decided to only focus on the EAD and compare the EAD values obtained from SA-CCR, IMM and our integrated simulation method, the Copula model.

Using the Copula model we can link the probability of default to exposures to make a new estimation for the EAD that takes into account the WWR. The Copula model is inspired by [23] and [25].

We are interested in the joint distribution of time to default and the exposure. Since the marginal default distributions are not normal we need to use a copula to preserve the marginal distributions while imposing a correlation structure to all random variables involved. For convenience we use the Gaussian copula approach. We say that the counterparty defaults at  $t$  if a latent variable  $Y$ , which represents the credit worthiness of the counterparty, falls below a threshold corresponding to the rating of this counterparty, i.e.,

$$Y \leq \Phi^{-1}(P(t)). \quad (5.16)$$

where  $\Phi$  is the standard normal cumulative distribution function and  $Y$  is a standard normal random variable and  $P(t)$  represents the unconditional probability of default, where for default time  $\tau$  we have:

$$P(t) = \mathbb{P}(\tau \leq t) = 1 - \exp^{-\int_0^t h(s) ds}. \quad (5.17)$$

Here we assume the hazard rate to be constant over time and it is based on the counterparty's creditworthiness, then the hazard rate can be easily obtained as in (5.15).

Furthermore,  $Y$  is constructed with two independent standard normal variables  $Z$  and  $\epsilon$ , where  $Z$  is a global parameter, indicating the global economic cycle for example, and  $\epsilon$  is the idiosyncratic risk factor of this counterparty,  $Y$  is then given by:

$$Y = \rho Z + \sqrt{1 - \rho^2} \epsilon. \quad (5.18)$$

We now say that we have  $M$  exposure paths indicated by  $\omega_i, i = 1, \dots, M$ . Then for every path we generate a random number  $Y_i$  using (5.18) and check if the inequality (5.16) holds. When the inequality holds, a default has occurred on time  $t$  of exposure path  $\omega_i$ .

We now want to simulate the WWR by introducing a market factor  $X$ . We can see  $X$  as the distribution of all exposure paths  $\omega_i, i = 1, \dots, M$ . The market factor  $X$  is then uniformly distributed since the probability of every event is equal to  $q_\omega = \frac{1}{M}$ .

We then order the exposures paths in descending order on their EPE (average exposure over time). Thus we have  $EPE(\omega_1) \geq EPE(\omega_2) \geq \dots \geq EPE(\omega_M)$ . We then have  $X = F_{EPE}^{-1}(-x)$ , where  $F_{EPE}(\cdot)$  is the cumulative distribution function of EPE, i.e.,:

$$F_{EPE}^{-1}(x) = \inf\{y : F_{EPE}(y) \geq x\}. \quad (5.19)$$

In other words,  $F^{-1}(-EPE(\omega_i)) = i/M$ . Since  $F_{EPE}^{-1}(-x)$  is a monotonically decreasing function in  $x$ , it holds that the exposures will be higher when  $X$  is low.

The model is then concluded by taking  $Z = \Phi^{-1}(X)$  in (5.18). Now  $Y$  and  $\Phi^{-1}(X)$  have a bivariate normal distribution with correlation  $\rho$ . Intuitively we can see that when  $Y$  is low, a default is more easily triggered. As a consequence,  $X$  is also low and the EPE of the corresponding exposure path is high. Thus, via the Gaussian copula we have correlated default probability with exposure.

### 5.2.1. Expected exposure

We now want to include the WWR into the calculations of EAD. When we find the EE with the WWR integrated, we can make an expression of the effective EE and the EEPE given by (3.25) and (3.26) respectively. The EAD is then given by only the EEPE, since we have integrated the WWR we do not need the alpha factor as in (3.24).

We can calculate this EE where the WWR is integrated in two ways, which we will name as the defaulted paths method and the exact method. For the exact method we will need the EE conditional on the counterparty's default at  $t$ , i.e. CEE. We can then write the EE where the WWR is integrated as:

$$CEE(t = \tau) = \mathbb{E}(E(t)|\tau = t) = \sum_{i=1}^M E(t, \omega_i) \mathbb{P}(\omega = \omega_i | \tau = t), \quad (5.20)$$

where, following the article [24], we can find an expression for the probability  $\mathbb{P}(\omega = \omega_i | \tau = t)$  using bivariate normal assumptions. For the derivation please refer to Appendix A.3. The analytical expression for the default probability is copied below:

$$\mathbb{P}(\omega = \omega_i | \tau = t) = \Phi\left(\frac{\Phi^{-1}(i/M) - \rho\Phi^{-1}(P(t))}{\sqrt{1 - \rho^2}}\right) - \Phi\left(\frac{\Phi^{-1}(i - 1/M) - \rho\Phi^{-1}(P(t))}{\sqrt{1 - \rho^2}}\right).$$

Another way of calculating the conditional expected exposure CEE is to use Monte Carlo simulation. In our Copula model, we do not distinguish at which time point per path the default happens. To be consistent to this set-up, we calculate CEE by taking the average of the exposures from defaulted paths only. To see the behaviour of this defaulted paths method we conduct a sensitivity analysis.

### 5.2.2. Sensitivity analysis Copula model

We have defined the main risk factors for our products as the interest and exchange rate and developed a Copula model that links the exposures to the probability of default of the concerned counterparty. The main risk factors involved in the Copula model are the spreads from which



the probability of default is bootstrapped, and the correlation  $\rho$  between  $Y$  and  $\Phi^{-1}(X)$ . In this subsection we perform sensitivity analysis on both risk factors with the use of the IRS with parameters described in section 4.6. First we will check the effect of the default correlation, the results can be observed in figure 5.2.

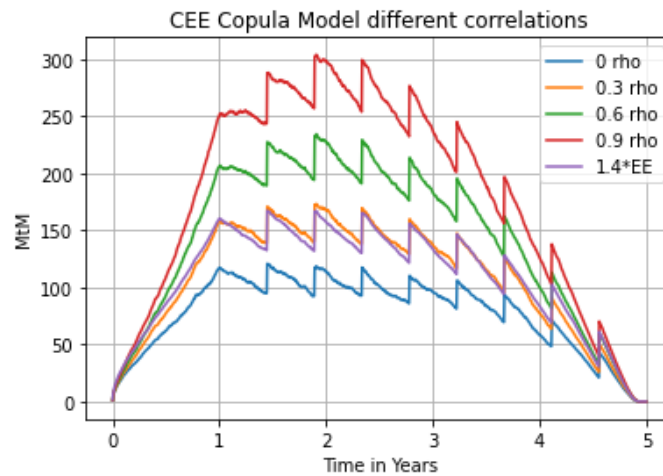


Figure 5.2: Conditional expected exposure using the Copula model with different values of  $\rho$ . We have taken a spread equal to 400 and IRS parameters from the calibration described in section 4.6.

We observe in figure 5.2 that, when we increase the correlation between  $Y$  and  $\Phi^{-1}(X)$ , the conditional expected exposure increases as well. This is as expected: the more the EPE and default are correlated, the higher the conditional EE, and thus, the higher the EAD. Furthermore, we can obtain a first comparison between the conditional EE profile and the one corresponding to IMM, which is based on the factor  $\alpha = 1.4$ . This comparison indicates that with  $\rho = 0.3$ , the EE where the WWR is integrated behaves similarly as the EE multiplied by  $\alpha$ .

Regarding the impact of the CDS spread on the defaulted path, the results are summarized in figure 5.3.

Figure 5.3 indicates that, the model gives a lower EE when we increase the spread. This is counter-intuitive because we connect high spreads with low credit ratings, using WWR we would expect the exposure to be higher when the spreads get higher. However, when we look at the corresponding default curves and expected loss given by the product of the CEE and probability curve, we observe the behaviour that we would expect. Namely, the expected loss increases when the spread increases.

The reason of the spread effect on the CEE is as follows: using higher spreads, more paths would go into default and the correlation impact is less prominent. When all paths go into default, the correlation impact is not present anymore. However, when a small amount of paths go into default we only use a small amount of paths in the simulation of the CEE and the correlation effect is much more prominent. To see this effect even better, we tested different values of the correlation for a high spread of 1000 and a low spread of 10. The results can be seen in figure 5.4.

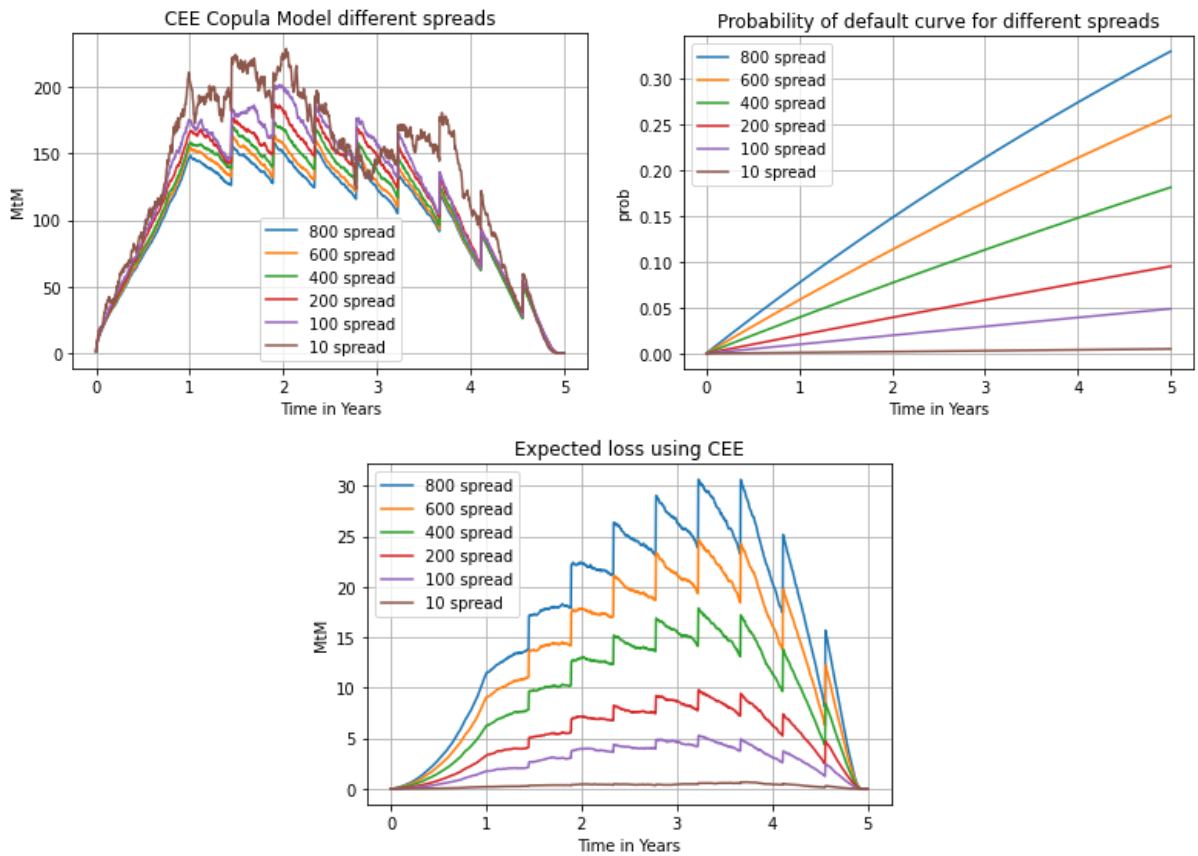


Figure 5.3: Conditional expected exposures using the Copula model for different spread values with corresponding probability of default curve and expected loss. We have taken  $\rho$  equal to 0.3 and IRS parameters from the calibration described in section 4.6

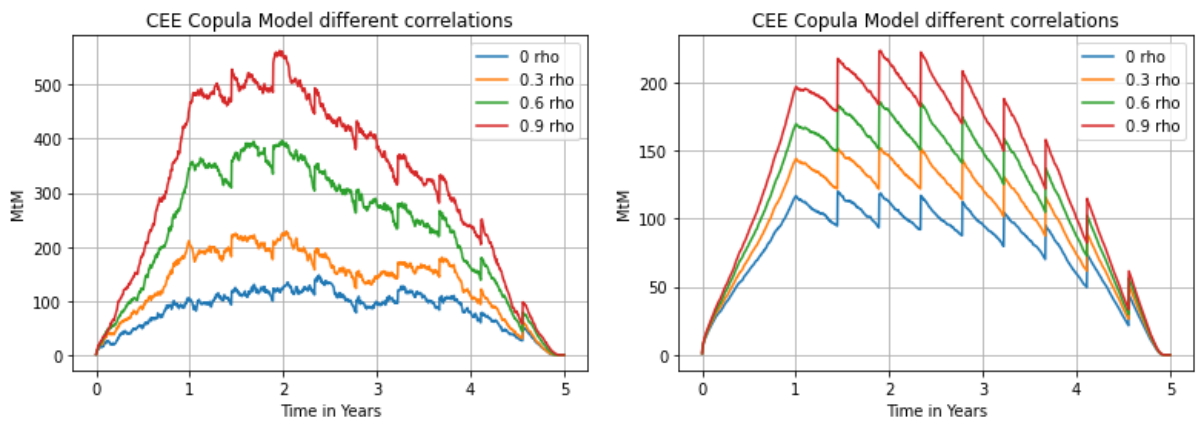


Figure 5.4: Comparison between the effect of  $\rho$  on the Copula model using a low spread of 10 on the left and a high spread of 1000 on the right.

# 6

## Numerical results

In this chapter, we conduct intensive tests to check: 1) how conservative the SA-CCR is compared to the IMM, and 2) if the Copula model can be a good benchmark for the IMM, or in other words, whether using an alpha factor of 1.4 is a good approximation method to account for WWR.

### 6.1. Comparing SA-CCR with IMM

First we use single trades to compare SA-CCR with IMM, via which we try to understand how the parameters are chosen for the SA-CCR by the regulators. Furthermore, we will check the impact of a stressed and normal calibration.

We first consider the case of three different portfolio's consisting of one single trade, then we test on the portfolio of all three trades combined. Using the products with parameter values calibrated in section 4.6, we can compare EAD from the SA-CCR with the EAD from IMM.

For the SA-CCR we have the following results:

Table 6.1: Parameters for SA-CCR.

Product	$V_0$	$d$	$\delta$	EN * SF
IRS	1.23	34485.73	1	172.43
XCS	-26.23	34485.73	1	172.43
FX	-503.98	10000	1	400

Furthermore,  $M_f$  is equal to 1 when there is no collateral and  $M_f$  is equal to (3.22) when the collateral is not zero. We can now calculate the EAD for SA-CCR as in (3.10). Note that for the single trade portfolio's we have  $PFE = multiplier * EN * SF$ . For the FX Forward and XCS products the multiplier is activated as these product start with a negative initial value. We also distinguish the cases, whereby we pay collateral with a threshold value of 60 and a minimal transfer amount of 10. The results are shown in table 6.2.

The results demonstrate that the SA-CCR is much more conservative than the IMM with normal parameters, i.e. from a calibration based on a long history of the past 12 years. However, when we use the IMM with parameters from a stressed calibration, i.e. calibration using the most stressful 3 years, we observe that the SA-CCR is less conservative. Indeed, the difference between SA-CCR and IMM with normal parameters ranges between a factor of 2 and 5, while for a stressed calibration, this difference ranges between a factor of 1 and 1.5.

Table 6.2: Comparison EAD from SA-CCR and EAD IMM for single trades.

Product	Without Collateral			With Collateral		
	SA-CCR	IMM	IMM stress	SA-CCR	IMM	IMM stress
IRS	243.12	86.85	167.84	170.13	44.58	68.6
XCS	223.76	125.36	206.98	154.027	49.39	75.7
FX	302.1	64.97	275.02	123.64	21.85	70.5
Portfolio	733.85	148.9	485.66	198.31	42.81	124.13

In the presence of collateral, it can be seen that the SA-CCR tends to be even more conservative than IMM, e.g. between 3 and 6 times more conservative than EAD from IMM with normal parameters; and between 1.5 and 2.5 times more conservative than EAD from IMM with stressed parameters.

The most conservative situation for SA-CCR is when we have all three types of products in a single portfolio. Since these three products are from different asset classes, the PFE add-ons of the single product type have to be aggregated to yield the PFE add-on for the portfolio. This additional layer of conservatism in SA-CCR is, however, not in IMM, i.e. when we add the values of the products together and take the exposure for the IMM these exposures within the same netting set can be netted.

To summarize, SA-CCR tends to be at least 1.5 times more conservative than IMM in the presence of collateral (which is usually the case in practice), even with stressed model parameters. And the level of conservatism is even higher because no diversification is allowed between different asset classes.

In the next section, we will set up the same tests for portfolios of multiple trades and also include the Copula model.

## 6.2. Comparison on testing portfolios

At first we compare three portfolios which consist of 10 trades of the same product. Then we test one portfolio with the 30 trades of three different product types combined, and in the end another portfolio with 90 trades of different product types. For each trade we assign a random starting date, maturity date, and notional. Note that per hedging set we constructed roughly the same amount of payer and receiver products. An overview of the portfolios is given in appendix B. For collateral we set  $TH = 200$  and  $MTA = 50$ . We set the value of the coupon rate of the IRS and XCS to  $K = r_0$ .

Lastly, we have to choose the parameters in the Copula model. We assume a credit spread of 200 basis points. Then, we conduct a sensitivity analysis similar as done in subsection 5.2.2 on the the IRS portfolio of 10 trades. We then choose a correlation of  $\rho = 0.4$  as this value seems to be a good benchmark for the comparison with IMM, which is different from the correlation that we observed in figure 5.2. We further test IMM using normal parameters (table 6.3) and using stressed parameters (table 6.4).

First, we again observe a clear difference between the IMM with normal parameters and stressed parameters. EAD's from the SA-CCR are a factor of 3 to 4.5 times of EAD's from the IMM when using normal parameters, whereas this ratio drops to 1 to 2.5 when using stressed parameters. Furthermore, we again observe that in the presence of collateral SA-CCR is even more conservative than IMM, e.g. between 4 and 8 times more conservative than EAD from IMM with normal parameters; and between 1.5 and 3.5 more conservative than EAD from IMM

Table 6.3: Comparison SA-CCR, IMM and Copula model using normal parameters.

Portfolio	Without Collateral			With Collateral		
	SA-CCR	IMM	Copula	SA-CCR	IMM	Copula
10 IRS trades	648.88	212.98	197.95	531.71	125.59	112.95
10 XCS trades	833.83	249.06	244.46	598.70	138.13	112.59
10 FX trades	257.80	63.92	60.17	374.22	45.16	48.07
30 trades	1644.21	518.56	445.81	822.50	201.31	171.57
90 trades	2210.98	502.26	397.35	925.17	213.51	177.59

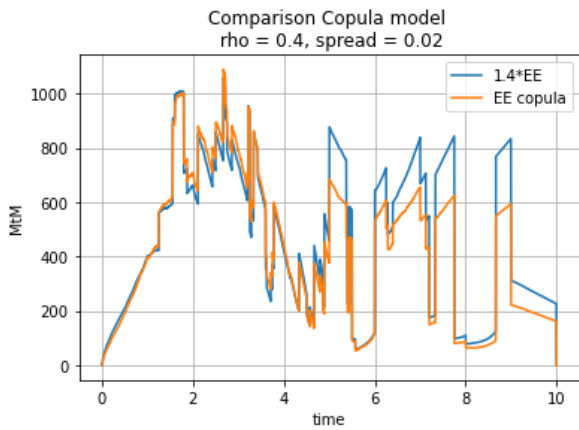
Table 6.4: Comparison SA-CCR, IMM and Copula model using stressed parameters.

Stressed Portfolio	Without Collateral			With Collateral		
	SA-CCR	IMM	Copula	SA-CCR	IMM	Copula
10 IRS trades	648.88	268.3	214.73	531.71	151.71	119.45
10 XCS trades	833.83	628.41	577.75	598.70	246.07	206.15
10 FX trades	257.80	247.25	216.68	374.22	126.31	110.32
30 trades	1644.21	1003.13	775.8	822.50	342.57	275.7
90 trades	2210.98	1977.38	1447.28	925.17	636.60	490.50

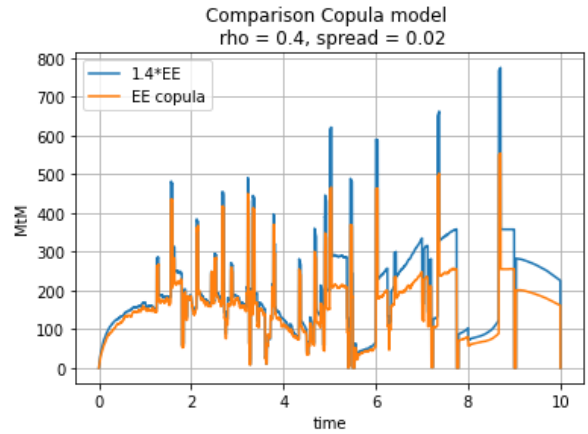
with stressed parameters. However, there seems no direct connection between the size of the portfolio and the conservative factor of SA-CCR over IMM, as this diverges per asset type and size of the portfolio.

To better observe the difference between the IMM and Copula model we plot the conditional EE of the Copula model with the EE of IMM multiplied with the  $\alpha$  factor of 1.4 for the normal parameters. The results can be observed in 6.1 and 6.2.

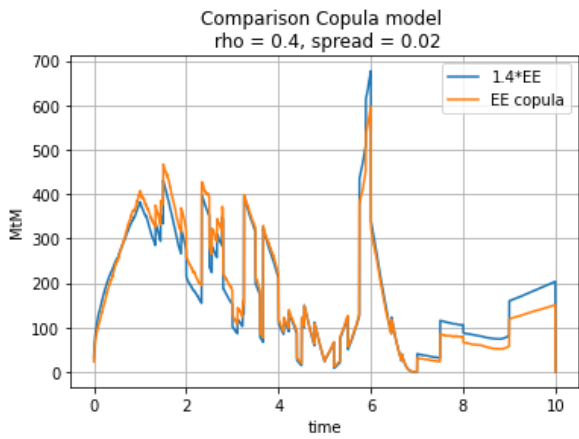
The parameters used in the Copula model are based on the sensitivity analysis of the IRS portfolio of 10 trades. With those parameter values, especially the default correlation of 40%, we can see that the Copula model gives comparable results to IMM for all testing portfolios. However, when we increase the size of the portfolios, the results of the Copula model tend to be less conservative than the results of the IMM, especially when we use the IMM with stressed parameters and fix the default correlation value. As seen in figure 5.2, different portfolios might need different correlations to have comparable EAD values with IMM. This indicates that the alpha value of 1.4 is a reasonable and not very conservative value to account for WWR in the framework of IMM.



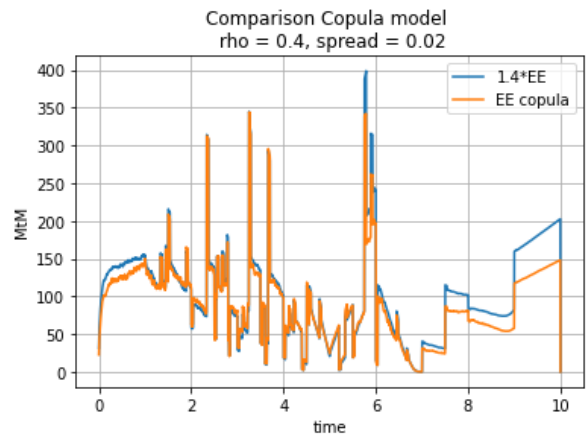
(a) Comparison expected exposure of 10 IRS products without collateral.



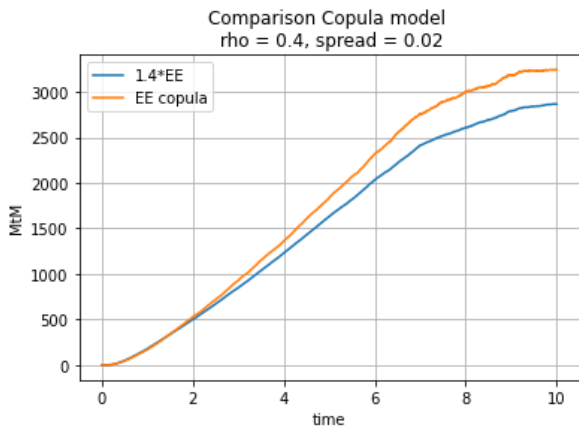
(b) Comparison expected exposure of 10 IRS products with collateral.



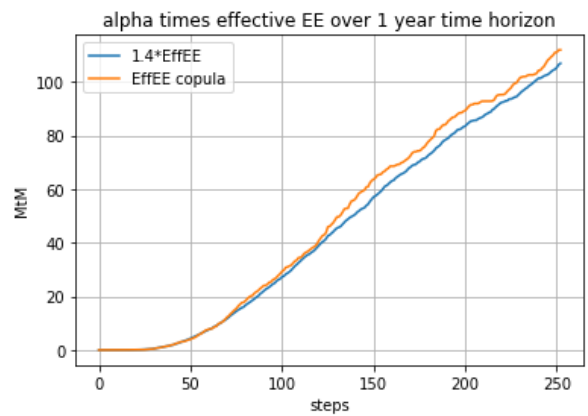
(c) Comparison expected exposure of 10 XCS products without collateral.



(d) Comparison expected exposure of 10 XCS products with collateral.

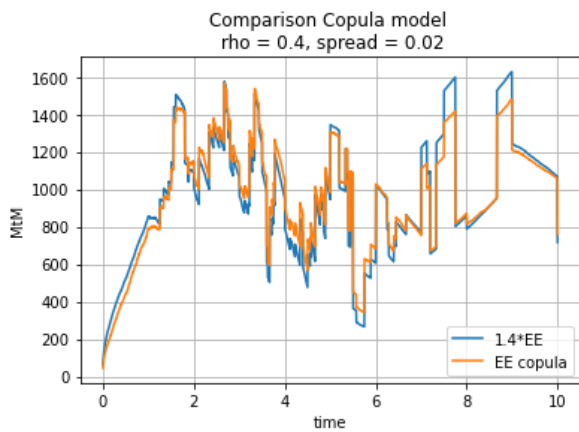


(e) Comparison expected exposure of 10 FX products without collateral.

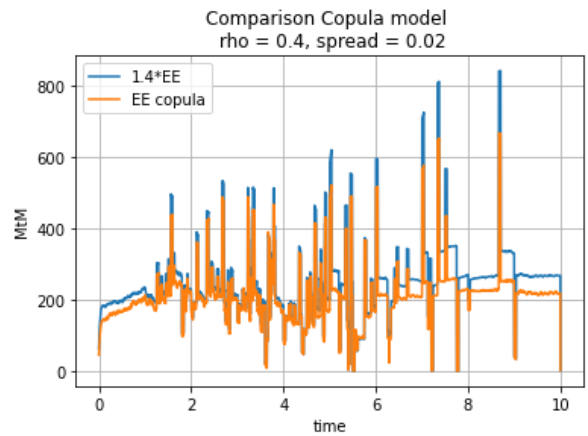


(f) Comparison expected exposure of 10 FX products with collateral.

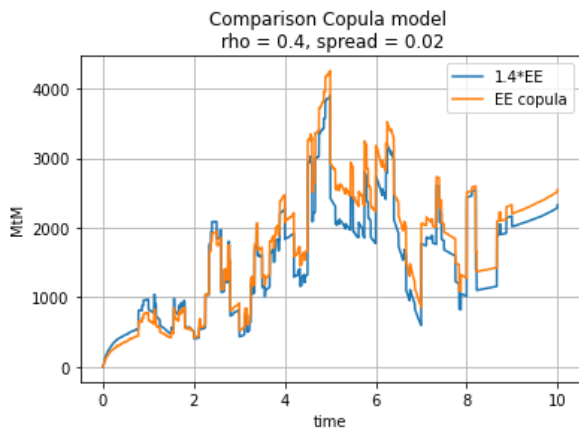
Figure 6.1: Comparison of Internal Model Method and Copula method for portfolio's consisting of 10 same type of products using normal parameters.



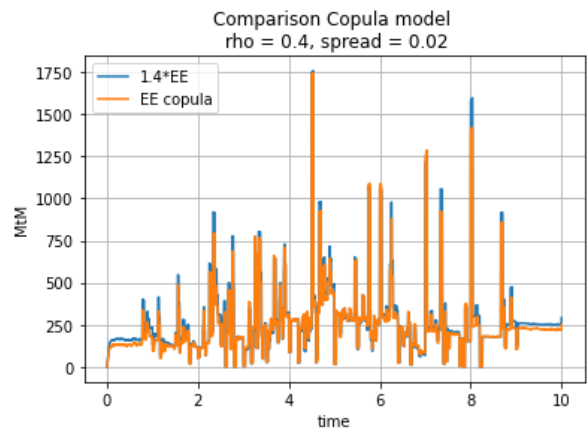
(a) Comparison expected exposure of 30 products without collateral.



(b) Comparison expected exposure of 30 products with collateral.



(c) Comparison expected exposure of 90 products without collateral.



(d) Comparison expected exposure of 90 products with collateral.

Figure 6.2: Comparison of Internal Model Method and Copula method for portfolio's consisting of 30 and 90 products using normal parameters.





## Conclusions and Discussion

In this thesis we developed a benchmark model that models WWR directly, and compared this model with both SA-CCR and IMM methods on the EAD metric for regulatory capital for CCR. This benchmark model is built on top of the IMM simulation framework but uses a Gaussian copula for integrating the WWR, i.e. the correlation between default and exposures is included in the EE simulation.

The research in this thesis focuses on the calculation of EADs. SA-CCR is implemented according to the Basel framework [5]. For EADs of IMM and EADs of the Copula model, we built a Monte Carlo simulation framework also following the corresponding Basel requirements [4]. For IR risk factors, we choose the Hull-White model and for FX a GBM. Next, a calibration framework is developed to estimate the parameters used in these models. We distinguished two types of calibration, one with a long period of the past 12 years and the other with a shorter, stressed period of 3 years. Furthermore, we established the pricing functions of IRS, XCS and FX Forward, which are the most liquid product types in the market and are also the product types included in our testing portfolios.

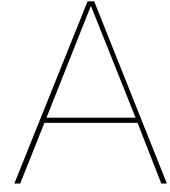
Both SA-CCR and IMM account for the WWR via an alpha factor of 1.4, which is to be multiplied to the EAD values for the regulatory capital calculations. Our integrated simulation framework based on copula takes into account the WWR directly inside the simulation. More precisely, we derived the formula for backing out the probability of default using CDSs and then, via a Gaussian copula, we include a correlation between the exposures and default probability. Lastly, we tested these three models on testing portfolios, using both the stressed and normal parameters and compared between with and without collateral.

The SA-CCR relies on a few variables which can be determined at the start date of a portfolio. The IMM and Copula model rely on the model assumptions and model parameter values of the risk factors. Testing results indicate that, SA-CCR tends to be at least 1.5 times more conservative than IMM in presence of collateral, even with stressed model parameters. And the level of conservatism is even higher because no diversification is allowed between different asset classes.

To include the WWR in the EAD without the factor  $\alpha$  we introduced the Copula model. The goal for this implementation was to see if we could find a method that captures the WWR directly and in case yes compare to the EADs from IMM multiplied by the  $\alpha$  factor. Via a copula, we directly link the EPE with the probability of default, such that the higher the WWR, the higher the resulting EE conditional on defaults. Using the parameter values either from the calibration

or as indicated by sensitivity analysis, we noticed that when we used larger test portfolios the Copula model values tend to be less conservative than the IMM. It is therefore hard to use one set of parameters for the Copula model so as to return comparable EAD values as EADs from IMM multiplied by 1.4 for different portfolios.

In the end, in view of the regulations, it is wise to have a conservative standard approach to account for the CCR. If a bank can show that it can provide an IMM that accounts for all the risks, they can be allowed to use this model instead. This can reduce the regulatory capital significantly as demonstrated by our analysis results. They would however have to use this  $\alpha$  factor of 1.4. Using the parameters backed out from calibration and a default correlation of about 30 to 40%, our integrated benchmark model based on copula returns more or less comparable EADs of IMM times the alpha factor of 1.4. The Copula model results indicate that the alpha factor of 1.4 is a reasonable value in general.



# Proofs

## A.1. Change of measure exchange rate

We have the exchange rate as a GBM in the real world measure  $\mathbb{P}$ .

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t)^{\mathbb{P}}. \quad (\text{A.1})$$

We will prove that the dynamics in the risk-neutral measure are given by:

$$dX(t) = (r_d - r_f)X(t)dt + \sigma X(t)dW_d^{\mathbb{Q}}(t). \quad (\text{A.2})$$

*Proof.* We take the money savings account in the foreign currency  $B_f(t)$ , where the value in the domestic currency is then given by  $X(t)B_f(t)$ . We know that for the risk-neutral measure, any discounted asset must be a martingale. The discounted value of the foreign saving account in domestic currency is then given by:

$$\chi(t) = \frac{B_f(t)}{B_d(t)}X(t). \quad (\text{A.3})$$

This must be a martingale under the domestic measure  $\mathbb{Q}$ . Using Ito's lemma we look at the dynamics and see:

$$d\chi(t) = (r_f - r_d)\frac{B_f(t)}{B_d(t)}X(t)dt + \frac{B_f(t)}{B_d(t)}dX(t) \quad (\text{A.4})$$

$$= (r_f - r_d)\frac{B_f(t)}{B_d(t)}X(t)dt + \frac{B_f(t)}{B_d(t)}\mu X(t)dt + \frac{B_f(t)}{B_d(t)}\sigma X(t)dW(t)^{\mathbb{P}} \quad (\text{A.5})$$

$$= (r_f - r_d)\chi(t)dt + \mu\chi(t)dt + \sigma\chi(t)dW(t)^{\mathbb{P}}. \quad (\text{A.6})$$

For  $\chi$  to be a martingale under the risk neutral probability the dynamics must be free of drift. To have this we must have:

$$dW(t)^{\mathbb{P}} = -\frac{((r_f - r_d) + \mu)}{\sigma}dt + dW^{\mathbb{Q}}. \quad (\text{A.7})$$

Filling this in for (A.1), we get the dynamics under the risk-neutral measure:

$$dX(t) = \mu X(t)dt + \sigma X(t)\left(-\frac{((r_f - r_d) + \mu)}{\sigma}dt + dW^{\mathbb{Q}}\right) \quad (\text{A.8})$$

$$= (r_d - r_f)X(t)dt + \sigma X(t)dW^{\mathbb{Q}}. \quad (\text{A.9})$$

□

## A.2. Change of measure foreign to domestic

We will do the proof for the risk neutral measure since we can then use the no-arbitrage principle. In the end, we can return to the real world measure for the short rate since we determine the parameters according to a calibration. Using the risk neutral dynamics for the foreign and domestic short rate:

$$\begin{cases} dX(t) &= (r_d - r_f)X(t) + \sigma X(t)dW_X^{\mathbb{Q}^d} \\ dr_d(t) &= \lambda_d(\theta_d(t) - r_d(t))dt + \eta_d dW_d^{\mathbb{Q}^d} \\ dr_f(t) &= \lambda_f(\theta_f(t) - r_f(t))dt + \eta_f dW_f^{\mathbb{Q}^f} \end{cases} \quad (\text{A.10})$$

we want to write the foreign interest rate dynamics of  $r_f$  in the domestic measure resulting in:

$$dr_f(t) = (\lambda_f(\theta_f(t) - r_f(t)) - \eta_f \sigma_X \rho_{fX})dt + \eta_f dW_f^{\mathbb{Q}^d}. \quad (\text{A.11})$$

*Proof.* Because of the no-arbitrage principle we can use the following expression for the Radon-Nikodyn derivative:

$$\frac{\mathbb{Q}_f}{\mathbb{Q}_d} \Big|_{\mathcal{F}_t} = \frac{B_f(t)X(t)}{B_d(t)X(0)}. \quad (\text{A.12})$$

Where for the money market accounts we can write:

$$B_d(t) = e^{\int_0^t r_d dt} \quad (\text{A.13})$$

$$B_f(t) = e^{\int_0^t r_f dt}. \quad (\text{A.14})$$

Using the dynamics of the exchange rate, we can find an expression for  $X(t)$  using Ito's lemma with  $f(X) = \ln(X)$ . We find:

$$\ln(X(t)) = \ln(X(0)) + \int_0^t \frac{1}{X(t)} dX(t) + \frac{1}{2} \int_0^t -\frac{1}{X^2} d[X(t)] \quad (\text{A.15})$$

$$= \ln(X(0)) + \int_0^t (r_d - r_f)dt + \int_0^t \sigma dW_X^{\mathbb{Q}^d} - \frac{1}{2} \int_0^t \sigma^2 dt \quad (\text{A.16})$$

$$= \ln(X(0)) + \int_0^t r_d - r_f - \frac{1}{2} \sigma^2 dt + \int_0^t \sigma dW_X^{\mathbb{Q}^d}. \quad (\text{A.17})$$

By taking the exponential on both sides we see:

$$X(t) = X(0) \exp\left(\int_0^t r_d - r_f - \frac{1}{2} \sigma^2 dt + \int_0^t \sigma dW_X^{\mathbb{Q}^d}\right). \quad (\text{A.18})$$

If we fill this in for our Radon Nikodyn derivative we have:

$$\frac{\mathbb{Q}_f}{\mathbb{Q}_d} \Big|_{\mathcal{F}_t} = \exp\left(\int_0^t r_f - r_d dt\right) \exp\left(\int_0^t r_d - r_f - \frac{1}{2} \sigma^2 dt + \int_0^t \sigma dW_X^{\mathbb{Q}^d}\right) \quad (\text{A.19})$$

$$= \exp\left(\int_0^t -\frac{1}{2} \sigma^2 dt + \int_0^t \sigma dW_X^{\mathbb{Q}^d}\right) \quad (\text{A.20})$$

$$= \exp\left(\sigma W_X^{\mathbb{Q}^d} - \frac{1}{2} \sigma^2 t\right) = \mathcal{E}(\sigma W_X^{\mathbb{Q}^d}), \quad (\text{A.21})$$

where  $\mathcal{E}$  is the stochastic exponential, because of Girsanov's theorem we can now define  $W_f^{\mathbb{Q}^f}$  a Brownian motion under the risk-neutral domestic measure by:

$$W_f^{\mathbb{Q}^f} = W_f^{\mathbb{Q}^d} - [W_f^{\mathbb{Q}^d}, \sigma W_X^{\mathbb{Q}^d}] = W_f^{\mathbb{Q}^d} - \rho_{X,f} \sigma t. \quad (\text{A.22})$$

We can fill this in for the dynamics of the foreign short rate and see:

$$dr_f(t) = \lambda_f(\theta_f(t) - r_f(t))dt + \eta_f(dW_f^{\mathbb{Q}^d} - \rho_{X,f}\sigma dt) \quad (\text{A.23})$$

$$= (\lambda_f(\theta_f(t) - r_f(t)) - \eta_f\rho_{X,f}\sigma)dt + \eta_f dW_f^{\mathbb{Q}^d}. \quad (\text{A.24})$$

□

### A.3. Copula model exact method

We want an expression for the probabilities for the events conditional on default exactly at each time point expressed by the probability  $\mathbb{P}(\omega = \omega_i | \tau = t)$ . Remember that when we order the exposure paths in descending order on their EPE, we choose event  $i$  when  $(i-1)/M < X < i/M$ . Then we know we have a default exactly at  $t$ , when  $Y = \Phi^{-1}(P(t))$ . This gives us the following expression for the probability:

$$\mathbb{P}(\omega = \omega_i | \tau = t) = \mathbb{P}((i-1)/M < X \leq i/M | Y = \Phi^{-1}(P(t))) \quad (\text{A.25})$$

$$= \mathbb{P}[\Phi^{-1}((i-1)/M) < \Phi^{-1}(X) \leq \Phi^{-1}(i/M) | Y = \Phi^{-1}(P(t))] \quad (\text{A.26})$$

$$= \mathbb{P}[\Phi^{-1}(X) \leq \Phi^{-1}(i/M) | Y = \Phi^{-1}(P(t))] \quad (\text{A.27})$$

$$= \mathbb{P}[\Phi^{-1}(X) \leq \Phi^{-1}((i-1)/M) | Y = \Phi^{-1}(P(t))]. \quad (\text{A.28})$$

We now will rewrite the inequality  $\Phi^{-1}(X) \leq \Phi^{-1}((i-1)/M)$  using that  $Y$  and  $\Phi^{-1}(X)$  have a bivariate normal distribution with  $\rho$  and we take  $Y = \Phi^{-1}(P(t))$ . We see  $\Phi^{-1}(X) = \rho\Phi^{-1}(P(t)) + \sqrt{1-\rho^2}\epsilon$  and we can write for the following inequality:

$$\Phi^{-1}(X) \leq \Phi^{-1}((i-1)/M) \quad (\text{A.29})$$

$$\rho\Phi^{-1}(P(t)) + \sqrt{1-\rho^2}\epsilon \leq \Phi^{-1}((i-1)/M) \quad (\text{A.30})$$

$$\epsilon \leq \frac{\Phi^{-1}((i-1)/M) - \rho\Phi^{-1}(P(t))}{\sqrt{1-\rho^2}}. \quad (\text{A.31})$$

From this we have:

$$\mathbb{P}[\Phi^{-1}(X) \leq \Phi^{-1}((i-1)/M) | Y = \Phi^{-1}(P(t))] = \Phi\left(\frac{\Phi^{-1}((i-1)/M) - \rho\Phi^{-1}(P(t))}{\sqrt{1-\rho^2}}\right). \quad (\text{A.32})$$

Combining this result with the expression for probability  $\mathbb{P}(\omega = \omega_i | \tau = t)$  we see:

$$\mathbb{P}(\omega = \omega_m | \tau = t) = \Phi\left(\frac{\Phi^{-1}(i/M) - \rho\Phi^{-1}(P(t))}{\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{\Phi^{-1}((i-1)/M) - \rho\Phi^{-1}(P(t))}{\sqrt{1-\rho^2}}\right). \quad (\text{A.33})$$



# B

## Portfolio's

In this section we will show the portfolio's used in chapter 6. Here we have tested the models first on 3 portfolio's consisting of the IRS, XCS and FX products respectively from the portfolio with 30 derivatives. After that, we have tested the model on the portfolio with 30 products and 90 products.

## B.1. Portfolio with 30 derivatives

TradeID	ProductType	PayOrRece	FixedLegDr	Notional	Coupon	StartDate	NumberOf	Maturity
1	IRS	1	True	10000		1	10	6
2	IRS	1	True	30000		2	8	5
3	IRS	1	True	5000		0	6	8
4	IRS	1	False	15000		2	7	10
5	IRS	1	False	25000		1	10	6
6	IRS	1	False	7000		0	5	5
7	IRS	-1	True	15000		2	8	7
8	IRS	-1	True	13000		1	9	8
9	IRS	-1	False	9000		0	6	9
10	IRS	-1	False	12000		1	5	10
11	XCCY	1	True	10000		1	10	5
12	XCCY	1	True	3000		0	7	8
13	XCCY	-1	True	15000		2	6	6
14	XCCY	-1	True	5000		1	9	9
15	XCCY	1	True	17000		2	5	7
16	XCCY	1	False	8000		0	7	9
17	XCCY	1	False	20000		1	4	5
18	XCCY	-1	False	2000		0	3	10
19	XCCY	-1	False	30000		2	9	6
20	XCCY	1	False	15000		2	10	7
21	FX	1		10000				5
22	FX	1		4000				8
23	FX	1		2000				7
24	FX	1		6000				6
25	FX	1		3500				9
26	FX	-1		8000				5
27	FX	-1		3500				8
28	FX	-1		2000				10
29	FX	-1		4000				6
30	FX	-1		6000				10



## B.2. Portfolio with 90 derivatives

TradeID	ProductTyf	PayOrRece	FixedLegDr	Notional	Coupon	StartDate	NumberOf	Maturity
1	IRS	1	True	23000		2	9	10
2	IRS	1	True	6000		1	6	9
3	IRS	1	True	16000		0	8	6
4	IRS	1	True	3000		0	10	5
5	IRS	1	True	18000		1	5	7
6	IRS	1	True	30000		1	7	5
7	IRS	1	True	15000		2	10	9
8	IRS	1	True	9000		1	8	10
9	IRS	-1	True	16000		2	5	8
10	IRS	-1	True	7000		1	7	5
11	IRS	-1	True	29000		2	5	6
12	IRS	-1	True	8000		0	8	7
13	IRS	-1	True	20000		1	5	10
14	IRS	-1	True	14000		1	10	9
15	IRS	-1	True	3000		1	6	6
16	IRS	1	False	7000		0	10	7
17	IRS	1	False	18000		2	8	8
18	IRS	1	False	27000		2	7	10
19	IRS	1	False	18000		0	10	7
20	IRS	1	False	4000		1	6	8
21	IRS	1	False	19000		2	9	5
22	IRS	1	False	20000		1	10	6
23	IRS	1	False	4000		1	3	10
24	IRS	-1	False	13000		0	6	5
25	IRS	-1	False	27000		1	6	5
26	IRS	-1	False	3000		0	8	6
27	IRS	-1	False	29000		2	10	9
28	IRS	-1	False	10000		2	5	5
29	IRS	-1	False	24000		1	5	5
30	IRS	-1	False	20000		2	7	8
31	XCCY	1	True	8000		1	9	10
32	XCCY	1	True	29000		2	5	7
33	XCCY	1	True	14000		0	9	9
34	XCCY	1	True	16000		1	5	8
35	XCCY	1	True	27000		1	9	9
36	XCCY	1	True	8000		0	7	6
37	XCCY	1	True	10000		1	10	9
38	XCCY	1	True	17000		2	6	8
39	XCCY	-1	True	8000		1	6	10
40	XCCY	-1	True	20000		2	7	9
41	XCCY	-1	True	17000		1	8	5
42	XCCY	-1	True	30000		2	9	6
43	XCCY	-1	True	6000		0	8	8
44	XCCY	-1	True	18000		1	6	10
45	XCCY	-1	True	16000		1	6	9
46	XCCY	-1	True	5000		0	9	10
47	XCCY	1	False	24000		1	4	5
48	XCCY	1	False	13000		0	5	6
49	XCCY	1	False	3000		1	10	10

50 XCCY	1 False	10000	0	10	8
51 XCCY	1 False	20000	1	5	6
52 XCCY	1 False	30000	2	5	7
53 XCCY	1 False	8000	0	3	8
54 XCCY	-1 False	4000	2	5	9
55 XCCY	-1 False	18000	0	7	7
56 XCCY	-1 False	9000	1	9	5
57 XCCY	-1 False	26000	2	8	8
58 XCCY	-1 False	17000	2	6	7
59 XCCY	-1 False	3000	1	5	10
60 XCCY	-1 False	24000	2	6	8
<hr/>					
61 FX	1	4000			8
62 FX	1	6500			7
63 FX	1	2500			8
64 FX	1	9000			10
65 FX	1	1500			9
66 FX	1	4500			4
67 FX	1	9500			4
68 FX	1	4000			9
69 FX	1	8500			8
70 FX	1	10000			6
71 FX	1	8000			3
72 FX	1	4500			10
73 FX	1	3500			8
74 FX	1	2000			5
75 FX	1	7500			6
76 FX	1	4500			3
<hr/>					
77 FX	-1	3500			3
78 FX	-1	7500			6
79 FX	-1	6000			5
80 FX	-1	5500			8
81 FX	-1	4500			4
82 FX	-1	9500			7
83 FX	-1	2500			9
84 FX	-1	7000			9
85 FX	-1	4000			8
86 FX	-1	10000			8
87 FX	-1	5500			9
88 FX	-1	4500			10
89 FX	-1	7000			7
90 FX	-1	6500			3

# Bibliography

- [1] Cre50 - counterparty credit risk definitions and terminology. URL [https://www.bis.org/basel\\_framework/chapter/CRE/50.htm](https://www.bis.org/basel_framework/chapter/CRE/50.htm).
- [2] Effective federal funds rate - federal reserve bank of new york. URL <https://www.newyorkfed.org/markets/reference-rates/effr>.
- [3] Federal reserve economic data | fred | st. louis fed. URL <https://fred.stlouisfed.org/>.
- [4] Cre53 - internal models method for counterparty credit risk. URL [https://www.bis.org/basel\\_framework/chapter/CRE/53.htm?inforce=20220101&published=20191215](https://www.bis.org/basel_framework/chapter/CRE/53.htm?inforce=20220101&published=20191215).
- [5] Cre52 - standardised approach to counterparty credit risk. URL [https://www.bis.org/basel\\_framework/chapter/CRE/52.htm](https://www.bis.org/basel_framework/chapter/CRE/52.htm).
- [6] Sonia interest rate benchmark | bank of england. URL <https://www.bankofengland.co.uk/markets/sonia-benchmark>.
- [7] Leif B. G. Andersen, Michael Pykhtin, and Alexander Sokol. Rethinking margin period of risk. *SSRN Electronic Journal*, 1 2016. doi: 10.2139/SSRN.2719964.
- [8] Phelim Boyle, Mark Broadie, and Paul Glasserman. Monte carlo methods for security pricing. *Journal of Economic Dynamics and Control*, 21:1267–1321, 6 1997. ISSN 0165-1889. doi: 10.1016/S0165-1889(97)00028-6.
- [9] D Brigo and F Mercurio. *Interest rate models: theory and practice*. Springer, 2001.
- [10] Damiano Brigo and Massimo Morini. Counterparty risk faq: Credit var, pfe, cva, dva, closeout, netting, collateral, re-hypothecation, wwr, basel, funding, ccds and margin lending. *SSRN Electronic Journal*, 11 2011. doi: 10.48550/arxiv.1111.1331.
- [11] CA de Kam Tijdschrift voor Openbare Financiën and undefined 2009. Van financiële crisis naar depressie? *wimdreesfonds.nl*, 2009. ISSN 1875-8401.
- [12] Marco Di Francesco. A general gaussian interest rate model consistent with the current term structure. *International Scholarly Research Notices*, 2012, 2012.
- [13] F.C.Park. Implementing interest rate models: A practical guide. *Capital markets Portfolio Research, Inc.*, 2004.
- [14] Jean-François Le Gall. *Brownian motion, martingales, and stochastic calculus*. Springer, 2016.
- [15] Brenda Faith Ginos. Parameter estimation for the lognormal distribution. *International Scholarly Research Notices*, 2012, 2012.

- [16] Kathrin Glau, Ricardo Pachon, and Christian Pötz. Speed-up credit exposure calculations for pricing and risk management. *Quantitative Finance*, 21:481–499, 2021. doi: 10.1080/14697688.2020.1781236.
- [17] Jong Gregory. *Counterparty Credit Risk and Credit Value Adjustment, Second Edition*. John Wiley Sons Ltd, 2012.
- [18] Allan Gut. *Probability: A Graduate Course*, volume 75. Springer New York, 2013. ISBN 978-1-4614-4707-8. doi: 10.1007/978-1-4614-4708-5.
- [19] John C. Hull and Alan White. Cva and wrong-way risk. *Financial Analysts Journal*, 68: 58–69, 9 2012.
- [20] Fsi Connect Financial Stability Institute. Counterparty credit risk in basel iii – executive summary. 2018.
- [21] Johann Jacobs and Gary van Vuuren. Is regulatory capital a legitimate, comparable and objective global standard? evidence from 51 institutions across 17 countries. *South African Journal of Economic and Management Sciences*, 17:266–283, 2014. ISSN 22223436. doi: 10.4102/SAJEMS.V17I3.568.
- [22] Cornelis W Oosterlee and Lech A Grzelak. *Mathematical Modeling Computation in Finance*. World Scientific Publishing Europe Ltd., 2020.
- [23] Dan Rosen and David Saunders. Cva the wrong way. *Journal of Risk Management in Financial Institutions*, 5(3):252–272, 2012.
- [24] Sheldon Ross. The multivariate normal distribution and copulas. *Simulation*, pages 97–109, 2013. doi: 10.1016/B978-0-12-415825-2.00006-1.
- [25] Anna Schlösser. One factor gaussian copula model. pages 95–127, 2011. doi: 10.1007/978-3-642-15609-0\_4.
- [26] P Shakdwipee, M Mehta Journal of New Technology, Research ..., and undefined 2017. From basel i to basel ii to basel iii. *academia.edu*.
- [27] J. Michael Steele. *Stochastic Calculus and Financial Applications*. Springer New York, 2001. doi: 10.1007/978-1-4684-9305-4.
- [28] Yuan Wen and Jacob Kinsella. Credit default swap-pricing theory, real data analysis and classroom applications using bloomberg terminal. *JOURNAL OF ECONOMICS AND FINANCE EDUCATION*, 2013.
- [29] Lawrence J. White. Markets: The credit rating agencies. *Journal of Economic Perspectives*, 24:211–26, 3 2010. ISSN 0895-3309. doi: 10.1257/JEP.24.2.211.
- [30] Steven H. Zhu and Michael Pykhtin. A guide to modeling counterparty credit risk. *GARP Risk Review*, July/August, 2007.