Modeling the interaction between micro-climate factors and moisture-related skin-support friction during patient repositioning in bed

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1 Introduction

Immobile patients that are limited to spending their time in bed predominantly are prone to skin breakdown as a consequence of moisture development between the skin and mattress. This wetness results from transpiration or urine. Due to wetting of the skin, the mechanical properties of the skin change and the friction between the skin and the mattress increases. This increase implies that the shear forces at the interface between the skin and mattress increase when a patient is moved or relocated on bed for daily care. This mechanism increases the likelihood of the development of a superficial pressure ulcer.

In this research, we will analyze, use and improve the phenomenological model developed by Gefen for the simulation of micro-climate factors. This model contains an interaction between the amount of transpiration and ambient temperature, increase of humidity, increase in the skin-support contact pressure. Furthermore, we will analyze and use a finite-element model for the mechanical support and equilibrium of tissue interacting with the mattress where the skin and subcutaneous tissue are incorporated. This interaction poses a contact problem where the surface of contact between the skin and mattress has to be determined. In this work, we will focus on the combination of the two models, where we aim at predicting the likelihood of the development of a superficial pressure ulcer in the course of time upon moving the patient over the surface of the mattress. This is done by using the finite-element method over the domain containing the tissue as well as the mattress. As an output parameter the shear strain will be important to estimate the probability that skin break-down (failure) occurs. Since the mechanical properties of skin change with local humidity, the skin will deteriorate in the course of time due to the build-up of moisture levels. In this MSc-thesis, we aim at a coupling of the micro-climate factors to the mechanical equilibrium which consists of a contact problem. Furthermore, skin behaves differently from rigid materials, hence most likely Hooke's Law will not be appropriate for the modeling.

The basics of this thesis lie in the two articles

- "How do microclimate factors affect the risk for superficial pressure ulcers: A mathematical modeling study" by Amit Gefen, and
- "Modeling the effects of moisture-related skin-support friction on the risk for superficial pressure ulcers during patient repositioning in bed" by Eliav Shaked and Amit Gefen.

These articles both describe a mathematical model regarding pressure ulcers in bed-bound patients. The first one assesses a patients risk of getting a pressure ulcer. Here a pressure ulcer is said to develop when the strength of the skin is smaller than the stress obtained by the movement. The second article describes a way of calculating the shear stress of the skin during movement using the finite element method.

Before a more detailed model can be formed from the two given models, a literature study needs to be done in order to fully understand all concepts and details included in these models. Also the field of contact mechanics has to be studied to be able to include this in the new model.

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2 Pressure ulcers

The official definition of a pressure ulcer is given by the European Pressure Ulcer Advisory Panel and says the following.

> "A pressure ulcer is localized injury to the skin and/or underlying tissue usually over a bony prominence, as a result of pressure, or pressure in combination with shear. A number of contributing or confounding factors are also associated with pressure ulcers; the significance of these factors is yet to be elucidated." – *http://www.epuap.org*

Such a pressure ulcer can occur after a large pressure has been applied to the skin for a short period of time, or when a small pressure is applied for a long period of time. Pressure ulcers, also referred to as "bedsores" or "pressure sores", usually occur at bony prominences, which are the parts of the body that are usually in direct contact with the underlying surface such as a mattress. Examples of the most common locations are the shoulders and the shoulder blades, back of the head, spine and tail bone.

The "European Pressure Ulcer Advisory Panel" (EPUAP) is a panel created to "support all European countries in their efforts to prevent and treat pressure ulcers". The overall mission of this panel is to

> " provide the relief of persons suffering from or at risk of pressure ulcers, in particular through research and the education of the public and by influencing pressure ulcer policy in all European countries towards an adequate patient centered and cost effective pressure ulcer care." – *http://www.epuap.org [1]*

In order to improve the communication between the different countries regarding pressure ulcers, the EPUAP has created a "Quick Reference Guide" which is translated into many different languages. In this reference guide guidelines are given that describe how a patients risk of pressure ulcers can be determined, and which factors should be taken into account. In this guide, the different types of pressure ulcers are also divided into four different categories. The categories and their (shortened) explanations are given below. The full explanation can be found on the EPUAP website [1].

Category/Stage I: Non-blanchable erythema Intact skin with non-blanchable redness of a localized area usually over a bony prominence. Darkly pigmented skin may not have visible blanching; its color may differ from the surrounding area. The area may be painful, firm, soft, warmer or cooler as compared to adjacent tissue. Category I may be difficult to detect in individuals with dark skin tones. May indicate at risk persons.

Category/Stage II: Partial thickness Partial thickness loss of dermis presenting as a shallow open ulcer with a red pink wound bed, without slough. May also present as an intact or open/ruptured serum-filled blister.Presents as a shiny or dry shallow ulcer without slough or bruising where bruising indicates deep tissue injury.

Category/Stage III: Full thickness skin loss Full thickness tissue loss. Subcutaneous fat may be visible but bone, tendon or muscle are not exposed. Slough may be present but does not obscure the depth of tissue loss. May include undermining and tunneling.The depth of a Category/Stage III pressure ulcer varies by anatomical location.

Category/Stage IV: Full thickness tissue loss Full thickness tissue loss with exposed bone, tendon or muscle. Slough or eschar may be present. Often includes undermining and tunneling.The depth of a Category/Stage IV pressure ulcer varies by anatomical location.

As can be seen in the definitions above, these different grades indicate the severity of the injury. In this thesis the main focus will lie on superficial pressure ulcers. According to Gefen ([3, 11]) these correspond to the pressure ulcers from Grade I and Grade II.

As mentioned in the definition of pressure ulcers, many different factors influence a patients risk at the injuries. These factors include among others the age of the patient, whether or not the patient is healthy, the wetness of the skin and the stiffness of the skin. Many of these factors lie close to each other, for example a patient who has diabetes often has a stiffer skin.
A lot of research has been done and is being done to investigate these factors and decrease patients risk at pressure ulcers. The models that are described in the articles that will be used in this thesis investigate the relation between the wetness of the skin (microclimate factors) and the risk of pressure ulcers. In the articles it is described that the temperature in the room has effect on the moisture level of the skin which has effect on the stiffness of the skin, hence again the factors are related.

3 Elasticity, Stress and Strain

In contact between solids the concepts of elasticity, stress and strain are very important. In the subject of pressure ulcers, especially stress and strain are importance. In this section there will therefore be given some attention to these topics. The knowledge used in this section and its subsections is acquired a.o. from the books Theory of Elasticity, by S. Timoshenko and J.N. Goodier [12], and Introduction to Finite Element Analysis Using MATLAB ®and Abaqus by Amar Khennane, chapter 5 [5].

3.1 Elasticity

The property of elasticity is something that all structural materials possess to a certain extent. It means that when external forces causes an object to deform up until a certain limit, the deformation will disappear when the external forces are removed. An object is said to be *perfectly elastic* when it resumes its initial form completely after the removal of all external forces.
Often when modeling elastic bodies it will be assumed that the matter of the body is *homogeneous*. This means that when taking a very small element of the body the same specific physical properties as the entire body will apply. Another assumption that is often made is the assumption that the body is *isotropic*. This means that the elastic properties are the same in all directions. Even though many structural materials do not satisfy these assumptions, experience has shown that the solutions of the theory of elasticity using these assumptions give very good results for these materials. When the elastic properties however are not the same in all directions, and also cannot be assumed to be the same, the condition of *anistropy* must be considered.

3.2 Stress and Strain

Stress and strain are the words most commonly used when talking about applying pressure on an object or when objects are deformed due to external forces.

Stress

If one applies pressure or other external forces on the outside of an object and this object is being restrained against rigid body movement, this pressure will be noted inside the object as internal forces are induced. These internal forces have a certain intensity, i.e. a certain amount of force per unit area of the surface on which they act. This intensity of the internal forces is called stress. The dimension of stress is pressure, hence it is mostly measured in terms of pascal (Pa).
When considering stress, it is usually resolved into two components: a normal stress which is perpendicular to the area one looks at, and a shearing stress which acts in the plane of this area. To denote stress the letters σ and τ are often used. Here σ denotes the normal stresses and τ denotes the shearing stresses.
To indicate the direction of the plane on which the stress is acting, subscripts to the components x, y and z are used. This means that when working in the Euclidean space the normal stresses are denoted by σ_x, σ_y and σ_z. The subscript x for example indicates that the stress is acting on a plane normal to the x-axis. It is agreed to take the normal stress positive when it produces tension and negative in the case it produces compression.

The shearing stresses are denoted by τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx} and τ_{zy} or simply by σ_{xy}, σ_{xz} σ_{yx}, σ_{yz}, σ_{zx} and σ_{zy}. Here $\tau_{ij} = \sigma_{ij}$. Note that the shearing stresses have two subscripts each. The first letter in the subscript indicates the direction of the normal to the plane under consideration. The second letter is then indicating the direction of the component of the stress. For example, considering the sides of a cube perpendicular to the z-axis, the component in the x-direction will be denoted by τ_{zx}.
In the paragraph above it becomes clear that stress has three symbols to describe the normal stresses (σ_x, σ_y and σ_z) and six symbols to describe the shearing stresses (τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx} and τ_{zy}).
Dividing the area one is looking at into very small elements, one can deduce that the shearing stress can be described using three symbols instead of one. This deduction can be done considering the equilibrium of the small elements. The following equations will be found.

$$\begin{array}{ccc} \tau_{xy} = \tau_{yx} & \tau_{xz} = \tau_{zx} & \tau_{yz} = \tau_{zy} \\ \sigma_{xy} = \sigma_{yx} & \sigma_{xz} = \sigma_{zx} & \sigma_{yz} = \sigma_{zy} \end{array} \tag{3.1}$$

Using these equations one finds that there are six components of stress, σ_x, σ_y, σ_z, $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$, at every point in the object.
These stress components are sometimes denoted in matrix style, which gives us the stress matrix

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}.$$

Using (3.1) it can be seen that the matrix above is symmetric. Sometimes in engineering a vector notation is used. In that case the stress is denoted as

$$\vec{\sigma} = \left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{array} \right\} = \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{array} \right\} = \left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{array} \right\}.$$

Strain

Besides inducing internal forces when one applies pressure or other external forces on the outside of an object while this object is being restrained against rigid body movement,material points inside the body can be displaced. When this displacement causes the distance between two points in the body to change one speaks of straining. Strain is dimensionless.
Similar to stress the strain in a certain point exists of different components. These components are again denoted using the subscripts x, y and z. Same as stress the strain is also divided into two parts; the unit elongations and the shearing strains. These are respectively denoted as ϵ_i and γ_{ij} or as ϵ_{ii} and ϵ_{ij}, where $\gamma_{ij} = \epsilon_{ij} + \epsilon_{ji}$. Here the unit elongation means that the distance between two points in the body only changes in one direction. In the shearing stresses the distance between the points will change in two different coordinates.
The displacements in the Euclidean space are denoted using the letters u, v and w for respectively displacements in the x-plane, y-plane and z-plane. The strain components are then given

by

$$\begin{array}{ccc} \epsilon_x = \frac{\partial u}{\partial x} & \epsilon_y = \frac{\partial v}{\partial y} & \epsilon_z = \frac{\partial w}{\partial z} \\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} & \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \end{array} \tag{3.2}$$

It can be easily seen that

$$\begin{array}{ccc} \gamma_{xy} = \gamma_{yx} & \gamma_{xz} = \gamma_{zx} & \gamma_{yz} = \gamma_{zy} \\ \epsilon_{xy} = \epsilon_{yx} & \epsilon_{xz} = \epsilon_{zx} & \epsilon_{yz} = \epsilon_{zy} \end{array} \tag{3.3}$$

Also the strain can be given in matrix form. One obtains the following matrix.

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_z \end{bmatrix}$$

Using equations (3.3) it can be seen that $\boldsymbol{\epsilon}$ is symmetric. Since the strain matrix consists of only six independent components, some engineers prefer to use a vector notation to represent the strain components. The following vector notations are commonly used in the literature.

$$\vec{\epsilon} = \left\{ \begin{array}{c} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{array} \right\} = \left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{array} \right\} = \left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} + \epsilon_{yx} \\ \epsilon_{yz} + \epsilon_{zy} \\ \epsilon_{xz} + \epsilon_{zx} \end{array} \right\} = \left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ 2\epsilon_{xy} \\ 2\epsilon_{yz} \\ 2\epsilon_{xz} \end{array} \right\}$$

or

$$\vec{\epsilon} = \left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{yz} \\ \epsilon_{xz} \end{array} \right\} = \left\{ \begin{array}{c} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \\ \epsilon_{23} \\ \epsilon_{13} \end{array} \right\} = \left\{ \begin{array}{c} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{array} \right\}.$$

3.3 Hooke's Law

In the previous section the different components for stress and strain are given. The relation between these components are given in Hooke's law. This law says the following.

$$\vec{\sigma} = \mathbf{D}\vec{\epsilon} \tag{3.4}$$

Here the matrix $\mathbf{D}$ is called the stiffness tensor.
Using the vectors $\vec{\sigma}$ and $\vec{\epsilon}$, which both contain only six elements due to symmetry, one can see that the matrix $\mathbf{D}$ has size 6 by 6. These elements contain information regarding the materials of the solids. This information is given using the coefficients of Elasticity. In total there are five different coefficients which are all related to one another. The coefficients are called the Lamé's constants (Lamé's first constant and the shear modules), Young's modulus, the Poisson's ratio and the Bulk modulus.
Of these parameters, three are moduli of elasticity. These are the shear modulus, Young's modulus and the Bulk modulus. All three describe the ratio of the stress to the strain, hence are equal to the slope of a stress-strain curve. The elasticity modulus is the mathematical description of an objects tendency to deform elastically when forces are applied to it.

Lamé's constants

The Lamé constants are two different constants described by the French mathematician Gabriel Lamé; Lamé's first parameter λ and the shear modulus denoted by μ or G.
The first parameter, λ, is an elastic modulus, but is often said to have no physical interpretation. The second parameter, denoted as μ or G, is mostly referred to as the shear modulus. Other names for this elastic modulus are rigidity or the modulus of rigidity. This parameter is defined to be the ratio of the shear stress to the shear strain. The modulus measures the stiffness of the material. It can be considered as measuring the response of a material to shear stress, for example cutting it with dull scissors. The SI unit of the shear modulus is the pascal (Pa = N/m^2).
Where the shear modulus always has to be positive, the first Lamé constant can be negative. For most materials however, the constant will be positive.

Young's modulus

Young's modulus is the most common elastic modulus, named after the British scientist Thomas Young. The modulus is also known as the modulus of elasticity, the elastic modulus or the Tensile modulus. This parameter measures the stiffness of an elastic isotropic material, and is therefore specific for the material. The modulus is defined as the ratio of the stress along an axis to the strain along that axis. It can be considered as the material's response to linear stress. Examples of such stress are pulling the ends of a wire and putting a weight on top of a column. Since the elastic modulus is defined as the ratio between stress and strain, it's SI unit is the same as the SI unit of the stress, pascal (Pa).
In anisotropic materials the Young's modulus can have different values for the different directions of the applied force with respect to the material's structure.

The value of the Young's modulus can be seen as a measure for the rigidness of the material. When the material has a high modulus, it is very rigid.

Poisson's ratio

One of the other elastic parameters is called the Poisson's ratio and is denoted using the Greek letter nu (ν). This parameter is defined as the ratio of the transverse strain to the longitudinal strain. Since strain has no dimension, the Poisson's ratio is also dimensionless. The ratio describes the material's response to when the object is squeezed (i.e. how much the material expands outwards) and the response to when the object is stretched (i.e. how much the material contracts).
The value of the ratio hence depends on the material of the object. When the material is incompressible, the ratio will have a value of approximately 0.5. A value equal to 0 means that the material does not expand radially when it is compressed. When the value of the ratio is negative it means that the material has an opposite response to compression, i.e. the material gets thinner when compressed. These materials are called *auxetic*.

Bulk modulus

The third elasticity modulus is called the Bulk modulus and is denoted as K or B. It measures the material's response to uniform pressure. An example of such uniform pressure is the pressure at the bottom of the ocean or a deep swimming pool. It can be defined as the ratio of the volume stress to the volume strain. In other words it can be described as "the ratio of the infinitesimal pressure increase to the resulting relative decrease of the volume". The modulus is measured in pascal (Pa).

In working with elasticity usually two of the above mentioned coefficients are used. In this thesis Young's modulus and the Poisson's ratio (E, ν) shall be used. In Table 1 the relations between the different coefficients are given.

Table 1: **Relationships between the Coefficients of Elasticity**
From [5]

	$(\boldsymbol{\lambda}, \boldsymbol{\mu})$	$(\boldsymbol{E}, \boldsymbol{\nu})$	$(\boldsymbol{E}, \boldsymbol{G})$
λ	λ	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{G(E-2G)}{3G-E}$
μ	μ	$\frac{E}{2(1+\nu)}$	G
E	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	E	E
ν	$\frac{\lambda}{2(\lambda+\mu)}$	ν	$\frac{E-2G}{2G}$
K	$\lambda + \frac{2}{3}\mu$	$\frac{E}{3(1-2\nu)}$	$\frac{GE}{3(3G-E)}$

Equation 3.4 gives Hooke's law as $\vec{\sigma} = \mathbf{D}\vec{\epsilon}$. For several situations the matrix $\mathbf{D}$ is actually known. (Note that the elements of this matrix will consist of factors of E and ν.) Writing Hooke's Law in index notation, one obtains

$$\sigma_{ij} = D_{ijkl}\epsilon_{kl}. \tag{3.5}$$

Here D_{ijkl} is called the stiffness tensor, which is a fourth order tensor with a total of 81 components. Equation 3.5 can also be written as a system of nine equations.

$$\left\{\begin{array}{lcl}
\sigma_{11} & = & D_{1111}\epsilon_{11} + D_{1112}\epsilon_{12} + D_{1113}\epsilon_{13} + D_{1121}\epsilon_{21} + D_{1122}\epsilon_{22} + D_{1123}\epsilon_{23} \\
 & & +D_{1131}\epsilon_{13} + D_{1132}\epsilon_{32} + D_{1133}\epsilon_{33} \\
\sigma_{12} & = & D_{1211}\epsilon_{11} + D_{1212}\epsilon_{12} + D_{1213}\epsilon_{13} + D_{1221}\epsilon_{21} + D_{1222}\epsilon_{22} + D_{1223}\epsilon_{23} \\
 & & +D_{1231}\epsilon_{13} + D_{1232}\epsilon_{32} + D_{1233}\epsilon_{33} \\
\sigma_{13} & = & D_{1311}\epsilon_{11} + D_{1312}\epsilon_{12} + D_{1313}\epsilon_{13} + D_{1321}\epsilon_{21} + D_{1322}\epsilon_{22} + D_{1323}\epsilon_{23} \\
 & & +D_{1331}\epsilon_{13} + D_{1332}\epsilon_{32} + D_{1333}\epsilon_{33} \\
\sigma_{21} & = & D_{2111}\epsilon_{11} + D_{2112}\epsilon_{12} + D_{2113}\epsilon_{13} + D_{2121}\epsilon_{21} + D_{2122}\epsilon_{22} + D_{2123}\epsilon_{23} \\
 & & +D_{2131}\epsilon_{13} + D_{2132}\epsilon_{32} + D_{2133}\epsilon_{33} \\
\sigma_{22} & = & D_{2211}\epsilon_{11} + D_{2212}\epsilon_{12} + D_{2213}\epsilon_{13} + D_{2221}\epsilon_{21} + D_{2222}\epsilon_{22} + D_{2223}\epsilon_{23} \\
 & & +D_{2231}\epsilon_{13} + D_{2232}\epsilon_{32} + D_{2233}\epsilon_{33} \\
\sigma_{23} & = & D_{2311}\epsilon_{11} + D_{2312}\epsilon_{12} + D_{2313}\epsilon_{13} + D_{2321}\epsilon_{21} + D_{2322}\epsilon_{22} + D_{2323}\epsilon_{23} \\
 & & +D_{2331}\epsilon_{13} + D_{2332}\epsilon_{32} + D_{2333}\epsilon_{33} \\
\sigma_{31} & = & D_{3111}\epsilon_{11} + D_{3112}\epsilon_{12} + D_{3113}\epsilon_{13} + D_{3121}\epsilon_{21} + D_{3122}\epsilon_{22} + D_{3123}\epsilon_{23} \\
 & & +D_{3131}\epsilon_{13} + D_{3132}\epsilon_{32} + D_{3133}\epsilon_{33} \\
\sigma_{32} & = & D_{3211}\epsilon_{11} + D_{3212}\epsilon_{12} + D_{3213}\epsilon_{13} + D_{3221}\epsilon_{21} + D_{3222}\epsilon_{22} + D_{3223}\epsilon_{23} \\
 & & +D_{3231}\epsilon_{13} + D_{3232}\epsilon_{32} + D_{3233}\epsilon_{33} \\
\sigma_{33} & = & D_{3311}\epsilon_{11} + D_{3312}\epsilon_{12} + D_{3313}\epsilon_{13} + D_{3321}\epsilon_{21} + D_{3322}\epsilon_{22} + D_{3323}\epsilon_{23} \\
 & & +D_{3331}\epsilon_{13} + D_{3332}\epsilon_{32} + D_{3333}\epsilon_{33}
\end{array}\right. \tag{3.6}$$

Using (3.1) and (3.3) it follows that the above equations can be simplified due to symmetry of the stiffness tensor, i.e.

$$D_{ijkl} = D_{ijlk} = D_{jikl} = D_{jilk}.$$

This means that instead of 81 different elements, the stiffness tensor only has 36 independent elastic coefficients. Using more simplifications the number of coefficients can even be reduced to 21.

3.3.1 Isotropic materials

In this thesis the skin and subcutaneous will be modeled as *isotropic* materials. Here isotropy means uniformity in all orientations, or in other words, the elastic properties of the material are the same in any direction and therefore do not depend on the choice of the coordinates system ([5]). Since none of the properties of the material depend on the orientation, the material is perfectly rotational and symmetric with respect to three orthogonal planes. Using this assumption the matrix $\mathbf{D}$ can be simplified to exist of only two independent coefficients, E and ν, obtaining the following stress-strain relationship for the elastic matrix.

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{pmatrix} \tag{3.7}$$

In equation (3.7) the relation between stress and strain is given for an isotropic material, with the stress a function of the strain. Here the stiffness tensor is called the elasticity matrix $\mathbf{D}$.

It is also possible to write this Hooke's Law differently, i.e. with the strain a function of the stress. The equation will then become $\vec{\epsilon} = \mathbf{C}\vec{\sigma}$. In this case instead of the elastic matrix one speaks of the compliance matrix, denoted by $\mathbf{C}$. Equation (3.8) shows the relation between stress and strain using this compliance matrix.

$$\begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{pmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{pmatrix} \tag{3.8}$$

3.4 Plane stress and Plane strain

Working with solids, hence in three dimensions, the vectors describing the stress and strain both contain six elements, and the stiffness tensor is six by six. This causes most problems to be quite large. Fortunately it is often possible to make some assumptions that lead to simplifications.

Plane stress

An example of a situation in which simplifying assumptions can be made is when working with a solid with one dimension relatively small compared to the two others and loaded in its plane. In such a situation the problem can be analyzed using the plane stress approach. This approach means that the stress on the small dimension is assumed to be zero throughout the entire solid. The only forces applied to the object will be parallel to the plate of this dimension. In other words, the stress vector is zero across a particular surface. This approach is usually taken when working with thin plates and beams. Look for example at Figure 1.

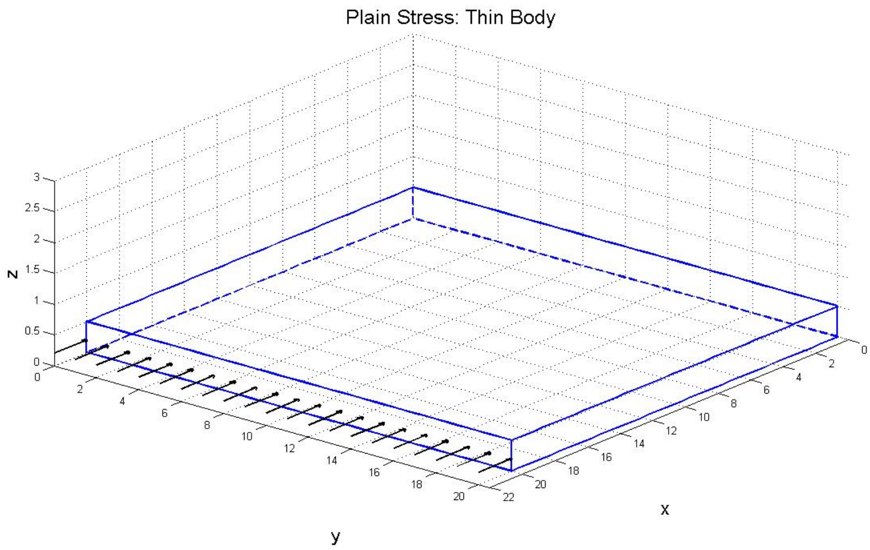

Figure 1: In plane stress one of the dimensions is very small compared to the others. In this thin body, the z-component is small.

In this figure one can see that the thickness of the beam (z-component) is small compared to the other two dimensions. It is also clear that the surfaces of the beam are free of forces. This leads to the stress components σ_{xz}, σ_{yz} and σ_{zz} being equal to zero. If the beam is thin, as it is shown in the figure, it can be assumed that these stress components are equal to zero throughout the entire thickness of the beam. Furthermore, it is reasonable to assume that the other stress components, σ_{xx}, σ_{yy} and σ_{xy} remain constant.
We find that in the case of plane stress the stress vector will only exist of three non-zero components, and using this the stress-strain relation using the elastic matrix shown in equation (3.7) will come down to the following.

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} \tag{3.9}$$

Furthermore, since σ_{zz} is equal to zero and using that

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}\left(\nu\epsilon_{xx} + \nu\epsilon_{yy} + (1-\nu)\epsilon_{zz}\right) \tag{3.10}$$

(from equation (3.7)) ϵ_{zz} can be determined. To do this, take

$$A = \frac{E}{(1+\nu)(1-2\nu)}.$$

Substituting $\sigma_{zz} = 0$ in (3.10) one obtains a result for ϵ_{zz}.

$$\begin{aligned} 0 &= A\nu(\epsilon_{xx} + \epsilon_{yy}) + A(1-\nu)\epsilon_{zz} \\ A(1-\nu)\epsilon_{zz} &= -A\nu(\epsilon_{xx} + \epsilon_{yy}) \\ \epsilon_{zz} &= \frac{-\nu}{1-\nu}(\epsilon_{xx} + \epsilon_{yy}) \end{aligned} \tag{3.11}$$

This expression can be rewritten further using equation (3.9). From Hooke's Law for plane stress it follows that

$$\begin{aligned} \sigma_{xx} &= \frac{E}{1-\nu^2}(\epsilon xx + \nu\epsilon_{yy}) \\ \sigma_{yy} &= \frac{E}{1-\nu^2}(\nu\epsilon xx + \epsilon_{yy}) \\ \sigma_{xx} + \sigma_{yy} &= \frac{E}{1-\nu^2}(1+\nu)(\epsilon_{xx} + \epsilon_{yy}) \\ \sigma_{xx} + \sigma_{yy} &= \frac{E}{1-\nu}(\epsilon xx + \epsilon_{yy}). \end{aligned} \tag{3.12}$$

Multiplying equation (3.12) by $\frac{-\nu}{E}$ and comparing this with equation (3.11) equation (3.13) can be derived.

$$\epsilon_{zz} = \frac{-\nu}{E}(\sigma_{xx} + \sigma_{yy}) \tag{3.13}$$

As before, Hooke's Law can also be given using the compliance matrix. This relation is given by equation (3.14).

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} \tag{3.14}$$

Plane strain

Another example in which certain assumptions can simplify the problem is called plane strain. In this case one of the dimensions of an object will be very large compared to the other two dimensions. In this case the loads are uniformly distributed with respect to the large dimension and act perpendicular to it. An example of this situation is shown in figure 2.

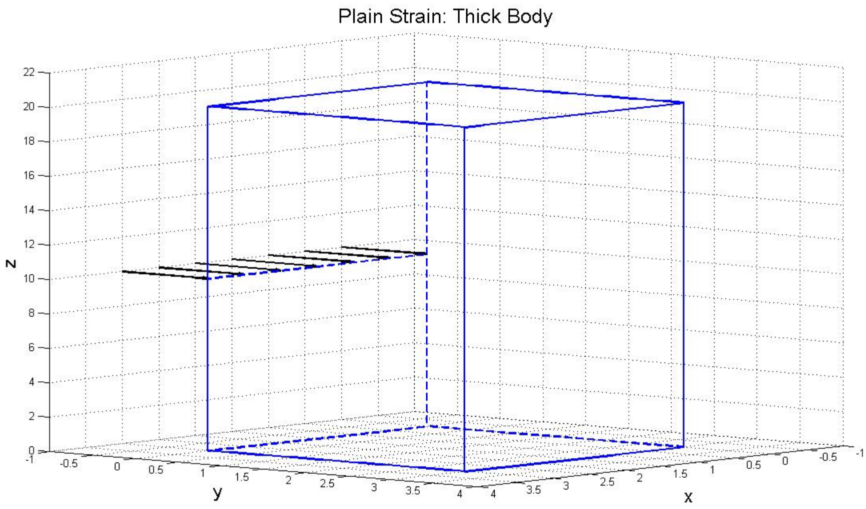

Figure 2: In plane strain one of the dimensions is very large compared to the others. In this thick body, the z-component is large.

Instead of the z-component being very small compared to the other two dimension as in Plain Stress, here the z-component is very large compared to the other two. In this case the strain components γ_{xz}, γ_{yz} and ϵ_{zz} are equal to zero. This holds throughout the beam because the displacements of all faces in the z-direction are kept equal to zero.
The strain components that are nonzero are ϵ_{xx}, ϵ_{yy} and γ_{xy}.

As in the case of plane stress we find that Hooke's law will be smaller, since the strain vector exists of only three nonzero components. The stress-strain relation using the elastic matrix shown in equation (3.7) will become the following.

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} \tag{3.15}$$

Hooke's Law can also be given the other way around, using the compliance matrix. This relation is given by equation (3.16).

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} \tag{3.16}$$

Furthermore, ϵ_{zz} is equal to zero, but σ_{zz} is not. Looking at Hooke's Law in equation (3.7) and writing zero for γ_{xz}, γ_{yz} and ϵ_{zz} the following equation is obtained.

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{pmatrix} \epsilon_x x \\ \epsilon_y y \\ 0 \\ \gamma_{xy} \\ 0 \\ 0 \end{pmatrix} \tag{3.17}$$

Equation (3.18) follows directly from this relation.

$$\sigma_{zz} = \nu(\epsilon_{xx} + \epsilon_{yy}) \tag{3.18}$$

Review Plane stress and Plane strain

In figure 3 the two states plane stress and plain strain are quickly compared.

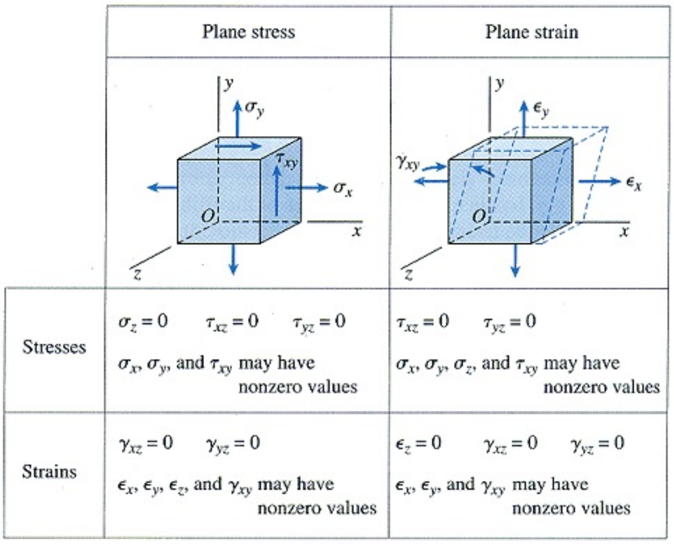

Figure 3: Comparison between plane stress and plane strain.
From: http://classes.mst.edu/civeng110/concepts/13/strain/plane_stress_vs_strain.gif

4 Contact Mechanics

Contact mechanics is the area having to do with situations in which multiple solids are in contact with one another. This area is very big, since there are many different options when looking at contacting solids. The solids themselves for example can be rigid or elastic, they can deform or stay the same as an effect of the contact, and the contact between the solids can be conforming or non-conforming. These latter options mean that without applying pressure the bodies either touch at multiple points (i.e. they "fit together") or they only touch at one point or one line (i.e. the shapes do not "fit together"). In the case of non-conforming contact, the contact area is very small compared to the sizes of the bodies, which causes the stresses to be high in this area. In this case the contact will also be called concentrated. In the case of a larger contact area the stresses will be more spread out and the contact will be called diversified.

In the general contact problem there are three components that can be of importance.

1. Due to the the load that presses the bodies together, deformation of the separate bodies will occur. The deformation depends on the material and structure of the body.
2. Secondly the bodies have an overall motion relative to each other. Possibilities are the bodies being at rest, approaching each other (after which impact follows), sliding and rolling over each other.
3. Thirdly there are the processes at the contact area: compression and adhesion in the direction perpendicular to the area, and friction and micro-slip in the tangential directions.

This last component can be described using conditions called the contact conditions.

- First of all the gap between the two bodies should always be greater than or equal to zero: $e_n \geq 0$, where equality holds in case of contact and inequality when the bodies are separated.
- Secondly, the normal stress acting on each body should also be greater than or equal to zero: $p_n \geq 0$, where equality means the bodies are separated and inequality holds when the bodies are in contact. In this latter case the normal stress is compressive.

  Note that:

  - The functions e_n and p_n depend on the location of the body surfaces.
  - The product of e_n with p_n will always be equal to zero: $e_n p_n = 0$.

The first important component of contact problems is, as given above, the deformation of the solids in contact. Researchers have been investigating this deformation done for a long time. In 1882, Hertz published an article called "On the contact of elastic solids". This article was one of the first steps in the research of contact mechanics. After this, many more models were created, such as the JKR model (Johnson, Kendall and Roberts), and the Bradley model.

There are many different models regarding contact between two solids. Some of these models are useable for contact problems of solids that are only pressed together (normal contact problems), while other models can be used when either or both solids are being moved with respect to the other (tangential contact problems). Four big models regarding normal contact problems

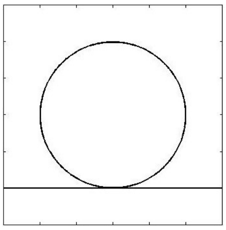

Figure 4: Contact with no force acting upon the contact.

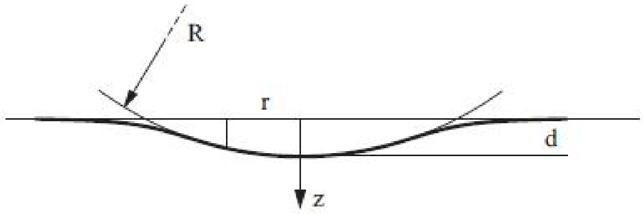

Figure 5: Contact with a normal force acting upon the contact.

are described in section 4.1. The contact problems with tangential movement are described in section 4.2.

4.1 Normal contact mechanics

The information described in this subsection is mostly obtained from the books Contact Mechanics by Johnson [4] and Contact Mechanics and Friction by Popov [9] and the website Wikipedia [13].

The following models will be described in this section. A more elaborate description of these models is given in respectively section 4.1.1 and 4.1.2.

Hertz fully elastic model

JKR fully elastic model considering adhesion in the contact zone

Bradley purely van der Waals model with rigid spheres

DMT fully elastic, adhesive and van der Waals model.

4.1.1 The Hertzian Theory of Elastic Deformations

The Hertzian Theory of Elastic Deformations is on of the first models regarding the geometrical effects on local elastic deformation properties. It was created around 1882 when Hertz solved the problem of contact between two elastic bodies with curved surfaces. The result described in the model forms a basis for contact mechanics today. The most common problem is called the *normal contact problem.* This problem revolves around two bodies which are brought into contact with another by forces perpendicular to their surfaces [9], or in other words, are being pressed together. The Hertzian Theory of Elastic Deformations considers such a normal contact problem between a rigid sphere and an elastic half-space. In the theory all adhesive forces are neglected. The information in this section closely follows the information from the book Contact Mechanics and Friction by V.L. Popov [9].

Figures 4 and 5 shows the contact between the elastic half-space and the rigid sphere schematically.

The original theory of Heinrich Hertz had three results

- the contact radius was determined,
- the maximum pressure was determined and
- the normal force of the contact was determined.

To obtain these results Hertz used the displacement of the points on the surface in the contact area between an originally even surface and a rigid sphere of radius R. This displacement is equal to[1]

$$u_z = d - \frac{r^2}{2R}, \tag{4.1}$$

where u_z denotes the surface displacement, d the indentation depth, $r = \sqrt{x^2+y^2}$ and R the radius of the sphere. The relation between the before mentioned parameters is shown in figure 5.

In solving the contact problem in order to obtain the three results a pressure distribution has to be assumed. The pressure distribution that is assumed in the theory of Hertz is $p = p_0(1-r^2/a^2)^n$, with $n = 1/2$. In this pressure distribution p_0 is the maximum pressure. This *Hertzian Pressure Distribution*

$$p = p_0\left(1-\frac{r^2}{a^2}\right)^{\frac{1}{2}}, \quad r^2 = x^2+y^2 \tag{4.2}$$

leads to a vertical displacement equal to

$$u_z = \frac{\pi p_0}{4E^*a}(2a^2-r^2), \quad r \leq a. \tag{4.3}$$

Equation (4.3) can be obtained as follows.[2] When working with a continuous distribution of the normal pressure $p(x,y)$, the displacement of the surface is calculated as

$$u_z = \frac{1}{\pi E^*}\int\int p(x',y')\frac{dx'dy'}{r} \quad \text{with } r = \sqrt{(x-x')^2+(y-y')^2} \quad \text{and } E^* = \frac{E}{1-\nu^2}. \tag{4.4}$$

Using a change of coordinates, taking $\alpha = a^2 - r^2$, $\beta = r\cos(\phi)$ and substituting equation (4.2), equation (4.4) becomes

$$u_z = \frac{1}{\pi E^*}\frac{p_0}{a}\int_0^{2\pi}\left(\underbrace{\int_0^{s_1}\left(\alpha^2-2\beta-s^2\right)^{\frac{1}{2}}ds}_{*}\right)d\phi. \tag{4.5}$$

The expression (*) can be calculated as

$$\begin{aligned}(*) &= \int_0^{s_1}\left(\alpha^2-2\beta-s^2\right)^{\frac{1}{2}}ds\\ &= \frac{1}{2}\alpha\beta + \frac{1}{2}(\alpha^2+\beta^2)\cdot\left(\frac{\pi}{2}-\arctan(\beta/\alpha)\right).\end{aligned}$$

[1]Only the z-component of the displacement is of interest within the framework of the half-space approximation in contact problems without friction [9].

[2]This derivation is explained in Appendix A of [9].

Integrating the above over ϕ from 0 to 2π, the terms $\alpha\beta$ and $\arctan(\beta/\alpha)$ vanish since $\beta = r\cos(\phi)$. This leads to the following result.

$$\begin{aligned}
u_z &= \frac{1}{\pi E^*}\frac{p_0}{a}\int_0^{2\pi}\left(\int_0^{s_1}\left(\alpha^2 - 2\beta - s^2\right)^{\frac{1}{2}} ds\right) d\phi \\
&= \frac{1}{\pi E^*}\frac{p_0}{a}\int\limits_0^{2\pi}\frac{\pi}{4}(\alpha^2+\beta^2)d\phi \\
&= \frac{1}{4E^*}\frac{p_0}{a}\int\limits_0^{2\pi} a^2 - r^2 + r^2\cos(\phi)d\phi \\
&= \frac{\pi p_0}{4E^* a}(2a^2 - r^2) \\
&= \text{equation (4.3)}
\end{aligned}$$

The total force of the contact is

$$F = \int\limits_0^a p(r)2\pi r dr = \frac{2}{3}p_0\pi a^2. \tag{4.6}$$

To solve the contact problem, one can now use the fact that both equation (4.1) and equation (4.3) describe the same vertical displacement and hence should be equal.

$$\frac{\pi p_0}{4E^* a}(2a^2 - r^2) = d - \frac{r^2}{2R}$$

From this equality the variables a and d can be derived. One obtains

$$a = \frac{\pi p_0 R}{2E^*} \quad \text{and} \quad d = \frac{\pi a p_0}{2E^*}. \tag{4.7}$$

Equation (4.7) leads to the first result of Hertz theory, the contact radius between the rigid sphere and the elastic half-space:

$$\textbf{1.} \quad a^2 = Rd. \tag{4.8}$$

Using equations (4.7) and (4.8) the second result can also be obtained, which is the maximum pressure.

$$\textbf{2.} \quad p_0 = \frac{2E^*}{\pi}\left(\frac{d}{R}\right)^{\frac{1}{2}} \tag{4.9}$$

Substituting both equations (4.8) and (4.9) in the equation of total force (4.6), the third result is obtained; the normal force.

$$\textbf{3.} \quad F = \frac{4}{3}E^*\left(\frac{d^3}{R}\right)^{\frac{1}{2}} \tag{4.10}$$

From this last result, the potential energy of the elastic deformation U can be determined using $-F = \partial U \partial d$.

$$U = \frac{8}{15}E^*(Rd^5)^{\frac{1}{2}}$$

The above results are explicitly for the contact between a rigid sphere and an elastic half space. The results can be used to obtain results for other scenarios, such as the contact between two elastic bodies, the contact between two spheres, the contact between two elastic cylinders, and more.

If the contact is for instance between two elastic bodies, the only difference with the previous result is that the expression of E^* must be changed into

$$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}, \tag{4.11}$$

where E_1 and E_2 are the moduli of elasticity of the two bodies, and ν_1 and ν_2 the respective Poisson's ratios.

4.1.2 Bradley's Van der Waals model, The JKR-theory and the DMT-theory

Fifty years after Hertz solved the normal contact problem without adhesion between elastic bodies in 1882, Bradley presented the solution for the normal contact problem with adhesion between a rigid sphere and a rigid plane.

In 1971 an article was written by K.L. Johnson, K. Kendall and A.D. Roberts. In this article a new contact mechanics model called the JKR-theory is described in which the contact adhesive interactions are taken into account. The JKR-theory is often referred of as the classical theory of adhesive contact.

The JKR-theory is based on the Hertzian theory. However, as has been noted before, Hertz did not include any adhesive forces in his model. It was found by Roberts and Kendall [6] that these contact forces are of little significance when two spheres are pressed together by a high load, but become more important when this load reduces. This means that for low loads the model of Hertz will be less accurate.

The DMT-theory is yet another theory that includes adhesive forces. This theory is created as a combination of the Hertzian theory and Bradleys model. When the two bodies are separated and significantly apart the DMT-theory will simplify to *Bradleys's Van der Waals model*.

All three models describe the adhesive normal contact problem, only between different types of bodies.

Bradley solved the adhesive normal contact problem between a rigid sphere and a rigid plane. The resulting adhesive force was found to be $F_A = 4\pi\gamma R$, with γ the surface energy and R the radius of the sphere.

JKR solved the adhesive contact problem between elastic bodies. They found the adhesive force to be equal to $F_A = 3\pi\gamma R$.

DMT described a different adhesive theory while they considered the case of deformable bodies by adding the adhesive force of Bradleys model to the theory of Hertz.

In 1976 Tabor realized that the above mentioned models were all valid for different scenarios. The DMT-Theory and JKR-Theory are both special cases of the general problem. He stated that the theories had only very small differences, but that ([9])

Bradleys model is correct for absolutely rigid bodies,

The JKR-Theory is valid for large, flexible spheres, and

The DMT-Theory is valid for small, rigid spheres.

4.2 Tangential Contact Problems

In the previous models, the two solids were only pressed together and both had absolutely smooth and frictionless surfaces. In these cases the shear forces in the contact area are equal to zero. In this section contacts are examined in which the point of contact is also loaded in the tangential direction. Now static and kinetic frictional forces will become interesting, and the shear forces will thus be nonzero. These Tangential Contact Problems belong to the field of Frictional contact mechanics, which is the study of the deformation of bodies in the presence of frictional effects.

In case of tangential contact problems there are additional contact conditions, coming from the fact that the shear stress should always be smaller than or equal to the so-called traction bound which depends on the position. This is called the local friction law. The friction law that is most commonly used is called Coulomb's law (see section 4.2.4), which states that $F_x \leq \mu F_N$. Here F_x is the tangential force, F_N is the normal force and μ is the coefficient of friction.
In this law equality holds in case of sliding and inequality holds in case of sticking.

Generally the contact area and the sticking and sliding parts are unknown in advance. If these were known, then the elastic fields in the two bodies could be solved independently from each other and the problem would not be a contact problem anymore.

4.2.1 Cattaneo problem

A commonly known tangential contact problem is called the Cattaneo problem. This contact problem is between an elastic sphere with radius R and an elastic plane (half space). The sphere is pressed onto the plane and then shifted over the plane's surface by a tangential force F_x.
When starting with only the normal force F_N the sphere will be pressed onto the plane. The contact point will turn into a contact area as both bodies deform and the center of the sphere moves down by a distance of δ_n called the approach (see figure 6). The contact area will be circular and a Hertzian normal pressure distribution arises.

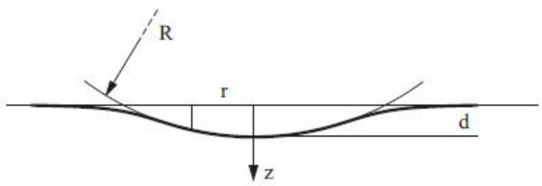

Figure 6: The center of the sphere moves down by a distance of $d = \delta_n$.
Source: CMaF

When both the sphere and the plane are from the same material (same elastic properties), the

Hertzian solution reads

$$p_n(x,y) = p_0\sqrt{1-r^2/a^2} \quad r = \sqrt{x^2+y^2} \le a \quad a = \sqrt{R\delta_n},$$
$$p_0 = \frac{2}{pi}E^*(\delta_n/R)^{1/2} \quad F_N = \tfrac{4}{3}E^* R^{1/2}\delta_n^{3/2} \quad E^* = \frac{E}{2(1-\nu^2)},$$

where E and ν are respectively the Young's modulus and the Poisson's ratio. This is the same as was shown earlier in section 4.1.1.
When the sphere and the plane are made of different materials, the same solution holds, only now using

$$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}. \tag{4.12}$$

Assumed before is that after the normal pressure is applied, a tangential force F_x will be applied that 'pushes' against the sphere.
When this force is lower than the Coulomb friction bound ($F_x < \mu F_N$) the center of the sphere will move sideways for a small distance δ_x, which is called the shift. An equilibrium will be obtained in which the bodies are deformed and frictional shear stresses occur. When the tangential force is removed the sphere will (mostly) shift back.

This problem was solved analytically by Cattaneo. In his solution he combined two Hertzian distributions which showed that there occurs partial sliding during the tangential loading (see section 4.2.3).
Before this combination of distributions is shown, some information regarding working with half-spaces, the deformations that will occur in these and the stress distributions causing these deformations are given.

4.2.2 Half-space approaches

It is often useful to work with half-spaces instead of fixed and bounded planes. We will therefore look into the deformations that will occur when a tangential stress distribution acts upon an elastic half-space.
As is done in the book of Valentin L. Popov [9] the problems will be considered using a half-space approximation. This means that "the gradient of the surface of he contacting bodies should be small in the vicinity relevant to the contact problem " ([9]).

A point is taken on the surface of this elastic half-space which is chosen to be the origin. A concentrated force acts on this origin, which for simplicity only has a component in the x-direction. When considering the surface $z = 0$ the following equations describe the displacements ([7][3])

$$\begin{aligned} u_x &= F_x\frac{1}{4\pi G}\left\{2(1-\nu) + \frac{2\nu x^2}{r^2}\right\}\frac{1}{r}, \\ u_y &= F_x\frac{1}{4\pi G}\cdot\frac{2\nu}{r^3}xy, \\ u_z &= F_x\frac{1}{4\pi G}\cdot\frac{(1-2\nu)}{r^2}x. \end{aligned} \tag{4.13}$$

In these equations G is the shear modulus also denoted by μ, described further in section 3.3 and table 1.

[3]referred to by [9]

To make the problem more realistic one can look at a tangential force distribution acting upon the displacement of the surface. Assuming this force acts in the x-direction it can be denoted by

$$\sigma_{zx}(x,y) = \tau(x,y).$$

Using this distribution, the displacement in the x-direction can be calculated using the integral

$$u_x = \frac{1}{4\pi G} \cdot 2 \iint\limits_A \left\{ \frac{1-\nu}{s} + \nu \frac{(x-x')^2}{s^3} \right\} \tau(x',y') dx'dy', \tag{4.14}$$

where

$$s^2 = (x-x')^2 + (y-y')^2.$$

It is obvious that a different force distribution will lead to a different displacement in the x-direction. The following possibilities are given in Chapter 8 of Contact Mechanics and Friction, by V.L. Popov [9].

- For example, a constant value of the displacement will be found if the following force distribution is taken

  $$\tau(x,y) = \tau_0 (1 - r^2/a^2)^{-1/2} \qquad \text{with} \qquad r^2 = x^2 + y^2 \leq a^2.$$

  Substituting this in equation (4.14) and integrating, the displacement inside the loaded area ($r \leq a$) is found to be

  $$u_x = \frac{\pi(2-\nu)}{4G} \tau_0 a = \text{constant}. \tag{4.15}$$

  Due to symmetry, in this case $u_y = 0$. The z-component of the displacement is however not equal to zero and can be calculated using equation (4.13). The total force F_x that acts on the contact area can be calculated as

  $$F_x = \int_0^a \tau(r) 2\pi r dr = 2\pi\tau_0 a^2. \tag{4.16}$$

- Another possible force distribution is the distribution

  $$\tau(x,y) = \tau_0 \left(1 - r^2/a^2\right)^{1/2}. \tag{4.17}$$

  Substituting this in equation (4.14) the x-displacement of the surface points in the loaded area ($r \leq a$) is obtained as

  $$u_x = \frac{\tau_0 \pi}{32 G a} \left[4(2-\nu)a^2 - (4-3\nu)x^2 - (4-\nu)y^2 \right], \tag{4.18}$$

  with the total force equal to

  $$F_x \frac{2}{3} \pi \tau_0 a^2. \tag{4.19}$$

- A third possibility would be that the force distribution acting in the x- direction upon an elastic body within a strip of width $2a$ is given by

$$\tau(x,y) = \tau_0(1 - x^2/a^2)^{1/2}. \tag{4.20}$$

In this the displacement of the surface points is given by [9]

$$u_x = \text{constant} - \tau_0 \frac{x^2}{aE^*}. \tag{4.21}$$

- The last case that is given in [9] is a special case. Now the tangential loading is presented as *torsion*. This phenomena occurs when working in a round contact area (radius a) and the tangential forces are directed perpendicular to the respective polar radius r. The stresses in this situation are give by

$$\sigma_{zx} = \tau(r)\sin(\phi) \quad \text{and} \quad \sigma_{zy} = \tau(r)\cos(\phi). \tag{4.22}$$

Here the force distribution τ is given as

$$\tau(r) = \tau_0 \frac{r}{a}\left(1 - \left(\frac{r}{a}\right)^2\right)^{-1/2}. \tag{4.23}$$

According to Johnson in his book Contact Mechanics ([4]), the displacement of the surface is given by (in polar components)

$$\begin{aligned} u_\phi &= \frac{\pi\tau_0 r}{4G}, \\ u_r &= 0, \\ u_z &= 0. \end{aligned} \tag{4.24}$$

Looking at these displacement components it is clear that the surface area turns, which happens if the chosen torsion is in fact the torsion of the rigid cylindrical indenter sticking to the surface. In this case the torsional moment is equal to [9]

$$M_z = \frac{4}{3}\pi a^3 \tau_0. \tag{4.25}$$

Now that some information is available regarding the deformations that occur due to different force distributions, the cases of sticking and sliding will be examined.

Complete Sticking - A contact problem without slip In the case of complete sticking, there exists no sliding in the contact. These type of problems are also called *tangential contact problems without slip.*Here the coefficient of friction (COF) between the two bodies is very high (tends to infinity), or the bodies are "glued together".
In most cases, however, the no-slip condition will not hold near the boundary, which means that relative sliding will occur. The fact that the no-slip condition often does not hold is due to the fact that in these cases the shear stress approaches infinity in these areas while the normal stress tends to zero [9].

A contact problem accounting for slip An example of such a problem is the Cattaneo problem described earlier. Many other examples can be considered as well. As said above, in most cases there will be slip in the boundary of the contact area. When sliding occurs in (part of) the contact area, the contact problem accounts for slip. In these problems one could deal with both sliding and sticking, or only sliding.
To get an idea of a contact problem accounting for slip, consider two bodies in contact where normal and tangential forces act simultaneously. As an example [9] two spheres are being pressed together with a normal force F_N while also pulled in the tangential direction with force F_x. The friction between the two bodies is assumed to be according to Coulomb's law of friction; the maximum static friction stress $\tau_{\max}$ is equal to the kinetic friction stress τ_k. Both are equal to the normal stress p multiplied with a constant coefficient of friction (COF) μ.

$$\tau_{\max} = \mu p \quad \text{and} \quad \tau_k = \mu p. \tag{4.26}$$

Now in the area where sticking occurs, the stress τ will have to be smaller than or equal to the normal stress multiplied with this coefficient of friction, i.e.

$$\tau \leq \mu p. \tag{4.27}$$

When one assumes the bodies completely adhere in the contact area, the following equations for the distributions of the normal and tangential stresses are obtained according to Valentin L. Popov [9].

$$\text{Normal stress} \qquad p = p_0 \left(1 - (r/a)^2\right)^{\frac{1}{2}} \quad , \quad F_N = \frac{2}{3} p_0 \pi a^2, \tag{4.28}$$

$$\text{Tangential stress} \qquad \tau = \tau_0 \left(1 - (r/a)^2\right)^{-\frac{1}{2}} , \quad F_x = 2\pi\tau_0 a^2. \tag{4.29}$$

Looking at these distributions it is clear that at the boundary of the area the normal stress p approaches zero, while the tangential stress τ tends to infinity. This means that here the sticking condition (4.27) will always be invalid and hence there will always be slip near the boundary of the contact area. Sticking will occur however inside of the area, when the tangential forces are sufficiently small (see figure 7. The sticking and sliding domains are separated by the boundary circle on which holds that $\tau = \mu p$.

It can be shown that the shear stress distribution given in equation (4.29) is only valid for contact without sliding. However, using this distribution one can prove that there will always be sliding at the boundary, which is a contradiction to the assumption.
A new distribution needs to be constructed, which is correct for a situation with both sliding and sticking. Such a distribution can be constructed as a combination of known distributions. This need for a new and better stress distribution is the same as the need in the Cattaneo problem.

4.2.3 Forming a new stress distribution

In Contact mechanics and Friction [9] it is described that in the case of sliding and sticking (such as in the Cattaneo problem) a distribution can be formed using two "Hertzian" stress distributions, obtaining

$$\begin{aligned} \tau &= \tau^{(1)} + \tau^{(2)} \\ &= \tau_1 (1 - r^2/a^2)^{1/2} - \tau_2 (1 - r^2/c^2)^{1/2}, \end{aligned} \tag{4.30}$$

where a is the contact radius and c is the radius of the sticking domain as is shown in figure 7.

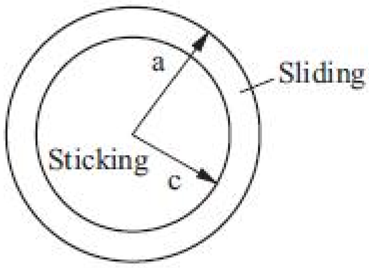

Figure 7: Sticking and sliding domains in a round tangential contact.
Source: [9, Ch. 8].

Since this stress distribution is of the form given in equation (4.17) the displacement will be similar to to one shown in equation (4.18). The following displacement can be obtained.

$$\begin{aligned} u_x &= \frac{\tau_0 \pi}{32Ga}\left[4(2-\nu)a^2 - (4-3\nu)x^2 - (4-\nu)y^2\right] \\ &\quad -\frac{\tau_0 \pi}{32Gc}\left[4(2-\nu)c^2 - (4-3\nu)x^2 - (4-\nu)y^2\right] \end{aligned} \tag{4.31}$$

Now combining this displacement with the fact that sticking occurs within the circle with radius c it is clear that the displacement in this area should be constant:

$$u_x(r) = \text{constant} \quad \text{if } r < c.$$

The fact that sliding occurs in the rest of the domain means that in that area Coulomb's law of friction is met:

$$\tau(r) = \mu p(r), \quad \text{if } c < r < a.$$

Using these conditions, the following stress distribution can be found.

$$\tau(r) = \mu p_0 (1 - r^2/a^2)^{1/2} - \mu p_0 \frac{c}{a}(1 - r^2/c^2)^{1/2} \quad \text{if } 0 \le r \le c \tag{4.32}$$

$$\tau(r) = \mu p(r) \quad \text{if } c \le r \le a \tag{4.33}$$

the displacement for the points in the sticking area and the sliding area can be determined. From these displacements, the tangential force can be given in terms of the normal force, using that $F_N = \frac{2}{3} p_0 \pi a^2$. One obtains

$$F_x = \mu F_N \left(1 - \left(\frac{c}{a}\right)^3\right). \tag{4.34}$$

Rewriting this equation, a radius for the static area is found.

$$\frac{c}{a} = \left(1 - \frac{F_x}{\mu F_N}\right)^{1/3} \tag{4.35}$$

From this relationship it can be seen that complete sliding occurs when

$$F_x = \mu F_N,$$

where μ is the coefficient of friction from Coulomb's Law of Friction.

4.2.4 Coulomb's Law of Friction

Coulomb's Law of Friction is a very simple model to describe the extremely complicated phenomenon of friction in case of dry friction (or Coulomb friction). Despite the simplicity of the law it is shown to be very wide applicable [9].
The law is given by the following inequality

$$F_f \leq \mu F_N, \tag{4.36}$$

where F_f is the frictional force, F_N the normal force and μ the coefficient of friction.

The coefficient of friction is a constant which can depend on a.o.

- the contact time,
- the normal force,
- the sliding speed,
- the surface roughness and
- the temperature.

In the article of Gefen on microclimate factors [3] the effect of the temperature of the room, the temperature of the patient and the production of sweat on the risk for pressure ulcers is being examined. It could therefore be important to include these factors in the contact model. A change in the temperature could for instance change the elastic properties of the materials (i.e. the skin, subcutaneous tissue and the mattress) and when sweat is produced it might be necessary to include the presence of fluid in the model.

4.2.5 Choosing a contact mechanics model

In the previous sections, several contact mechanics models have been described. In this section they will briefly be compared after which a choice will be made as to which model to use in this thesis.

In this thesis the contact is between a human body, which is flexible, and a hospital mattress which also is elastic. The human body is pressed against the mattress, but is simultaneously moved along the mattress. The body is not glued to the mattress, and the coefficient of friction between the patient and the bed does not tend to infinity, which means that sliding will occur during the movement.
These factors lead to the conclusion that the problem is in fact a tangential contact problem accounting for slip. For this reason, the contact models describing the normal contact problem (see section 4.1) can be discarded. The information given in section 4.2, such as the solving process of the Cataneo problem, can be used to solve the problem described in this thesis. However, since other factors are of interest too the model needs to be expended. For instance,

this thesis will also include the effect of moisture on the risk of pressure ulcers. This means that the presence of fluids might have to be included in the contact model. Also, the effect of the temperature on the risk of pressure ulcers is being examined. Therefore it might be important to include the temperature changes in the contact model.

Very important is that the contact is between a human body and a hospital mattress. This means that the exact contact area is not known as opposed to all examples shown above. This in turn means that the integrals used in the models above, in which is integrated over the contact area, can not be calculated as simple as shown. This is similar to the **Signorini problem** which is explained in section 4.3. We will have to determine the contact area using a sort of trial and error system in which the contact area is estimated and improved until it is obtained right.

4.3 The Signorini Problem

The Signorini problem is a problem posed in 1959 regarding the equilibrium configuration of an elastic body resting on a rigid surface. In this contact only the mass forces on the body were taken into account.
The problem is to find the elastic equilibrium configuration of this elastic body subject to only its mass forces. In other words, the problem is to find the deformation of the body, only subject to its body forces. The difficulty in the problem is that the contact area between the elastic body and the sphere is not known prior to solving the problem. Due to this the problem originally was named the problem with ambiguous boundary conditions. These ambiguous boundary conditions consist of both equalities and inequalities and represent the difference between contact and separation. Every point in the body has to satisfy one of the two sets of boundary conditions, i.e. it will either be in the contact area or in the separation.

Antonio Signorini posed the problem asking his students whether the problem is well-posed or not in a physical sense, i.e. if its solution exists and is unique or not. This eventually was solved by one of his students, Gaetano Fichera, who named the problem after his teacher.

Fichera, as opposed to Signorini, did not consider only incompressible bodies and a plane rest surface, which made the problem more general.
The goal of the problem is to [14] "find the displacement vector from the natural configuration $\vec{u}(\vec{x}) = (u_1(\vec{x}), u_2(\vec{x}), u_3(\vec{x}))$ of an anisotropic non-homogeneous elastic body that lies in a subset A of the three dimensional euclidean space, whose boundary δA and whose interior normal is the vector $\vec{n}$, resting on a rigid frictionless surface whose contact surface (or contact set) is Σ and subject only to its body forces $\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), f_3(\vec{x}))$, and surface forces $\vec{g}(\vec{x}) = (g_1(\vec{x}), g_2(\vec{x}), g_3(\vec{x}))$ applied on the free surface $\delta A \backslash \Sigma$: the set A and the contact surface Σ characterize the natural configuration of the body and are known a priori. Therefore the body has to satisfy the general equilibrium equations:

$$\frac{\delta \sigma_{ik}}{\delta x_k} - f_i = 0 \qquad \text{for } i = 1, 2, 3 \tag{4.37}$$

the ordinary boundary conditions on $\delta A \backslash \Sigma$

$$\sigma_{ik} n_k - g_i = 0 \qquad \text{for } i = 1, 2, 3 \tag{4.38}$$

and the following two sets of boundary conditions on Σ, where $\vec{\sigma} = \vec{\sigma}(\vec{u})$ is the Cauchy stress tensor."

As said before, each point has to satisfy on of two sets of ambiguous boundary conditions. These sets are the following.

$$\left\{\begin{array}{lcl} u_i n_i & = & 0 \\ \sigma_{ik} n_i n_k & \geq & 0 \\ \sigma_{ik} n_i \tau_k & = & 0 \end{array}\right. \qquad \text{or} \qquad \left\{\begin{array}{lcl} u_i n_i & > & 0 \\ \sigma_{ik} n_i n_k & = & 0 \\ \sigma_{ik} n_i \tau_k & = & 0 \end{array}\right. , \tag{4.39}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ is a tangent vector to the contact set Σ.

Looking at these sets of boundary conditions it can be seen ([14]) that points which satisfy the first set of conditions are the points which do not leave the contact set Σ in the equilibrium configuration. This area is called the area of support. The points which satisfy the second set of conditions are those which do leave this contact set, and are referred to as the area of separation.

As mentioned before, the problem posed by Signorini only asked whether the problem was well-posed and solvable. Actually solving the problem was not part of this. This has been done later on, and is reported in the literature.

5 Risks on pressure ulcers due to Microclimate factors

In this thesis a model will be created describing a patients risk on pressure ulcers while being moved on a hospital bed. In this model the effect of microclimate factors will also be included. The basics of this part are described in the article ”How do microclimate factors affect the risk for superficial pressure ulcers: A mathematical modeling study.” by Amit Gefen [3].
In this section the assumptions, calculations and results of this article will be described.

In the article the risk of superficial pressure ulcers (SPUs) is being examined. Here superficial pressure ulcers will mean ”skin damage associated with sustained mechanical loading”.
The research described in the article continues on the idea that thermodynamic conditions within and around the skin tissue (i.e. the skin being wet) influences the risk of a patient getting a SPU. The term *microclimate* is used here to describe factors like the local temperature and moisture conditions of the skin. The area of interest will be the parts of the human body that are considered the weight-bearing regions ([8][4]). Previous papers described the effect of surface temperature, humidity, moisture and air movement as risks factors on the patients susceptibility. All these papers however, were based on purely experimental research. The article written by Gefen creates a mathematical model to prove that the microclimate factors are indeed risk factors.

In figure 8 the part of the human body that is considered is shown. This *region of interest* (ROI) is ”a small region of contact between the skin an a support (e.g. mattress or cushion), possibly with a covering sheet, some clothing or stocking in-between the skin and support”.

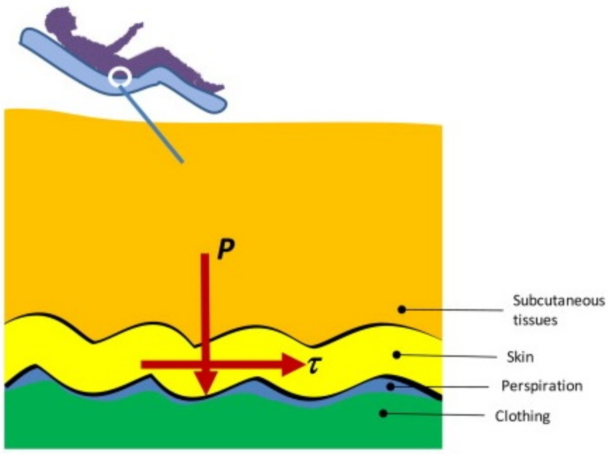

Figure 8: The model will consider a small weight-bearing part of the human body.
Source: [3]

Perspiration

The first step that is taken in the article is to assume the expression of the perspiration accumulated over a certain time period within the available space.

[4]Referred to by Gefen in [3].

The following denotations are used.

Notation	Factor
ΔV	volume of perspiration
t	time
V	available space between the skin and the contact materials at the ROI
$\dot{S}$	rate of production of perspiration by the sweat glands contained on the ROI
$\dot{D}$	rate of drainage of perspiration out of the ROI via the contact materials
$\dot{E}$	rate of evaporation of perspiration.

With the factors above, the accumulated perspiration over time t within V is assumed to be

$$\frac{\Delta V(t)}{V} = \int_0^t (\dot{S} - \dot{E} - \dot{D})dt' \tag{5.1}$$

Now the rate of production of perspiration can be assumed to start with an ambient temperature T_a (temperature within the ROI) of 30 °C. It can also ion is be assumed that the production is proportional to the temperature gradient $T_a - 30\,°\mathrm{C}$. Using this $\dot{S}$ can be formulated as

$$\dot{S} = \alpha \frac{T_a - 30\,°\mathrm{C}}{T_a^{\max} - T_s^{\min}}. \tag{5.2}$$

Here α is a dimensionless proportionality constant, $T_a^{\max}$ is the maximal ambient temperature and is equal to 40 °C and $T_s^{\min}$ is the minimal skin temperature, equal to 30 °C.

In a similar way the evaporation rate is formulated.

$$\dot{E} = \beta \frac{T_a - T_s}{T_a^{\max} - T_s^{\min}}(1 - RH) \tag{5.3}$$

Here β is another dimensionless proportionality constant, T_s is the skin temperature and $RH = 1 - \frac{\Delta V(t)}{V}$ is the relative humidity at the liquid free-space of the ROI. In the article a more detailed definition is given.

> The RH is defined as the ratio between the amount of water vapor at the ROI and the maximum amount ow water vapor that the ROI can hold, and hence, the RH ranges between 0 and 1. - Amit Gefen, [3]

Lastly an expression for the drainage of perspiration $\dot{D}$ is given. This is simply given as a single dimensionless effective permeability coefficient

$$\dot{D} = \gamma. \tag{5.4}$$

This constant weighs together the contributions of permeabilities of all contact materials. If for instance $\gamma = 0$, there is no drainage of perspiration at all.

To establish a model that is simple enough mathematically speaking to solve, the assumption is made that the ambient temperature, skin temperature and relative humidity do not change in time, and thus are independent of t. With these assumptions and using equations (5.2), (5.3) and (5.4)equation (5.1) becomes

$$\frac{\Delta V(t)}{V} = \left[\alpha \frac{T_a - 30\,^{\circ}\mathrm{C}}{T_a^{\max} - T_s^{\min}} + \beta \frac{T_a - T_s}{T_a^{\max} - T_s^{\min}} (1 - RH) + \gamma \right] \cdot t, \tag{5.5}$$

with t such that $0 \leq \Delta V(t)/V \leq 1$.

The coefficient of friction

Another factor in the model described in the article is the *coefficient of friction* (COF) between the skin and a contacting covering sheet or clothing. This coefficient strongly depends on the volume of perspiration accumulated over the skin.
For instance, for the contact between dry skin and common hospital textiles the COF is equal to approximately 0.4. For contact between wet skin and the same textiles the COF will increase to approximately 0.9. Using this, an expression for the COF (denoted as μ) between the skin and the covering sheet or clothing in the ROI is described for the model.

$$\mu = 0.5 \frac{\Delta V(t)}{V} + 0.4 \tag{5.6}$$

This equation shows that the accumulation of perspiration on the skin will consequently increase the shear forces f between the skin and the contact materials over time.
For the shear forces f it holds that $f = \mu N$ where N is the bodyweight force applied perpendicularly to the skin-support or skin-clothing contact area at the weight-bearing region. This bodyweight force N is assumed to be constant over time since the patient is not moving. Despite this fact, μ does increase with time as can be obtained from equation (5.6). As a consequence the shear stress between the skin and the contact materials will increase over time, as the amount of perspiration increases. This shear stress τ is equal to the shear force normalized by the contact area A which gives $\tau = \mu N/A$. Because the pressure P delivered to the skin from the support surface at the skin-support or clothing-support region of contact is given as $P = N/A$ the shear stress can be written in terms of this pressure, that is $\tau = \mu P$. Substituting the expression of the COF (equation (5.6) into this relationship the following equation holds for the ROI.

$$\tau = \left(0.5 \frac{\Delta V(t)}{V} + 0.4 \right) \cdot P \tag{5.7}$$

Here the pressure P depends on the stiffness of the support, and will rise as the stiffness of the support increases ([2][5]). Since τ is linearly proportional to P, the same dependency on the stiffness of the support will hold.

Skin Breakdown

When the shear stress applies on the skin (given by equation (5.7) exceeds the shear strength of the skin, skin break down will occur (figure 9). It was shown before that the shear stress will increase over time as perspiration accumulates. The shear strength of the skin will however decline. A reference is given to ([10]) regarding the fact that the shear strength reduces "by a

[5]Referred to by Gefen in [3].

factor 5 for a completely hydrated skin with respect to dry skin." Using this an expression of the shear strength of the skin τ_w^s is given.

$$\tau_w^s = \left(1 - 0.8\frac{\Delta V(t)}{V}\right)\tau_0^s \tag{5.8}$$

Here τ_0^s is the shear strength of dry skin.

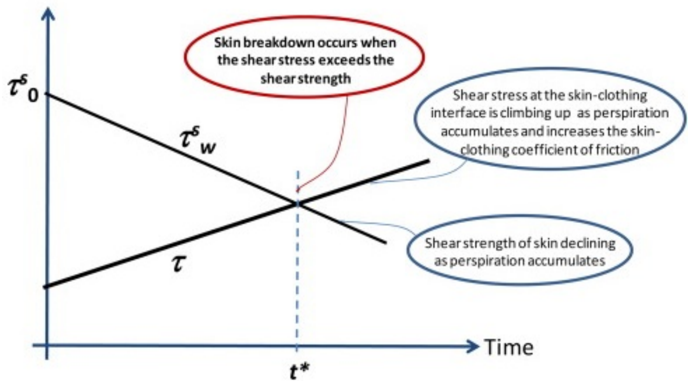

Figure 9: Skin break down will occur when the shear stress applied on the skin exceeds the shear strength of the skin.
Source: [3]

Since the skin breaks down when the shear stress applied on the skin exceeds the shear strength of the skin, the next step is to find the time t^* for which the shear stress is equal to the shear strength of the skin, hence where $\tau = \tau_w^s$. This equality yields

$$t^* = \frac{\tau_0^s - 0.4P}{(0.5P + 0.8\tau_0^s)\left\{\frac{[\alpha-\beta(1-RH)]T_a+\beta(1-RH)T_s-\alpha\cdot 30\,^\circ\mathrm{C}}{T_a^{\max}-T_s^{\min}} - \gamma\right\}} \tag{5.9}$$

With this equation it is possible to examine the effect of the different factors on this critical time t^*.

5.1 Calculations

In the article, the effect of the microclimate factors T_a, RH and T_s as well as interacting factors P and permeability γ on the critical for skin breakdown is examined. In order to study the effects of these factors on the critical time, several plots were made in which the factors took different values. In every plot, the critical time (t^*) is plotted against the skin temperature (T_s) and one of the other factors is being varied. In Table 2 the values for all parameters are given. Note that whenever one of the factors is being varied, the others are equal to the values given in this table ([3]).

The Matlab code used to repeat the calculations can be found in Appendix (A).

Table 2: These parameter values were used in the plots shown in figure 10 .

Parameters	τ_0^s	P	α	β	γ	T_a	RH
Value	70 kPa	7 kPa	2	1	0.1	35 °C	0.5

5.2 Results

In the article the following plots are given as the results.

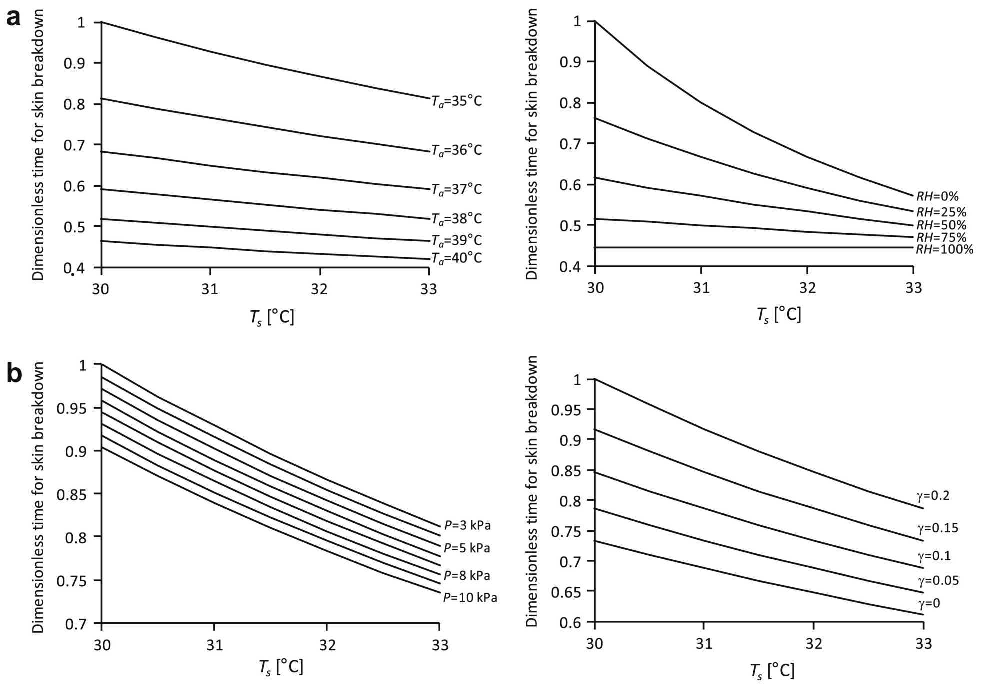

Figure 10: The calculated dimensionless critical times for skin breakdown versus the skin temperature (T_s) for different values of (a) the microclimate parameters of ambient temperature (T_a) (left panel) and relative humidity (RH) (right panel), and (b) the interacting parameters of pressure delivered from the support (P) (left panel) and permeability to perspiration (γ) of the materials contacting the skin or being in close proximity to the skin (right panel). The following values were assigned to the model variables in these simulations:$\tau_0^s = 70$ kPa, $P = 7$ kPa*, $\alpha = 2$, $\beta = 1$, and $\gamma = 0.13$, $T_a = 35\,°\text{C}^*$ and $RH = 0.53$. * denotes; where not altered as detailed in the specific panel.

As can be seen in the figures, all the factors that have been examined do have effect on the critical time.

6 The human body, contact mechanics and the finite element method

In the article of Gefen [11] they created a model of a human body resting on a hospital mattress. In working with the body they used the finite element method.

In this section the article written by Eliav Shaked and Amit Gefen will be reproduced. In this article the look at the interaction between the skin and subcutaneous tissue of a patient and the hospital mattress. The model of the skin in interaction of the hospital mattress is shown in Figure 11. So far I have not been able to repeat the calculations that are done in the article, hence this will be done in the official thesis report.

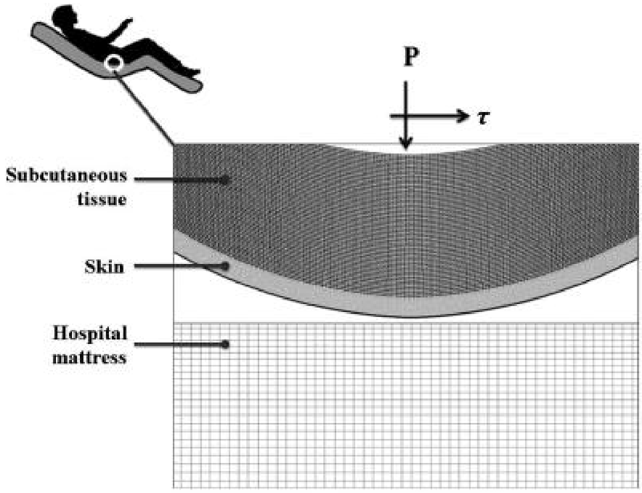

Figure 11: The model of the skin in interaction of the hospital mattress
Source: [11]

The relevant values for the model parameters are listed in Table 3.

Table 3: **Values of the physical and mechanical properties, and the numerical characteristics used in this study**
Source: [11]

Parameter	**Skin**	**Subcutaneous tissue**	**Hospital mattress**
Density (kg/m^3)	1100	971	30
Poisson's ratio (-)	0.49	0.48	0.3
Elastic modulus (kPa)	15.2/50/100	2	10
Thickness (mm)	2	15	50
Length (mm)	60	60	400
Number of elements (-)	8515	24300	20000

In the article several modeling assumptions and boundary conditions are named.

- A finite element model is used to analyze the skin stresses under a weight-bearing bony prominence while this region of interest slides frictionally over the support surface, as occurs during repositioning.
- The computations were carried out in a plane stress analysis.
- The skin and subcutaneous tissues were modeled as linearly elastic isotropic nearly incompressible materials.

- A pressure boundary condition was applied on the top edge of the model, in order to simulate the load over the ROI (region of interest), generated by the relative body-weight imposed to the bony prominence and downwards to the outer tissue layers.
- Pressure under the bony prominence was estimated elsewhere, and was set for all simulations at the level of 130kPa which corresponds to a male with a normal body mass index.
- The hospital mattress was constrained of any movement (translations and rotations) on the sides and the bottom. Constraining the mattress on the sides was needed in order to simulate the resistance to deformation from the lateral mattress parts outside the ROI (that is, which were not modeled).
- Displacement was applied to the top edge of the model in a standard lateral turning, assuming repositioning regime of 10cm horizontal sliding along and 1 cm toward (i.e., immersion into) the mattress.
- The aforementioned 130kPa pressure represented the static weight-bearing of the patient, and the 1-cm displacement toward the mattress represented the additional loading applied by a caregiver to reposition the patient.

So far I have not been able to fully repeat the calculations which are done in this article. This will be done in the final thesis report.

A Effect of microclimate factors on the patients risk of pressure ulcers - Matlab code

```
% This Matlab code repeats the calculation which are done in the article by
% Gefen.
clear all;

taus = 70;
P = 7;
alpha = 2;
beta = 1;
gamma = 0.1;
RH = 0.5;
Ts = 30:0.5:33;

% Start with the subplot(2,2,1), which plots different values of the
% ambient temperature Ta
x = zeros(1,6);
matrix = zeros(6,7);
for Ta = 35:40;
    counter = taus-0.4*P;
    nD1 = (0.5*P+0.8*taus);
    nD2 = alpha * ((Ta-30)/10);
    nD3 = beta * ((Ta-Ts)/10)*(1-RH);
    nD4 = gamma;
    tX = counter ./ (nD1.*(nD2- nD3 - nD4));
    matrix(Ta-34,:) = tX;
    x(Ta-34) = tX(1);
 end;
 matrix = matrix./max(x);
 subplot(2,2,1);
 plot(Ts,matrix);
 axis([30,33,0.4,1]);
 xlabel('T_s[^oC]');
 ylabel('Dimensionless time for skin breakdown');
 hold on;

 % Continue with the second subplot, which takes Ta=35 and plots for
 % different values of RH.
 x = zeros(1,5);
 matrix = zeros(5,7);
 Ta = 35;
clear RH;
for RH = 0:0.25:1;
    counter = taus-0.4*P;
    nD1 = (0.5*P+0.8*taus);
    nD2 = alpha * ((Ta-30)/10);
```

```
    nD3 = beta * ((Ta-Ts)/10)*(1-RH);
    nD4 = gamma;
    tX = counter ./ (nD1.*(nD2- nD3 - nD4));
    matrix(RH*4+1,:) = tX;
    x(RH*4+1) = tX(1);
end;
matrix = matrix./max(x);
subplot(2,2,2);
plot(Ts,matrix);
axis([30,33,0.4,1]);
xlabel('T_s[^oC]');
ylabel('Dimensionless time for skin breakdown');
hold on;

 % In the third subplot, RH is taken to be 0.5, and different values of P
 % are plotted.
x = zeros(1,8);
matrix = zeros(8,7);
RH =0.5;
clear P;
for P = 3:10;
    counter = taus-0.4.*P;
    nD1 = (0.5.*P+0.8*taus);
    nD2 = alpha * ((Ta-30)/10);
    nD3 = beta * ((Ta-Ts)/10)*(1-RH);
    nD4 = gamma;
    tX = counter ./ (nD1.*(nD2- nD3 - nD4));
    matrix(P-2,:) = tX;
    x(P-2) = tX(1);
end;
matrix = matrix./max(x);
subplot(2,2,3);
plot(Ts,matrix);
axis([30,33,0.7,1]);
xlabel('T_s[^oC]');
ylabel('Dimensionless time for skin breakdown');
hold on;

% In the final subbplot P=7 and gamma is being changed.
x = zeros(1,5);
matrix = zeros(5,7);
P=7;
clear gamma;
for gamma = 0:0.05:0.2;
    counter = taus-0.4.*P;
    nD1 = (0.5.*P+0.8*taus);
    nD2 = alpha * ((Ta-30)/10);
    nD3 = beta * ((Ta-Ts)/10)*(1-RH);
```

```
    nD4 = gamma;
    tX = counter ./ (nD1.*(nD2- nD3 - nD4));
    matrix(gamma*20+1,:) = tX;
    x(gamma*20+1) = tX(1);
end;
matrix = matrix./max(x);
subplot(2,2,4);
plot(Ts,matrix);
axis([30,33,0.6,1]);
xlabel('T_s[^oC]');
ylabel('Dimensionless time for skin breakdown');
hold on;
```

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