Msc thesis proposal:

Fast valuation of barrier options under Lévy process

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**Background**

Barrier options are path-dependent exotics that are similar in some ways to ordinary options. You can call or put in American, Bermudan, or European exercise style. But they become activated (or extinguished) only if the underlying breaches a predetermined level (the barrier).

"In" options only become active in the event that a predetermined knock-in barrier price is breached:

* If the barrier price is far from being breached, the knock-in option will be worth slightly more than zero.
* If the barrier price is close to being breached, the knock-in option will be worth slightly less than the corresponding vanilla option.
* If the barrier price has been breached, the knock-in option will trade at the exact same value as the corresponding vanilla option.

"Out" options start their lives active and become null and void in the event that a certain knock-out barrier price is breached:

* If the barrier price is far from being breached, the knock-out option will be slightly less than the corresponding vanilla option.
* If the barrier price is close to being breached, the knock-out option will be worth slightly more than zero.
* If the barrier price has been breached, the knock-out option will trade at the exact value of zero.

Some variants of "Out" options compensate the owner for the knock-out by paying a cash fraction of the premium at the time of the breach.

The four main types of barrier options are:

* Up-and-out: spot price starts below the barrier level and has to move up for the option to be knocked out.
* Down-and-out: spot price starts above the barrier level and has to move down for the option to become null and void.
* Up-and-in: spot price starts below the barrier level and has to move up for the option to become activated.
* Down-and-in: spot price starts above the barrier level and has to move down for the option to become activated.

In this thesis, we focus on down-and-out barrier options. However, the generalization of the same approach to other barrier types is trivial.

Various numerical methods can be applied to price barrier options, such as Monte Carlo simulation, PIDE-based methods that directly solve the corresponding PIDE, Fourier methods including the COS method and other FFT-based methods (on the basis of e.g. Fast Gauss or the Hilbert Transform). A brief literature review on this can be found in [2]. The COS method developed in [2] is among the fastest numerical solutions to pricing barrier options.

Lately, there have been efforts devoted in using machine learning methods to directly approximate the option pricing function, e.g. [4] for American options.

**Challenge**

The application of the COS method for the valuation of barrier and Bermudan options in [2] is still not fast enough for daily calibration purpose, especially for continuously monitored Barrier options.

The main idea in [2] is that, under Lévy processes, the option between two monitored dates is of European type. Hence, the option price can be solved backwards in time based on a recursive formula.

The computational complexity of the method presented in [2] is with *M* being the number of monitoring dates and *N* being the number of leading cosine terms in the COS method. This dependency on is the bottle neck of computational performance of this method. Also, as the number of monitoring dates increases, it gets difficult to determine the truncation range for the COS method.

To fulfill the needs of daily calibration in practice, the calculation speed of the daily-monitored or continuously monitored barrier options, especially those defined on FX rates, has to be done much faster e.g. within a fraction of a second.

**The goal and content of this thesis**

The goal of this Msc thesis is to develop a much more efficient method for pricing daily-monitored or continuously monitored barrier options under Lévy process assumption. Below we provide two candidate solutions. Let’s focus on the down-and-out barrier options.

**Approach 1 (preferred)**

Consider the situation when the forward price process does not fall below a given nonnegative lower boundary (we refer to it as the barrier hereafter). Let be the first time that the process hits the lower boundary . Under the -forward martingale measure, pricing the derivative can be achieved by calculating an expectation of its payoff (see, e.g., formula (2.20), Brigo and Mercurio, 2006). Therefore, under the assumption that the forward price does not hit the barrier before time , a down‐and‐out option price with payoff function can be represented as follows:

If we further assume that the marginal survival density of the forward price exists, then the down‐and‐out option price is

which is a one‐dimensional integral of the payoff function and the survival density. Denote

as the marginal survival transition density from time‐ state to the marginal state at time conditional on that the realized path does not cross the barrier from above. Once the barrier is reached before time , the process stays at 0 thereafter. Thus, the probability density function at the barrier is zero at any time t before the maturity time .

If we have the formula for or its Fourier transform, i.e. characteristic function, then we have the closed‐form formulas for the prices of barrier options. Therefore, we concentrate on deriving an analytical approximation to the marginal survival density .

This is the starting point of the derivation in [5]. They first formulate the survival transition density as the solution to a backward Kolmogorov partial differential equation (PDE) with boundary and terminal (initial) conditions. After that, they obtain an approximate solution of the survival density function as well as error estimation. However, their focus is in the so-called SABR model, which is a stochastic volatility model.

We could follow the same line of derivations as in [5] to derive the survival density function for Lévy processes.

**Approach 2 (might be difficult)**

Alternatively, we can use a discretely monitored barrier option price to approximate the continuously monitored barrier option price, i.e.

The set of monitoring time points is with In other words,

which is a one‐dimensional integral of the payoff function on the survival probability.

Therefore, we are interested to get the distribution of conditional on the survival at all monitoring dates.

Applying Bayes law, we have

Now the problem boils down to deriving the survival distribution function with and without the knowledge of the underlying price at maturity. Let us first focus on the unconditional survival probability function, i.e.,

Under the assumption that follows a Markov process, we have

Note that the highlighted part can be solved analytically by inserting the COS representation of the transition density function in the integrand. And, if following the derivations as in [2] and by making use of the properties of the cos() and sin() functions, we can explore 1) whether the nested integrals above can be eventually solved (semi-)analytically, or 2) whether we can derive the limit of the above nested integral as we increase the number of monitoring dates to infinity.

Similarly, for the conditional survival probability, the target is to find a (semi-)analytical solution.

**Reference**

1. A novel option pricing method based on Fourier-cosine series expansions. F. Fang and C. W. Oosterlee. SIAM J. Sci. Comput.,31(2):826-848, 2008
2. Pricing early-exercise and discrete barrier options by Fourier-cosine series expansions. F. Fang and C. W. Oosterlee. Numer. Math. DOI 10.1007/s00211-009-0252-4, 2009.
3. Efficient Pricing of European-Style Asian Options under Exponential Lévy Processes Based on Fourier Cosine Expansions, B. Zhang and C. W. Oosterlee, SIAM J. FINANCIAL MATH., Vol. 4, pp. 399–426, 2013.
4. On Calibration Neural Networks for extracting implied information from American options, Shuaiqiang Liu, Álvaro Leitao, Anastasia Borovykh, Cornelis W. Oosterlee, 31 Jan 2020.
5. Pricing Continuously Monitored Barrier Options under the SABR Model: A Closed‐Form Approximation, Nian Yang, Yanchu Liu and Zhenyu Cui, JMSE 2017, 2(2), 116–131, 2017.

**Contact**

If you are interested to enter the field of quantitative risk analysis, this is a very good starting point. Please feel free to contact me directly if this topic is of your interest, or if you would like to learn more details: fang.fang@ffquant.nl or f.fang@tudelft.nl

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