Limiting discontinuous Galerkin solutions using multiwavelets

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Literature review

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Outline

1 Motivation
2 Discontinuous Galerkin
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5 Summary and further research
Climate modelling: simulation of the mean temperature change
Motivation

A computer simulation of high velocity air flow around the Space Shuttle during re-entry.
Discontinuous Galerkin: discretization in space

Linear advection equation on \([-1, 1]\):

\[
 u_t + u_x = 0, \quad x \in [-1, 1], \quad t \geq 0,
\]

\[
 u(x, 0) = u^0(x), \quad x \in [-1, 1],
\]

\[ u = u(x, t), \text{ periodic boundary conditions.} \]

Exact solution: \[ u(x, t) = u^0(x - t). \]

Discretize in space: \[ x_j = -1 + (j + \frac{1}{2})\Delta x, \quad j = 0, \ldots, N. \]

(B. Cockburn, Springer, 1998)
Discontinuous Galerkin: approximations

Approximate $u(x, t)$ by a piecewise polynomial of degree $k$ (polynomial on each $I_j$), using

- Taylor expansion: linear combination of $1, x, x^2, \ldots, x^k$;
- Now: scaled Legendre polynomials $\phi_0, \phi_1, \phi_2, \ldots, \phi_k$.

Example of scaled Legendre polynomials $\phi_0, \ldots, \phi_4$
Discontinuous Galerkin: approximation space

\[ u_h(x, t) = \sum_{\ell=0}^{k} u_j^{(\ell)}(t) \phi_\ell(\xi), \text{ on element } I_j, \]

\[ \xi = \frac{2}{\Delta x} (x - x_j). \]

\[ \int_{I_j} (u_t + u_x) v dx = 0, \]

use \( u_h(x, t) \) and \( v_h(x) = \phi_m(\xi), m \in \{0, \ldots, k\} \).
Discontinuous Galerkin: weak form

\[ \int_{I_j} (u_{h,t} + u_{h,x}) v_h \, dx = 0. \]

- Integration by parts;
- Coordinate transformation to \( \xi = \frac{2}{\Delta x} (x - x_j) \);
- Orthonormal property of scaled Legendre polynomials;

\[
\frac{\Delta x}{2} \frac{d u_j^{(m)}(t)}{dt} = \sum_{\ell=0}^{k} u_j^{(\ell)}(t) \int_{-1}^{1} \phi_{\ell}(\xi) \frac{d}{d\xi} \phi_{m}(\xi) \, d\xi + \]

\[
- \hat{u}_h(x_j + \frac{1}{2}) v_h(x_j + \frac{1}{2}) + \hat{u}_h(x_j - \frac{1}{2}) v_h(x_j - \frac{1}{2}).
\]
Discontinuous Galerkin: fluxes

Required: \( \hat{u}_h(x_{j+\frac{1}{2}})v_h(x_{j+\frac{1}{2}}) \) and \( \hat{u}_h(x_{j-\frac{1}{2}})v_h(x_{j-\frac{1}{2}}) \)

\[
\begin{array}{ccc}
\downarrow & \uparrow \\
x_{j-\frac{1}{2}} & I_j & x_{j+\frac{1}{2}} \\
\end{array}
\]

Boundaries of element \( I_j \)

- \( u_h \): exact solution \( u(x, t) = u^0(x - t) \): initial condition advects to the right \( \rightarrow u_h(x_{j-\frac{1}{2}}^-) \) and \( u_h(x_{j+\frac{1}{2}}^-) \);

- \( v_h \): inside element \( \rightarrow v_h(x_{j-\frac{1}{2}}^+) \) and \( v_h(x_{j+\frac{1}{2}}^+) \).
Discontinuous Galerkin: differential equation

\[ \frac{\Delta x}{2} \frac{du_j^{(m)}}{dt} = \sum_{\ell=0}^{k} u_j^{(\ell)} \int_{-1}^{1} \phi_\ell(\xi) \frac{d}{d\xi} \phi_m(\xi) d\xi + \]

\[ - \left( \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_\ell(1) \right) \phi_m(1) + \left( \sum_{\ell=0}^{k} u_{j-1}^{(\ell)} \phi_\ell(1) \right) \phi_m(-1). \]

\[
\hat{u}_h(x_{j+\frac{1}{2}}) v_h(x_{j+\frac{1}{2}})\]

\[
\hat{u}_h(x_{j-\frac{1}{2}}) v_h(x_{j-\frac{1}{2}})\]

Let \( u_j = [u_j^{(0)}(t) \ u_j^{(1)}(t) \ \ldots \ u_j^{(k)}(t)]^\top \), then

\[ M \frac{d}{dt} u_j = S_1 u_j + S_2 u_{j-1}, j = 0, \ldots, N. \]
Discontinuous Galerkin: time stepping

- Time stepping: total variation diminishing RK, $O(\Delta t)^3$;
- Approximation space with polynomials $\phi_0, \ldots, \phi_k$: $O(\Delta x)^{k+1}$ if function is smooth.

<table>
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<th>$N + 1$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
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<td></td>
<td>$|u - u_h|_\infty$</td>
<td>order</td>
</tr>
<tr>
<td>10</td>
<td>0.1659</td>
<td>-</td>
</tr>
<tr>
<td>20</td>
<td>0.0435</td>
<td>1.9329</td>
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<td>40</td>
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<td>80</td>
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<td>1.8997</td>
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<tr>
<td>160</td>
<td>8.9903e-04</td>
<td>1.9532</td>
</tr>
</tbody>
</table>

Norms of errors and orders, $T = 0.5$, $u^0(x) = \sin(2\pi x)$, using $\phi_0, \ldots, \phi_k$
Limiters needed

Approximations, discontinuous initial condition, $\phi_0, \ldots, \phi_k$, $T = 0.5$
Limiters: overview

Currently used limiters for DG are, for example:
- Minmod limiters (Shu);
- Projection limiters (Cockburn and Shu);
- Moment limiters (Krivodonova);
- WENO limiters (Qiu and Shu);
- Multiwavelet limiters (Iacono, Hovhannisyan).

Drawbacks: reduce to low order around discontinuities, multidimensional case.

Research project focuses on new type of multiwavelet limiter.
Multiwavelets

Theory of multiwavelets uses several levels to approximate a function \( f \in L^2(-1, 1) \).

\[ \vdots \quad V_n \quad \text{Level } n \]

\[ I_0^2 \quad I_1^2 \quad I_2^2 \quad I_3^2 \quad V_2 \quad \vdots \]

\[ I_0^1 \quad I_1^1 \quad V_1 \quad \text{Level } 1 \]

\[ -1 \quad I_0^0 \quad 1 \quad V_0 \quad \text{Level } 0 \]

Subdividing intervals in their midpoint

Level \( n \) contains \( 2^n \) intervals.

(Alpert, SIAM, 1993)
Multiwavelets: scaling functions

Scaling functions on level $n$: $\phi^n_{\ell j}, \ell = 0, \ldots, 4, j = 0, \ldots, 2^n - 1$, $\phi_{\ell}$ is scaled Legendre polynomial of degree $\ell$.

Scaling functions $\phi_{\ell}$, level 0 ($V_0$)  
Scaling functions $\phi^1_{\ell j}$, level 1 ($V_1$)  
Dilation and translation of $\phi_{\ell}$
Multiwavelets

\[ V_0 \oplus W_0 = V_1. \]

Scaling functions \( \psi_0 \), \( \psi_1 \), \( \psi_2 \), \( \psi_3 \), \( \psi_4 \)

Limiting DG solutions using multiwavelets
Multiwavelets: approximation of functions

\[ V_n = V_0 \oplus W_0 \oplus \ldots \oplus W_{n-1}. \]

Orthogonal projection of \( f \in L^2(-1,1) \) on \( V_n \), using a combination of scaling functions and multiwavelets:

\[ P_n^k f(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(\xi) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^m \psi_{\ell j}^m(\xi). \]

- Projection on \( V_0 \)
- Projection on \( W_m \)
Multiwavelets: decomposition

\[ P^k_n f(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(\xi) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell 0}^j \psi_{\ell 0}^j(\xi) \]

- **Level 0**
  - \( s_{\ell 0}^0 \)
  - \( d_{\ell 0}^0 \)

- **Level 1**
  - \( s_{\ell j}^1 \)
  - \( d_{\ell j}^1 \)

- **Level 2**
  - \( s_{\ell j}^2 \)
  - \( d_{\ell j}^2 \)

- **Level \( n-1 \)**
  - \( s_{\ell j}^{n-1} \)
  - \( d_{\ell j}^{n-1} \)

- **Level \( n \)**
  - \( s_{\ell j}^n \)
  - \( d_{\ell j}^n \)

**Approximation on level \( n \), decomposed into coefficients \( s_{\ell 0}^0, d_{\ell 0}^0, \ldots, d_{\ell j}^{n-1} \)**

\[ P^k_0 f(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(\xi) \rightarrow \text{standard DG approximation} \]
Multiwavelets: \( f(x) = \sin(2\pi x), k = n = 3 \)

\[
P^k_n f(x) = \sum_{\ell=0}^{k} s^0_{\ell0} \phi_\ell(\xi) + \sum_{m=0}^{n-1} 2^{m-1} \sum_{j=0}^{k} d^m_j \psi^m_j(\xi)
\]
Multiwavelets: \( f(x) = \sin(2\pi x), \quad k = n = 3 \)

\[
P^k_n f(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_{\ell} (\xi) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^m \psi_{\ell j}^m (\xi)
\]

\[
m = 0 : \quad \sum_{\ell=0}^{k} d_{\ell j 0}^0 \psi_{\ell j 0} (\xi), \mathcal{O}(10^{-5})
\]
Multiwavelets: \( f(x) = \sin(2\pi x), k = n = 3 \)

\[
P_n^k f(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(\xi) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^m \psi_{\ell j}^m(\xi)
\]

\( m = 1 : \sum_{j=0}^{1} \sum_{\ell=0}^{k} d_{\ell j}^1 \psi_{\ell j}^1(\xi) \quad \mathcal{O}(10^{-6}) \)

\( m = 2 : \sum_{j=0}^{3} \sum_{\ell=0}^{k} d_{\ell j}^2 \psi_{\ell j}^2(\xi) \quad \mathcal{O}(10^{-7}) \)
Discontinuous function, $k = n = 3$

$$P_n^k f(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_{\ell} (\xi) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m - 1} \sum_{\ell=0}^{k} d_{\ell j}^m \psi_{\ell j}^m (\xi)$$

$$\sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_{\ell} (\xi)$$
Discontinuous function, \( k = n = 3 \)

\[
P_n^k f(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(\xi) + \sum_{m=0}^{n-1} 2^m - 1 \sum_{j=0}^{k} \sum_{\ell=0}^{d_{\ell j}^m} \psi_{\ell j}^m(\xi)
\]

\( m = 0 : \quad \sum_{\ell=0}^{k} d_{\ell j}^0 \psi_{\ell j}^0(\xi), \mathcal{O}(10^{-2}) \)
Discontinuous function, \( k = n = 3 \)

\[
P_n^k f(x) = \sum_{\ell=0}^{k} s^0_{\ell 0} \phi_{\ell}(\xi) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d^m_{\ell j} \psi^m_{\ell j}(\xi)
\]

\[
m = 1: \sum_{j=0}^{1} \sum_{\ell=0}^{k} d^1_{\ell j} \psi^1_{\ell j}(\xi) \quad \mathcal{O}(10^{-3})
\]

\[
m = 2: \sum_{j=0}^{3} \sum_{\ell=0}^{k} d^2_{\ell j} \psi^2_{\ell j}(\xi) \quad \mathcal{O}(10^{-3})
\]
Summary and further research

Combine DG with multiwavelet limiter:
- Detect and manage discontinuities;
- Relation between degree of multiwavelet basis, $k$, and number of levels, $n$;
- High-order and multidimensional case.