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LOAD FLOW LITERATURE SURVEY

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Load Flow Literature Survey

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1 Introduction

This document contains an overview of the literature survey, as conducted for a PhD study on the subject on load flow problems for power systems. The PhD study is done in the Numerical Analysis group of the Delft Institute of Mathematics, at the University of Technology, Delft.

In Section 2, the concept of a power system is introduced, as well as power system basics that are needed for the treatment of the load flow problem. This includes voltage, current, power, impedance and admittance, as well as Kirchoff's circuit laws.

Next, in Section 3, we deal with the mathematical model for a power system. Generators, loads, transmission lines and other devices are treated, as well as the admittance matrix that relates the line currents to the node voltages in the power system.

In Section 4 we define the load flow problem, and introduce the Newton-Raphson method to solve the problem. Further, the Fast Decoupled Load Flow method is derived and analysed.

Sections 5 is dedicated to the research questions that we plan to attack initially, and finally, in Section 6, we present some conclusions drawn from the treated material.

2 Power Systems

A system of hardware that provides for the generation of power, and the transmission to the consumers, is called a Power System. This section contains an introduction to voltage, current, and power, for the steady state of a power system operating on alternating current (AC).

2.1 Voltage and Current

The voltage and current, in a power system in steady state, can be assumed to be sinusoidal functions of time with constant frequency ω. It is conventional to use the cosine function to describe these quantities, i.e.,

$$
\begin{aligned}
v\,(t) &= V_{\max}\cos\,(\omega t + \delta_V) &&= \Re\left(V_{\max}e^{j\delta_V}e^{j\omega t}\right), \\
i\,(t) &= I_{\max}\cos\,(\omega t + \delta_I) &&= \Re\left(I_{\max}e^{j\delta_I}e^{j\omega t}\right),
\end{aligned}
$$

where j is the imaginary unit, and $\Re$ is the operator that takes the real part.

The complex numbers $V = V_{\max}e^{j\delta_V}$ and $I = I_{\max}e^{j\delta_I}$ are called the phasor representation of the voltage and current respectively, and are used to represent the voltage and current in circuit

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theory. In power system theory, instead the effective phasor representation is used:

$$V = |V|e^{j\delta_V}, \text{ with } |V| = \frac{V_{\max}}{\sqrt{2}}, \tag{1}$$

$$I = |I|e^{j\delta_I}, \quad \text{with } |I| = \frac{I_{\max}}{\sqrt{2}}. \tag{2}$$

Note that $|V|$ and $|I|$ are the RMS values of $v(t)$ and $i(t)$, and that the effective phasors differ from the circuit theory phasors by a factor $\sqrt{2}$.

As we are dealing with power system theory, we use V and I to denote the effective voltage and current phasors, as defined above. Further, as usual in the treatment of power systems, we use per unit quantities, and represent the balanced three-phase network of a power system as its equivalent single-phase system. For more details see for example [5] or [1].

2.2 Complex Power

Assume that t is chosen such that the voltage is $v(t) = V_{\max}\cos(\omega t)$, and the current is $i(t) = I_{\max}\cos(\omega t - \phi)$. The value $\phi = \delta_V - \delta_I$ is called the power factor angle, and $\cos\phi$ the power factor. Then the instantaneous power $p(t)$ is given by

$$
\begin{aligned}
p(t) &= v(t)\,i(t) \\
&= \sqrt{2}|V|\cos(\omega t)\sqrt{2}|I|\cos(\omega t - \phi) \\
&= 2|V||I|\cos(\omega t)\cos(\omega t - \phi) \\
&= 2|V||I|\cos(\omega t)\cos(\omega t - \phi) \\
&= 2|V||I|\cos(\omega t)\left[\cos\phi\cos(\omega t) + \sin\phi\sin(\omega t)\right] \\
&= |V||I|\left[2\cos\phi\cos^2(\omega t) + 2\sin\phi\sin(\omega t)\cos(\omega t)\right] \\
&= |V||I|\cos\phi\left[2\cos^2(\omega t)\right] + |V||I|\sin\phi\left[2\sin(\omega t)\cos(\omega t)\right] \\
&= |V||I|\cos\phi\left[1 + \cos(2\omega t)\right] + |V||I|\sin\phi\left[\sin(2\omega t)\right] \\
&= P\left[1 + \cos(2\omega t)\right] + Q\left[\sin(2\omega t)\right],
\end{aligned}
$$

where $P = |V||I|\cos\phi$, and $Q = |V||I|\sin\phi$.

Thus the instantaneous power is the sum of a unidirectional component, that is sinusoidal with average value P, and amplitude P, and a component of alternating direction, that is sinusoidal with average value 0, and amplitude Q. Note that integrating the instantaneous power over one time period $T = \frac{2\pi}{\omega}$ gives

$$\int_0^T p(t)\,dt = P.$$

The magnitude P is called the active power, or also real power or average power, and is measured in W (watts). The magnitude Q is called the reactive power, or also imaginary power, and is measured in VAr (voltamperes reactive).

Using the complex representation of voltage and current, we can write

$$P = |V||I|\cos\phi = \Re\left(|V||I|e^{j(\delta_V - \delta_I)}\right) = \Re\left(V\overline{I}\right),$$

$$Q = |V||I|\sin\phi = \Im\left(|V||I|e^{j(\delta_V - \delta_I)}\right) = \Im\left(V\overline{I}\right),$$

where $\overline{I}$ is the complex conjugate of I. Thus we can extend Joule's law to AC circuits, as

$$S = V\overline{I}. \tag{3}$$

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Note that strictly speaking VA and VAr are the same unit as W, however it is very useful to use descriptive units, to distinguish between the measured quantities.

2.3 Impedance and Admittance

Impedance is the extension of the notion resistance, to AC circuits, and is thus a measure of opposition to a sinusoidal current. The impedance is denoted by $Z = R + jX$, and measured in ohms (Ω). We call $R \geq 0$ the resistance, and X the reactance. If $X > 0$ the reactance is called inductive and we can write $jX = j\omega L$, where $L > 0$ is called the inductance. If $X < 0$ the reactance is called capacitive and we write $jX = \frac{1}{j\omega C}$, where $C > 0$ is called the capacitance.

The admittance $Y = G + jB$ is the inverse of impedance, i.e., $Y = \frac{1}{Z} = \frac{R}{|Z|^2} - j\frac{X}{|Z|^2}$, and is measured in siemens (S). We call $G = \frac{R}{|Z|^2} \geq 0$ the conductance, and $B = -j\frac{X}{|Z|^2}$ the susceptance.

The voltage drop over an impedance Z is equal to $V = ZI$. This is the extension of Ohm's law to AC circuits. Alternatively, using the admittance, we can write

$$I = YV. \tag{4}$$

The power consumed by the impedance is $S = V\bar{I} = ZI\bar{I} = |I|^2 Z = |I|^2 R + j|I|^2 X$.

2.4 Kirchhoff's circuit laws

To calculate the voltage and current in an electrical circuit, we use Kirchoff's circuit laws.

Kirchhoff's current law (KCL)
At any point in the circuit that does not represent a capacitor plate, the sum of currents flowing towards that point is equal to the sum of currents flowing away from that point, i.e., $\sum_k I_k = 0$.

Kirchhoff's voltage law (KVL)
The directed sum of the electrical potential differences around any closed circuit is zero, i.e., $\sum_k V_k = 0$.

3 Power System Model

Power systems are modeled as a network of buses (nodes) and lines (edges). At each bus i four electrical magnitudes are of importance:

$|V_i|$, the voltage amplitude,

δ_i, the voltage phase angle,

P_i, the injected active power,

Q_i, the injected reactive power.

Each bus can consist of a number of electrical devices. The bus is named according to the electical magnitudes specified by its devices,

generator bus or PV-bus: P_i and $|V_i|$ specified, Q_i and δ_i unknown,

load bus or PQ-bus: P_i and Q_i specified, $|V_i|$ and δ_i unknown.

3.1 Generators, Loads, and Transmission Lines

Generally, a physical generator has P and $|V|$ controls and thus specifies these magnitudes. Likewise, a load will have specified negative injected active power P, and specified reactive power Q. However, the name of the bus does not necessarily indicate what type of devices it consists of. A wind turbine, for example, is a generator but does not have PV controls. A wind turbine is instead modeled as a load, with positive injected active power P. If a PV generator and a PQ load are connected to the same bus, the result is a PV-bus with a voltage amplitude equal to that of the generator, and an active power equal to the sum of the active power of the generator and the load. Also there may be buses without a generator or load connected, such as transmission substations, which are modeled as load with $P = Q = 0$.

In any practical power system there are system losses. These losses have to be taken into account, but since they depend on the power flow they are not known in advance. Therefore one generator bus has to be assigned to supply these unknown losses. This bus is generally called the slack bus. Obviously it is no longer possible to specify the real power P for this bus. Instead the voltage magnitude $|V|$ and angle δ are specified. Note that δ does not have any practical meaning, it is merely the reference phase to which the other phase angles are related. As such, for the slack bus it is generally specified that $\delta = 0$.

Lines are the network representation of the transmission lines, that connect the buses in the power system. From a modeling viewpoint, lines define how to relate buses through Kirchhoff's circuit laws. Lines can incur losses on the transported power and must be modeled as such.

A transmission line from bus i to bus j has some impedance. The total impedance over the line is modeled as a single impedance z_{ij} of the line. The admittance of that line is $y_{ij} = \frac{1}{z_{ij}}$. Further there is a shunt admittance from the line to the neutral ground. The total shunt admittance y_s of the line is modeled to be evenly split between bus i and bus j, see Figure 1.
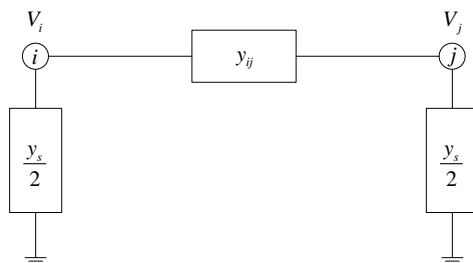

Figure 1: Transmission line model

It is usually assumed that there is no conductance from the line to the ground. This means that the shunt admittance is due only to the electrical field between line and ground, and is thus a capacitive susceptance, i.e., $y_s = jb_s$, with $b_s > 0$. For this reason, the shunt admittance y_s is also sometimes referred to as the shunt susceptance b_s. See also the notes about modeling a shunt in Section 3.2. For details on how to calculate y_s for a given line, we refer to [5].

3.2 Shunts, Tap Transformers, and Phase Shifters

Three other devices, that are also commonly found in power systems, are shunts, tap transformers and phase shifters. Shunt capacitors can be used to inject reactive power, resulting in a higher node voltage, whereas shunt inductors consume reactive power, thus lowering the node voltage.

A shunt is modeled as a reactance $z_s = jx_s$ between the bus and the ground, see Figure 2. The shunt admittance is thus $y_s = \frac{1}{z_s} = -j\frac{1}{x_s} = jb_s$. If $x_s > 0$ the shunt is inductive, and if $x_s < 0$

the shunt is capacitive. Note that the shunt susceptance b_s has the opposite sign of the shunt reactance x_s.
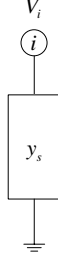

Figure 2: Shunt model

A tap transformer is a transformer that can be set to different turns ratios. The taps are the connection points along the transformer winding on one side, that allow a certain number of turns to be selected. Tap transformers are generally used to control the voltage magnitude, dealing with fluctuating industrial and domestic demands, or with the effects of switching out a circuit for maintenance. Note that transformers without taps do not need to be modeled specifically, as the per unit system reduces their numerical effect to that of a series impedance.

Phase shifters are devices that can change the voltage phase angle, while keeping the voltage magnitude constant. As such they can be used to control the active power.

Both a tap transformer, and a phase shifter, can be modeled as in Figure 3. The transformer ratio is given by $T : 1$. For a tap transformer T is a positive real number, typically between 0.8 and 1.2. In the case of a phase shifter, T is a rotation in the complex plane, i.e., $T = e^{j\delta_T}$, where δ_T is the phase shift.
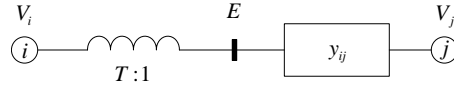

Figure 3: Transformer model

3.3   Admittance Matrix

The admittance matrix Y, is a matrix that relates the injected current I at each bus to bus voltages V, such that

$$I = YV, \tag{5}$$

where I is the vector of injected currents at each bus, and V is the vector of bus voltages. This is in fact Ohm's law (4) in matrix form. As such we can also define the impedance matrix $Z = Y^{-1}$.

To calculate the admittance matrix Y, we look at the injected current I_i at each bus i. If $I_i > 0$ power is generated, if $I_i < 0$ power is consumed, and if $I_i = 0$ there is no current injected, for example at a transmission substation. Let I_{ij} denote the current flowing from bus i, in the direction of bus j, or to the ground in case of a shunt. Applying Kirchoff's current law now gives

$$I_i = \sum_k I_{ik}. \tag{6}$$

Let y_{ij} denote the admittance of the line between bus i and j, with $y_{ij} = 0$ if there is no line between these buses. For a simple transmission line from bus i to bus j, without shunt admittance,

Ohm's law states that

$$I_{ij} = y_{ij} (V_i - V_j), \text{ and } I_{ji} = -I_{ij}, \tag{7}$$

or in matrix notation:

$$\begin{bmatrix} I_{ij} \\ I_{ji} \end{bmatrix} = y_{ij} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} V_i \\ V_j \end{bmatrix}. \tag{8}$$

Note that, if a power system would consist of such simplified lines only, the admittance matrix for that system is a Laplacian matrix (see Appendices A) given by

$$Y_{ij} = \begin{cases} \sum_{k \neq i} y_{ik} & \text{if } i = j, \\ -y_{ij} & \text{if } i \neq j. \end{cases} \tag{9}$$

Indeed, then we have

$$I_i = \sum_k I_{ik} = \sum_k y_{ik} (V_i - V_k) = \sum_{k \neq i} y_{ik} V_i - \sum_{k \neq i} y_{ik} V_k = \sum_k Y_{ik} V_k = (YV)_i. \tag{10}$$

Now suppose there is a shunt s connected to bus i. Then, according to equation (6), an extra term I_{is} is added to the injected power I_i. From Figure 2, it is clear that

$$I_{is} = y_s (V_i - 0) = y_s V_i. \tag{11}$$

This means that in the admittance matrix, an extra term y_s has to be added to Y_{ii}. Recall that $y_s = jb_s$, and that the sign of b_s depends on the shunt being inductive or capacitive.

Knowing how to deal with shunts, it is now easy to incorporate the line shunt admittance model, as depicted in Figure 1. In the admittance matrix, for a transmission line between the buses i and j, the halved line shunt admittance $\frac{y_s}{2}$ of that line, has to be added to both Y_{ii} and Y_{jj}. For a transmission line with shunt admittance y_s, we thus find

$$\begin{bmatrix} I_{ij} \\ I_{ji} \end{bmatrix} = \left(y_{ij} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + y_s \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} V_i \\ V_j \end{bmatrix}. \tag{12}$$

The influence on the admittance matrix, of a device t between the buses i and j, that is either a tap transformer or a phase shifter, can be derived from the model as depicted in Figure 3. Let E be the voltage induced by t, then

$$V_i = TE. \tag{13}$$

Thus the current from bus j to t, and thus in the direction of bus i, is

$$I_{ji} = y_{ij} (V_j - E) = y_{ij} \left(V_j - \frac{V_i}{T} \right). \tag{14}$$

Conservation of power gives

$$V_i \overline{I}_{ij} = -E \overline{I}_{ji} \Rightarrow T \overline{I}_{ij} = -\overline{I}_{ji} \Rightarrow \overline{T} I_{ij} = -I_{ji}. \tag{15}$$

and thus for the current from bus i to t, and thus in the direction of j, we find

$$I_{ij} = -\frac{I_{ji}}{\overline{T}} = y_{ij} \left(\frac{V_i}{|T|^2} - \frac{V_j}{\overline{T}} \right). \tag{16}$$

If the device t that connects bus i to bus j is a tap transformer, then $\overline{T} = T$, and $|T|^2 = T^2$, and thus instead of the admittance matrix values from equation (12), we find

$$\begin{bmatrix} I_{ij} \\ I_{ji} \end{bmatrix} = y_{ij} \begin{bmatrix} \frac{1}{T^2} & -\frac{1}{T} \\ -\frac{1}{T} & 1 \end{bmatrix} \begin{bmatrix} V_i \\ V_j \end{bmatrix}. \tag{17}$$

If instead t is a phase shifter, then $\overline{T} = e^{-j\delta_T} = \frac{1}{T}$, and $|T|^2 = 1$, and we find

$$\begin{bmatrix} I_{ij} \\ I_{ji} \end{bmatrix} = y_{ij} \begin{bmatrix} 1 & -T \\ -\overline{T} & 1 \end{bmatrix} \begin{bmatrix} V_i \\ V_j \end{bmatrix}. \tag{18}$$

4   Load Flow

The load flow problem, or power flow problem, is the problem of computing the flow of electrical power in a power system in steady state. In practice, this amounts to calculating all node voltages and line currents in the power system. The load flow is regarded the most important network computation. The problem arises in many applications in power system analysis, and is treated in many books on power systems, see for example [5], [1] and [4].

The mathematical equations describing the load flow problem can be obtained by combining Ohm's law from equation (5), with Joule's law:

$$S_i = V_i \overline{I}_i = V_i \left(\overline{YV} \right)_i = V_i \sum_{k=1}^{N} \overline{Y}_{ik} \overline{V}_k. \tag{19}$$

Note that the admittance matrix Y is easy to obtain, and generally very sparse. Therefore a formulation using the admittance matrix has preference over using the impedance matrix, which is generally a lot harder to obtain and not sparse.

Further note that for each bus i that has no injected power, i.e., $S_i = 0$, we also have that injected current $I_i = 0$. Therefore, we can use the linear equation $(YV)_i = 0$, to eliminate the variable V_i from the problem. This method is called Kron reduction, see [1] Section 9.3. The elimination can be done by simple Gaussian elimination, but in the case of many such buses, using a Schur complement may provide better results.

Recall that voltage and current are complex numbers, and substitute $V_i = |V_i| e^{j\delta_i}$, $Y = G + jB$, and $\delta_{ij} = \delta_i - \delta_j$ into equation (19). This gives

$$S_i = |V_i| e^{j\delta_i} \sum_{k=1}^{N} (G_{ik} - jB_{ik}) |V_k| e^{-j\delta_k} = \sum_{k=1}^{N} |V_i| |V_k| \left(\cos \delta_{ik} + j \sin \delta_{ik} \right) (G_{ik} - jB_{ik}) \tag{20}$$

Define the real vector of the voltage variables of the load flow problem as

$$\mathbf{V} = [\ \delta_1, \ldots, \delta_N, |V_1|, \ldots, |V_N|\ ]^T. \tag{21}$$

For the purpose of notational comfort, we further define

$$P_{ij}(\mathbf{V}) = |V_i| |V_j| \left(G_{ij} \cos \delta_{ij} + B_{ij} \sin \delta_{ij} \right), \tag{22}$$
$$Q_{ij}(\mathbf{V}) = |V_i| |V_j| \left(G_{ij} \sin \delta_{ij} - B_{ij} \cos \delta_{ij} \right). \tag{23}$$

Then we can write equation (20) as

$$S_i = \sum_{k=1}^{N} P_{ik}(\mathbf{V}) + j \sum_{k=1}^{N} Q_{ik}(\mathbf{V}). \tag{24}$$

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Splitting the real and imaginary parts of the equation, and defining $P_i\left(\mathbf{V}\right) = \sum_{k=1}^{N} P_{ik}\left(\mathbf{V}\right)$ and $Q_i\left(\mathbf{V}\right) = \sum_{k=1}^{N} Q_{ik}\left(\mathbf{V}\right)$, we have

$$P_i = P_i\left(\mathbf{V}\right), \tag{25}$$
$$Q_i = Q_i\left(\mathbf{V}\right). \tag{26}$$

Equations (25)–(26) relate the complex power in each node to the node voltages, using the admittance matrix of the power system network. It consists of $2N$ non-linear real equations. Each node has four variables, $|V_i|$, δ_i, P_i and Q_i, two of which have a specified value, as discussed in section 3. Thus we have $2N$ non-linear real equations in $2N$ real unknowns, commonly known as the load flow equations, or power flow equations.

4.1   Newton-Raphson

It is generally not possible to solve a system of non-linear equations analytically. However, there are many iterative techniques to approximate the solution. The problem of solving a system of equations is trivially equivalent to finding the simultaneous roots of a set of functions in the same variables. As such, we can deploy root finding algorithms as a tool to solve a system of equations. Of these algorithms, Newton's method, also known as the Newton-Raphson method, is widely accepted to be the best for many applications. Newton's method is the base algorithm of choice when solving load flow equations.

First, define a real vector of the voltage variables of the load flow problem:

$$\mathbf{V} = \left[\ \delta_1, \ldots, \delta_N, |V_1|, \ldots, |V_N|\ \right]^T. \tag{27}$$

Then write the load flow equations as a set of functions $\mathcal{F}_i$, whos simultaneous roots are the solution of the load flow equations:

$$\mathcal{F}_i\left(\mathbf{V}\right) = \left[\begin{array}{c} \Delta P_i\left(\mathbf{V}\right) \\ \Delta Q_i\left(\mathbf{V}\right) \end{array}\right] = \left\{\begin{array}{ll} P_i - P_i\left(\mathbf{V}\right), & i = 1 \ldots N, \\ Q_i - Q_i\left(\mathbf{V}\right), & i = N + 1 \ldots 2N, \end{array}\right. \tag{28}$$

where $S_i = P_i + jQ_i$ are the desired values of the power, whereas $S_i\left(\mathbf{V}\right) = P_i\left(\mathbf{V}\right) + jQ_i\left(\mathbf{V}\right)$ are the values dictated by the bus voltages and network admittance matrix, i.e., $S_i\left(\mathbf{V}\right)$ is the right-hand side in equation (20). The function $\mathcal{F}$ is called the power mismatch.

Next, we start with some guess or approximation $\mathbf{V}^0$, and update it iteratively by the rule

$$\mathbf{V}^{k+1} = \Phi\left(\mathbf{V}^k\right), \tag{29}$$

where the function Φ is such that

$$\Phi\left(\mathbf{V}\right) = \mathbf{V} \Leftrightarrow \mathcal{F}\left(\mathbf{V}\right) = \mathbf{0}, \tag{30}$$

with $\mathcal{F}$ the vector of functions $\mathcal{F}_i$. Condition (30) is always satisfied, if we choose

$$\Phi\left(\mathbf{V}\right) = \mathbf{V} - A\left(\mathbf{V}\right)^{-1} \mathcal{F}\left(\mathbf{V}\right), \tag{31}$$

with $A\left(\mathbf{V}\right)$ a non-singular matrix. Different choices of $A\left(\mathbf{V}\right)$ correspond to different methods. For example, $A = I$ leads to the Gauss-Seidel method.

Newton's method is based on the first order Taylor expansion of $\mathcal{F}$, and as such uses $A\left(\mathbf{V}\right) = J\left(\mathbf{V}\right)$, with J the Jacobian of $\mathcal{F}$. Thus, Newton's method is characterized by the update rule

$$\mathbf{V}^{k+1} = \mathbf{V}^k + \mathbf{\Delta V}^k, \tag{32}$$

where $\mathbf{\Delta V}^k$ is the solution of the linear system of equations

$$-J\left(\mathbf{V}^k\right)\mathbf{\Delta V}^k = \mathcal{F}\left(\mathbf{V}^k\right). \tag{33}$$

Convergence of Newton's method is not guaranteed, however, when the starting approximation is sufficiently near the solution, the convergence is quadratic. Note that the Jacobian generally differs in each iteration. We will refer to this method as full Newton, as opposed to inexact Newton methods which make some kind of approximation, often in the Jacobian. Inexact Newton methods generally have linear convergence.

4.2 Generator Buses

When solving a linear system of the form $A\mathbf{x} = \mathbf{b}$, we assume the coefficient matrix A, and right-hand side vector $\mathbf{b}$ to be known, and the variable vector $\mathbf{x}$ to be unknown. In the case of the linear system from equation (33), each bus i of the power system makes for two equations, begin row i and row $N + i$. If bus i is a load bus, then indeed all coefficients, and the right-hand side value, of row i and row $N + i$ are known, whereas δ_i and $|V_i|$ are unknown. However, if bus i is a generator bus, then we have a different situation.

First let us consider the slack bus, of which every power system has exactly one, and assume that it is located at bus 1. Then, δ_1 and $|V_1|$ are known, whereas P_1 and Q_1 are not. As a result, we can eliminate the variables δ_1 and $|V_1|$ from the system, by substituting their values into the coefficient matrix, and bringing the resulting values to the right-hand side. Also, we can take row 1, which corresponds to the equation for P_1, and row $N + 1$, which corresponds to the equation for Q_1, out of the system. Note that the resulting coefficient matrix, is a principal minor of the original matrix.

Once the Newton-Raphson process has converged, P_1 and Q_1 are easily calculated by substituting the found solution into the original equations 1 and $N + 1$, that were taken out.

Now consider a generator bus i that are not the slack bus. Then P_i and $|V_i|$ are known, whereas Q_i and $\delta)i$ are unknown. With $|V_i|$ known, we can eliminate this variable, and column $N + i$ from the linear system. And since Q_i is unknown, we also take row $N + i$ out of the system, like we did in the case of a slack bus. Note that again, the resulting coefficient matrix is a principal minor of the original matrix.

4.3 Full Newton Jacobian

Applying Newton's method to the load flow problem, in each iteration we have to solve the linear system of equations (33), and update the approximate solution according to equation (32). All terms in these equations are straightforward to calculate, except for the Jacobian.

In this section we will derive the explicit form of the Jacobian of the load flow equations. The general structure of the Jacobian is

$$J\left(\mathbf{V}\right) = -\begin{bmatrix} \frac{\partial P_1}{\partial \delta_1} & \cdots & \frac{\partial P_1}{\partial \delta_{N_1}} & \frac{\partial P_1}{\partial |V_1|} & \cdots & \frac{\partial P_1}{\partial |V_{N_2}|} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial P_{N_1}}{\partial \delta_1} & \cdots & \frac{\partial P_{N_1}}{\partial \delta_{N_1}} & \frac{\partial P_{N_1}}{\partial |V_1|} & \cdots & \frac{\partial P_{N_1}}{\partial |V_{N_2}|} \\ \frac{\partial Q_1}{\partial \delta_1} & \cdots & \frac{\partial Q_1}{\partial \delta_{N_1}} & \frac{\partial Q_1}{\partial |V_1|} & \cdots & \frac{\partial Q_1}{\partial |V_{N_2}|} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial Q_{N_2}}{\partial \delta_1} & \cdots & \frac{\partial Q_{N_2}}{\partial \delta_{N_1}} & \frac{\partial Q_{N_2}}{\partial |V_1|} & \cdots & \frac{\partial Q_{N_2}}{\partial |V_{N_2}|} \end{bmatrix}, \tag{34}$$

where the dependance of P_i and Q_i of $\mathbf{V}$ has been left out for readability. The dimensions N_1 and N_2 are due to the elimination of the slack bus, and other generator buses.

Below the first order derivates, of which the Jacobian exists, are derived. Note that we assume that $i \neq j$ wherever applicable.

$$\frac{\partial P_i(\mathbf{V})}{\partial \delta_j} = |V_i||V_j|\left(G_{ij}\sin\delta_{ij} - B_{ij}\cos\delta_{ij}\right) = Q_{ij}(\mathbf{V}) \tag{35}$$

$$\frac{\partial P_i(\mathbf{V})}{\partial \delta_i} = \sum_{k \neq i} |V_i||V_k|\left(-G_{ik}\sin\delta_{ik} + B_{ik}\cos\delta_{ik}\right) = -Q_i(\mathbf{V}) - |V_i|^2 B_{ii} \tag{36}$$

$$\frac{\partial Q_i(\mathbf{V})}{\partial \delta_j} = |V_i||V_j|\left(-G_{ij}\cos\delta_{ij} - B_{ij}\sin\delta_{ij}\right) = -P_{ij}(\mathbf{V}) \tag{37}$$

$$\frac{\partial Q_i(\mathbf{V})}{\partial \delta_i} = \sum_{k \neq i} |V_i||V_k|\left(G_{ik}\cos\delta_{ik} + B_{ik}\sin\delta_{ik}\right) = P_i(\mathbf{V}) - |V_i|^2 G_{ii} \tag{38}$$

$$\frac{\partial P_i(\mathbf{V})}{\partial |V_j|} = |V_i|\left(G_{ij}\cos\delta_{ij} + B_{ij}\sin\delta_{ij}\right) = \frac{P_{ij}(\mathbf{V})}{|V_j|} \tag{39}$$

$$\frac{\partial P_i(\mathbf{V})}{\partial |V_i|} = 2|V_i|G_{ii} + \sum_{k \neq i} |V_k|\left(G_{ik}\cos\delta_{ik} + B_{ik}\sin\delta_{ik}\right) = \frac{P_i(\mathbf{V})}{|V_i|} + |V_i|G_{ii} \tag{40}$$

$$\frac{\partial Q_i(\mathbf{V})}{\partial |V_j|} = |V_i|\left(G_{ij}\sin\delta_{ij} - B_{ij}\cos\delta_{ij}\right) = \frac{Q_{ij}(\mathbf{V})}{|V_j|} \tag{41}$$

$$\frac{\partial Q_i(\mathbf{V})}{\partial |V_i|} = -2|V_i|B_{ii} + \sum_{k \neq i} |V_k|\left(G_{ik}\sin\delta_{ik} - B_{ik}\cos\delta_{ik}\right) = \frac{Q_i(\mathbf{V})}{|V_i|} - |V_i|B_{ii} \tag{42}$$

Summarizing, again leaving out the dependance on $\mathbf{V}$:

$\dfrac{\partial P_i}{\partial \delta_j} = Q_{ij}$	$\vert V_j\vert \dfrac{\partial P_i}{\partial \vert V_j\vert} = P_{ij}$
$\dfrac{\partial P_i}{\partial \delta_i} = -Q_i - \vert V_i\vert^2 B_{ii}$	$\vert V_i\vert \dfrac{\partial P_i}{\partial \vert V_i\vert} = P_i + \vert V_i\vert^2 G_{ii}$
$\dfrac{\partial Q_i}{\partial \delta_j} = -P_{ij}$	$\vert V_j\vert \dfrac{\partial Q_i}{\partial \vert V_j\vert} = Q_{ij}$
$\dfrac{\partial Q_i}{\partial \delta_i} = P_i - \vert V_i\vert^2 G_{ii}$	$\vert V_i\vert \dfrac{\partial Q_i}{\partial \vert V_i\vert} = Q_i - \vert V_i\vert^2 B_{ii}$

4.4   Fast Decoupled Jacobian

With respect to convergence, full Newton load flow is as good as it gets. However there is a major drawback to this method. Because the Jacobian depends on the current approximation of the solution, it has to be recalculated every iteration. Moreover, because the Jacobian is not constant, we cannot use the information from the previous iterations to solve the Jacobian system fast.

If we can approximate the Jacobian such that it is the same in each iteration, we can use the factorization, the preconditioner, or whatever information we calculate in the first iteration, in each subsequent iteration to solve the Jacobian system quickly.

Fast decoupled load flow (FDLF) provides such an approximation, based on the practical properties of load flow problems. We will follow the original derivation of the method from [6], and discuss the important additions and insights given in [7] and [3].

In FDLF the assumption is made that, for all i, j,

$$\delta_{ij} \approx 0, \tag{43}$$
$$|V_i| \approx 1. \tag{44}$$

For the original derivation in [6], it is further assumed that

$$G_{ij} \ll B_{ij}. \tag{45}$$

Using assumptions (43) we can approximate

$$P_{ij}\left(\mathbf{V}\right) = |V_i||V_j|\left(G_{ij}\cos\delta_{ij} + B_{ij}\sin\delta_{ij}\right) \approx +|V_i||V_j|G_{ij}, \tag{46}$$
$$Q_{ij}\left(\mathbf{V}\right) = |V_i||V_j|\left(G_{ij}\sin\delta_{ij} - B_{ij}\cos\delta_{ij}\right) \approx -|V_i||V_j|B_{ij}. \tag{47}$$

Note that for $i = j$ these approximations are exact, since $\delta_{ii} = 0$.

Now, using assumption (45) it follows that

$$P_{ij}\left(\mathbf{V}\right) \ll Q_{ij}\left(\mathbf{V}\right). \tag{48}$$

This leads to the idea of decoupling, i.e., neglecting the off-diagonal blocks of the Jacobian matrix, as they are small compared to the diagonal blocks.

Thus, by the above assumptions, we can approximate the Jacobian as follows. Note that in the first two equations we use assumption (44), though only on $|V_j|$.

$$\frac{\partial P_i\left(\mathbf{V}\right)}{\partial \delta_j} = Q_{ij}\left(\mathbf{V}\right) \approx -|V_i||V_j|B_{ij} \approx -|V_i|B_{ij} \tag{49}$$

$$\frac{\partial P_i\left(\mathbf{V}\right)}{\partial \delta_i} = -\sum_{k\neq i} Q_{ik}\left(\mathbf{V}\right) \approx \sum_{k\neq i} |V_i||V_k|B_{ik} \approx |V_i|\sum_{k\neq i} B_{ik} \tag{50}$$

$$\frac{\partial Q_i\left(\mathbf{V}\right)}{\partial \delta_j} = -P_{ij}\left(\mathbf{V}\right) \approx 0 \tag{51}$$

$$\frac{\partial Q_i\left(\mathbf{V}\right)}{\partial \delta_i} = \sum_{k\neq i} P_{ik}\left(\mathbf{V}\right) \approx 0 \tag{52}$$

$$\frac{\partial P_i\left(\mathbf{V}\right)}{\partial |V_j|} = \frac{P_{ij}\left(\mathbf{V}\right)}{|V_j|} \approx 0 \tag{53}$$

$$\frac{\partial P_i\left(\mathbf{V}\right)}{\partial |V_i|} = 2|V_i|G_{ii} + \sum_{k\neq i} \frac{P_{ik}\left(\mathbf{V}\right)}{|V_i|} \approx 0 \tag{54}$$

$$\frac{\partial Q_i\left(\mathbf{V}\right)}{\partial |V_j|} = \frac{Q_{ij}\left(\mathbf{V}\right)}{|V_j|} \approx -|V_i|B_{ij} \tag{55}$$

$$\frac{\partial Q_i\left(\mathbf{V}\right)}{\partial |V_i|} = -2|V_i|B_{ii} + \sum_{k\neq i} \frac{Q_{ik}\left(\mathbf{V}\right)}{|V_i|} \approx -2|V_i|B_{ii} - \sum_{k\neq i} |V_k|B_{ik} \tag{56}$$

The last equation (56) still requires some work. To this purpose, we define the negated rowsum of the imaginary part B of the admittance matrix $Y = G + jB$:

$$\mathcal{D}_i = \sum_k -B_{ik} = (-B_{ii}) - \sum_{k\neq i} B_{ik}. \tag{57}$$

Note that, assuming that the diagonal elements of B are negative, and the off-diagonal elements are non-negative, $\mathcal{D}_i$ is also equal to the diagonal dominance of row i. In a system with only

generators, loads and transmission lines, $\mathcal{D}_i$ is directly related to the shunt admittance of the lines.

Now we can use assumption (44) to approximate $|V_k|$ with $|V_i|$ for all k, and find that

$$\frac{\partial Q_i\left(\mathbf{V}\right)}{\partial |V_i|} \approx -2|V_i|B_{ii} - \sum_{k \neq i} |V_k|B_{ik} \approx -2|V_i|B_{ii} - |V_i|\sum_{k \neq i} B_{ik}$$

$$= |V_i|\sum_{k \neq i} B_{ik} - 2|V_i|\left(B_{ii} + \sum_{k \neq i} B_{ik}\right) = |V_i|\sum_{k \neq i} B_{ik} + 2|V_i|\mathcal{D}_i. \tag{58}$$

The only term left in the approximated Jacobian matrix, that depends on the current approximation of the solution, is $|V_i|$. Because of assumption (44) we can just set this term to 1. However, it is also common practice to divide row i of the approximated Jacobian matrix, and element i of the right-hand side vector of the Jacobian system, by $|V_i|$ for each i.

In either case, the coefficient matrix of the system to solve is the same in each iteration, and can be characterized by

$$
\begin{array}{|c|c|}
\hline
\dfrac{\partial P_i}{\partial \delta_j} = -B_{ij} & |V_j|\dfrac{\partial P_i}{\partial |V_j|} = 0 \\[2ex]
\dfrac{\partial P_i}{\partial \delta_i} = \displaystyle\sum_{k \neq i} B_{ik} & |V_i|\dfrac{\partial P_i}{\partial |V_i|} = 0 \\[2ex]
\hline
\dfrac{\partial Q_i}{\partial \delta_j} = 0 & |V_j|\dfrac{\partial Q_i}{\partial |V_j|} = -B_{ij} \\[2ex]
\dfrac{\partial Q_i}{\partial \delta_i} = 0 & |V_i|\dfrac{\partial Q_i}{\partial |V_i|} = \displaystyle\sum_{k \neq i} B_{ik} + 2\mathcal{D}_i \\[2ex]
\hline
\end{array}
$$

As introduced in [6], we will refer to the upper diagonal block as B', and to the lower diagonal block as B''. Note that, in a system with only generators, loads and transmission lines, B' is equal to $-B$ without line shunts incorporated, while B'' is equal to $-B$ with double line shunt values.

Summarizing, FDLF calculates the update, for the approximate solution in iteration k, by solving the following linear systems

$$B'\Delta\delta^k = \Delta P^k, \tag{59}$$

$$B''\Delta|V|^k = \Delta Q^k, \tag{60}$$

with $\Delta P_i^k = P_i\left(\mathbf{V}^k\right) - P_i$ and $\Delta Q_i^k = Q_i\left(\mathbf{V}^k\right) - Q_i$, or $\Delta P_i^k = \frac{P_i\left(\mathbf{V}^k\right) - P_i}{|V_i|}$ and $\Delta Q_i^k = \frac{Q_i\left(\mathbf{V}^k\right) - Q_i}{|V_i|}$.

4.4.1 Shunts, Tap Transformers, and Phase Shifters

A few added notes can be made with respect to shunts, tap transformers and phase shifters. Shunts have the same influence on the system as transmission line shunts, i.e., they only change the diagonal entries of the admittance matrix. Thus they are left out in B', and doubled in B''. Since tap transformers predominantly influence the voltage magnitude $|V|$, they are generally left out in B'. Likewise, since phase shifters change the voltage angle δ, they are left out in B''.

4.4.2 BB, XB, BX, and XX

The FDLF method derived above is commonly refered to as the BB version, because for both B' and B'' we use the susceptance values

$$B = \Im\left(\frac{1}{R + jX}\right) = \frac{-X}{R^2 + X^2}. \tag{61}$$

Stott and Alsac [6] already reported slightly improved convergence, if the series resistance R was neglected in B', that is, if for B' the values

$$B_X = \Im\left(\frac{1}{jX}\right) = \frac{-1}{X} \tag{62}$$

are used instead of the full susceptance. Thus B' is derived from the reactance values X, and B'' from the susceptance values B. Therefore, this method is often referred to as the XB version.

Van Amerongen [7] reported that the BX scheme yields convergence that is comparable to XB in most cases, and considerably better in some. This is the version where B' is based on the full susceptance values B, whereas B'' is made from B_X. Further, he noted that an XX scheme is never better than BX or XB.

Monticelli, et al. [3] presented mathematical support for the good results obtained with the XB and BX schemes. Their idea is the following. Starting with assumptions (43) and (44), the full Newton system is approximated by

$$\begin{bmatrix} -B & G \\ -G & -B \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta|V| \end{bmatrix} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}. \tag{63}$$

Using block Gaussian elimination, we get

$$\begin{bmatrix} -B & G \\ 0 & -\left(B + GB^{-1}G\right) \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta|V| \end{bmatrix} = \begin{bmatrix} \Delta P \\ \Delta Q - GB^{-1}\Delta P \end{bmatrix}. \tag{64}$$

First calculate the partial voltage angle update

$$-B\Delta\delta_B^k = \Delta P\left(\delta^k, |V|^k\right) \Rightarrow \Delta\delta_B^k = -B^{-1}\Delta P\left(\delta^k, |V|^k\right), \tag{65}$$

and note that using the first order Taylor expansion we can write

$$\Delta Q\left(\delta^k + \Delta\delta_B^k, |V|^k\right) \approx \Delta Q + \frac{\partial\Delta Q}{\partial\delta}\Delta\delta_B^k = \Delta Q - GB^{-1}\Delta P, \tag{66}$$

where $\Delta P = \Delta P\left(\delta^k, |V|^k\right)$ and $\Delta Q = \Delta Q\left(\delta^k, |V|^k\right)$. Thus we can approximate the lower block of linear equations in (64) by the system

$$-\left(B + GB^{-1}G\right)\Delta|V|^k = \Delta Q\left(\delta^k + \Delta\delta_B^k, |V|^k\right). \tag{67}$$

Next calculate a second partial voltage angle update

$$B\Delta\delta_G^k = G\Delta|V|^k \Rightarrow \Delta\delta_G^k = B^{-1}G\Delta|V|^k. \tag{68}$$

Then the solution of the upper block of linear equations in (64) is given by

$$\Delta\delta^k = -B^{-1}\Delta P\left(\delta^k, |V|^k\right) + B^{-1}G\Delta|V|^k = \Delta\delta_B^k + \Delta\delta_G^k. \tag{69}$$

The next step would be to go to the next iteration, and cacluate

$$-B\Delta\delta_B^{k+1} = \Delta P\left(\delta^{k+1}, |V|^{k+1}\right) \Rightarrow \Delta\delta_B^{k+1} = -B^{-1}\Delta P\left(\delta^{k+1}, |V|^{k+1}\right). \tag{70}$$

However, note that

$$\Delta\delta_B^{k+1} + \Delta\delta_G^k = -B^{-1}\Delta P\left(\delta^{k+1}, |V|^{k+1}\right) + B^{-1}G\Delta|V|^k \tag{71}$$

$$= -B^{-1}\left(\Delta P\left(\delta^k + \Delta\delta_B^k + \Delta\delta_G^k, |V|^{k+1}\right) - G\Delta|V|^k\right) \tag{72}$$

$$\approx -B^{-1}\left(\Delta P\left(\delta^k + \Delta\delta_B^k, |V|^{k+1}\right) + B\Delta\delta_G^k - G\Delta|V|^k\right) \tag{73}$$

$$= -B^{-1}\Delta P\left(\delta^k + \Delta\delta_B^k, |V|^{k+1}\right), \tag{74}$$

where we used a first order Taylor expansion to go from equation (72) to (73). Thus instead of calculating δ_G^k, updating the angle update $\Delta\delta^k$ with it, and then calculating δ_B^{k+1} from equation (70), we can instead calculate $\Delta\delta_B^{k+1} + \Delta\delta_G^k$ in one go from equation (74).

The above observations lead to the following iterative scheme:

$$\begin{array}{ll} \text{Solve} & -B\Delta\delta_B = \Delta P\left(\delta, |V|\right) \\ \text{Update} & \delta = \delta + \Delta\delta_B \\ \text{Solve} & -\left(B + GB^{-1}G\right)\Delta|V| = \Delta Q\left(\delta, |V|\right) \\ \text{Update} & |V| = |V| + \Delta|V| \end{array}$$

Now write B and G as $A^T dBA$ and $A^T dGA$, where A is the node-incidence matrix (see Appendix A) of the associated network, and the matrices dB and dG are the diagonal matrices of edge susceptances and conductances respectively. Then, if A is non-singular, i.e., the network is radial, or if $dB = bI$ and $dG = gI$, i.e., the R/X ratio is equal on all lines, we can write

$$-\left(B + GB^{-1}G\right) = -A^T dBA - A^T dGA\left(A^T dBA\right)^{-1}A^T dGA$$

$$= -A^T dBA - A^T\left(\frac{dG^2}{dB}\right)A$$

$$= -A^T\left(dB + \frac{dG^2}{dB}\right)A$$

$$= -A^T\left(\frac{dB^2 + dG^2}{dB}\right)A = -A^T\left(dX^{-1}\right)A \tag{75}$$

For general networks, if the R/X ratios do not vary a lot, we can thus use the matrix constructed from the inverse reactances X^{-1}, as an approximation for the Schur complement $\left(B + GB^{-1}G\right)$. This leads to the BX scheme for the fast decoupled load flow.

Similarly, starting with the system of equations (63), and applying Gaussion elimination from the bottom up, leads to the XB scheme. However, when there are PV buses, the convergence of this scheme becomes less reliable that that of the BX scheme. This can be understood by analysing what happens to the XB scheme if all buses are PV buses. In this case, the system from equation (63) will reduce to

$$[-B][\Delta\delta] = [\Delta P]. \tag{76}$$

In the BX scheme, this is indeed the system that will be solved. However, in the XB scheme, instead of $-B$ the coefficient matrix $-X^{-1}$ will be used, leading to unnecessary approximations.

Summarizing, the assumptions that $\delta \approx 0$ and $|V| \approx 1$, and the assumption that the R/X ratio doesn't vary too much between different lines in the network, lead to the BX scheme of the fast

decoupled load flow method. The assumption on the R/X ratios replaces the original assumption (45). The BX scheme is not decoupled in the orginal meaning of the term, as the off-diagonal blocks are not disregarded, but are fully incorporated in the method. As such it has better convergence properties than the BB scheme.

4.4.3 Matrix Analysis

As noted in Section 3.3, the admittance matrix Y in equation (9) is a Laplacian matrix. Line shunts, tap transformers, and phase shifters, generally change the values in the admittance matrix in such a way that it is no longer Laplacian. Let us assume that we have no such devices, nor any line shunt admittances, and thus that the admittance matrix is Laplacian. For details on Laplacian matrices, see Appendices A and B.

As described above, the FDLF method involves repeatedly solving linear systems with coefficient matrices B' and B'', both derived from the imaginary part B of the admittance matrix Y. If Y is a Laplacian matrix, then the imaginary part B is also Laplacian. Thus it has non-positive off-diagonal entries, positive diagonal entries, and all row sums are 0.

As discussed in Section 4.2, due to the removal of the equation for the slack bus, and possibly other generator buses, B' and B'' are principal minors of B. In the case of a BX or XB scheme, the coefficient matrix corresponding to the X, is a principal minor of the imaginary part of Y in which the resistance has been neglected. In either case, the coefficient matrices B' and B'' are principal minors of a Laplacian matrix. As such, they still have non-positive off-diagonal entries and positive diagonal entries. However, instead of all row sums being 0, there is at least one row that has a positive row sum. Such a matrix is called an M-matrix, see for example [2]. Any M-matrix is positive definite, and thus B' and B'' are symmetric positive definite M-matrices.

As described in Section 3.3, shunts lead to an extra term added to the diagonal of the admittance matrix. If the shunt is inductive, the shunt will only add to the diagonal dominance, of both the real and imaginary part of the admittance matrix. However, if the shunt is capacitive, the admittance matrix may lose its diagonal dominance. This can lead to the matrices B' and B'' no longer being M-matrices. Line shunts are by always capacitive, and thus will always have a negative effect on the diagonal dominance.

From equation (17) we can see that, for $T \neq 1$, tap transformers also have a negative effect on the diagonal dominance. Because if $T > 1$, then $\frac{1}{T^2} < \frac{1}{T}$, whereas if $T < 1$, then $1 < \frac{1}{T}$.

For a phase shifter, from equation (18) it is not clear how it will effect diagonal dominance of the real or imaginary part of the admittance matrix. However it is clear that, where all other devices retained the symmetry of the admittance matrix, a phase shifter generally does make the admittance matrix non-symmetric.

5 Research Questions

The general direction into which we want to take this research, is to develop efficient iterative solvers for large load flow problems.

The first step is to do an in-depth analysis of the coefficient matrices of the linear systems that arise when solving load flow problems, both for the full Newton method, and the FDLF schemes, treated in Section 4. The spectrum of eigenvalues, the sparsity pattern, the effect of the structure of the network, the effect of special devices like shunts, tap transformers, and phase shifters, and other properties, are essential for the choice and performance of iterative solvers.

The next step is to incorporate existing iterative methods in a load flow solver, to test the quali-

tative and quantative performance on existing problems, and compare the iterative methods with each other, and with direct solvers. From this research we can identify promising candidates and directions for further research, to develop new methods to efficiently deal with load flow problems. Some ideas are the development of specific preconditioners, for example FDLF as preconditioner for full Newton, multigrid, either algebraic, or exploiting information on the power system network, domain decomposition, deflation, model order reduction, and complex iterative solvers, as opposed to splitting the problem into real equations.

Scalability is an important property due to the ever growing power systems, both from the coupling of existing power systems, as the need to calculate the load flow all the way down to the consumers, instead of just the high voltage network. Therefore, the scalability of the developed techniques will be explored extensively.

The above described research questions should not only bear with a toolbox of iterative solvers for load flow problems, but also provide us with a lot of knowledge on the use of iterative solvers for load flow problems. With these insights, we can formulate new research questions to attack. Currently, an important candidate for further research is the efficient solution of many load flow problems with only relatively small differences, like changes in the load at certain busses, or removing one or more transmission lines from the system. Related to this problem, is the idea to bring stochastics into the system, for example to deal with fluctuating power demands of consumers, or fluctuating power generation due to wind turbines or solar panels.

6 Conclusions

In this literature survey, we have treated the basic concepts that define the load flow problem. The load flow problem is a large system of non-linear equations, and as such the Newton-Raphson method is start to tackle the problem.

The full Newton method ensures the fastest convergence, however each iteration is costly, because the coefficient matrix of the linear system changes. The FDLF method is an inexact Newton method, that sacrifices some of the fast convergence, to make each iteration a lot cheaper. Over the years, this method has proven very powerful in practice.

For a simplified power system, the coefficient matrices used in the FDLF method, are symmetric positive definite M-matrices. Systems with such coefficient matrices are well solved by numerous techniques. With the current size of load flow problems, in practice generally direct solvers are used. For larger system iterative techniques, that deal well with symmetric M-matrices, like Algebraic Multigrid (AMG) may prove a strong alternative. In Section 5, the current plans for further research on the subject have been presented.

A Network Matrices

Let $G = (V, E)$ be a graph with N vertices $v_k \in V$ and M edges $e_k \in E$, with $e_k = (v_i, v_j)$ denoting an edge from v_i to v_j.

The node-incidence matrix A of a graph $G = (V, E)$ is an $M \times N$ matrix in which each row i represents an edge $e_i = (p, q)$, and is defined as

$$a_{ij} = \begin{cases} -1 & \text{if } p = v_j, \\ 1 & \text{if } q = v_j, \\ 0 & \text{otherwise.} \end{cases} \tag{77}$$

In other words, if $e_i = (v_p, v_q)$, then row i has value -1 at index p and value 1 at index q.

The Laplacian matrix L of a graph $G = (V, E)$ is an $N \times N$ matrix in which each off-diagonal entry l_{ij} is -1 if the vertices v_i and v_j are connected, and 0 otherwise. Each diagonal entry l_{ii} is equal to the degree d_i of vertex v_i, i.e., the number of edges connected to v_i. So L is defined by

$$l_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -1 & \text{if } v_i \text{ and } v_j \text{ are connected,} \\ 0 & \text{otherwise.} \end{cases} \tag{78}$$

The Laplacian matrix L can also be defined as

$$L = A^T A. \tag{79}$$

The weighted Laplacian matrix L is a Laplacian matrix whose off-diagonal entries are negated edge weights, and whose diagonal entries are the sum of the weights of all edges connected to the vertex, instead of its degree. Let $w_{ij} = w_{ji}$ be the weight of the edge between v_i and v_j. Edge weights can be real valued, but are also allowed to be complex valued. Then

$$l_{ij} = \begin{cases} \sum_{k \neq i} w_{ik} & \text{if } i = j, \\ -w_{ij} & \text{if } v_i \text{ and } v_j \text{ are connected,} \\ 0 & \text{otherwise.} \end{cases} \tag{80}$$

The weighted Laplacian matrix L can also be defined as

$$L = A^T W A, \tag{81}$$

where W is a diagonal matrix such that w_{kk} is the weight of edge e_k. Note that if all weights are 1, the weighted Laplacian is identical to the non-weighted Laplacian matrix. Further note that $w_{kk} = 0$ corresponds to removing edge e_k from the graph.

In this document, with Laplacian matrix we mean a weighted Laplacian matrix. Obviously all properties that hold for weighted Laplacian matrices, also hold for non-weighted Laplacian matrices.

B Laplacian Matrix Properties

In this appendix we will explore some properties of (weigted) Laplacian matrices.

B.1 Symmetric Positive Semi-Definite

Let L be the Laplacian matrix of a graph $G = (V, E)$. Then L is symmetric positive semi-definite, but never positive definite.

From definition (80) it is obvious that L is symmetric, since $w_{ij} = w_{ji}$, and therefore all eigenvalues are real. Also from definition (80) it is easy to verify that

$$L\mathbf{1} = \mathbf{0}, \tag{82}$$

where $\mathbf{1}$ is a vector of just ones, and similarly $\mathbf{0}$ is a vector of only zeros. Thus $\lambda = 0$ is an eigenvalue of L with corresponding eigenvector $\mathbf{1}$.

Using Gershgorin's circle theorem, we find that all eigenvalues λ_i of L lie in the union of circles

$$|z - l_{ii}| \leq \sum_{j \neq i} |l_{ij}|, \ z \in \mathbb{C}.$$

Since we know that all eigenvalues are real, and that $\sum_{j \neq i} |l_{ij}| = l_{ii}$, this means they lie in the union of intervals

$$l_{ii} \leq z - l_{ii} \leq l_{ii}, \ z \in \mathbb{R}.$$

Thus we find

$$\lambda_i \in \left[0, 2 \max_k l_{kk}\right]. \tag{83}$$

With all eigenvalues being non-negative, it immediately follows that L is positive semi-definite. From (82) we also know that $\lambda_0 = 0$, and thus L is not positive definite. As a result L is not invertible, and there are vectors $\mathbf{b} = (b_1, \ldots, b_N) \in \mathbb{R}^N$ for which the system $L\mathbf{x} = \mathbf{b}$ has no solution.

B.2   Algebraic Connectivity

Let L be the Laplacian matrix of a graph $G = (V, E)$. The smallest non-trivial eigenvalue λ_1 of L is called the algebraic connectivity. A small positive value for λ_1 means that G is connected, but involves two loosely coupled subgraphs. It is known from spectral graph theory[1] that removing edges from G will make λ_1 become smaller, and that $\lambda_1 = 0$ if we have subgraphs that are disconnected from each other, which we also derive below.

Suppose G has K connected subgraphs G_i that are disconnected from all other subgraphs, then we can reorder the vertices of G such that

$$L = \begin{bmatrix} L_1 & & 0 \\ & \ddots & \\ 0 & & L_K \end{bmatrix},$$

where L_i is the weighted Laplacian matrix of subgraph G_i. If we partition

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_K \end{bmatrix},$$

corresponding to the partition of L, then it is clear that for each vector $\mathbf{x}$ with $\mathbf{x}_i = \mathbf{1}$ and $\mathbf{x}_j = \mathbf{0}$ for all $j \neq i$, we have $L\mathbf{x} = \mathbf{0}$. These K vectors $\mathbf{x}_i$ span the kernel of L. Therefore the 0 eigenvalue of L has multiplicity K, with corresponding eigenvectors $\mathbf{x}$ as described.

Note that if we partition $\mathbf{b}$ the same way as we did $\mathbf{x}$, then we can solve $L\mathbf{x} = \mathbf{b}$ by solving $L_i\mathbf{x}_i = \mathbf{b}_i$ for $i = 1, \ldots, K$. Since each matrix L_i is a (weighted) Laplacian matrix of a connected graph, they all have eigenvalue $\lambda_0 = 0$ with multiplicity 1. Therefore, solving $L\mathbf{x} = \mathbf{b}$, we may always assume to be solving one or more systems with a Laplacian matrix with eigenvalue $\lambda_0 = 0$ with multiplicity 1.

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[1]See http://en.wikipedia.org/wiki/Spectral_graph_theory for a definition and references.

B.3 Solution and Least Squares Solution

Let L be the Laplacian matrix of a graph $G = (V, E)$, and let $\lambda_0 = 0$ be the only eigenvalue with value 0. Then the linear system $L\mathbf{x} = \mathbf{b}$ has solutions if and only if $\sum_i b_i = 0$. Moreover, the solution of the system $L\mathbf{x} = \mathbf{b} - \frac{\sum_i b_i}{N}\mathbf{1}$ is the least squares solution of the system $L\mathbf{x} = \mathbf{b}$. Note that $\mathbf{b} - \frac{\sum_i b_i}{N}\mathbf{1}$ is the projection of $\mathbf{b}$ onto the orthogonal complement of the nullspace. Further note that, due to (82), if $\mathbf{x}$ is a solution of the system $L\mathbf{x} = \mathbf{b}$, then $\mathbf{x} + \alpha\mathbf{1}$ is also a solution for all $\alpha \in \mathbb{R}$.

Let $\lambda_0 \leq \ldots \leq \lambda_{N-1}$ again be the eigenvalues of L, and let $\mathbf{v}_0, \ldots, \mathbf{v}_{N-1}$ be the corresponding eigenfactors, where $\lambda_0 = 0$ and $\mathbf{v}_0 = \mathbf{1}$. As the eigenvectors span $\mathbb{R}^N$ we can write

$$\mathbf{x} = \sum_{i=0}^{N-1} \xi_i \mathbf{v}_i, \text{ with } \xi_i = \frac{\langle \mathbf{v}_i, \mathbf{x} \rangle}{\langle \mathbf{v}_i, \mathbf{v}_i \rangle}\mathbf{v}_i. \tag{84}$$

and

$$\mathbf{b} = \sum_{i=0}^{N-1} \beta_i \mathbf{v}_i, \text{ with } \beta_i = \frac{\langle \mathbf{v}_i, \mathbf{b} \rangle}{\langle \mathbf{v}_i, \mathbf{v}_i \rangle}\mathbf{v}_i. \tag{85}$$

We then have

$$L\mathbf{x} = L\sum_{i=0}^{N-1} \xi_i \mathbf{v}_i = \sum_{i=0}^{N-1} \xi_i L\mathbf{v}_i = \sum_{i=0}^{N-1} \xi_i \lambda_i \mathbf{v}_i = \sum_{i=1}^{N-1} \xi_i \lambda_i \mathbf{v}_i = \sum_{i=0}^{N-1} \beta_i \mathbf{v}_i = \mathbf{b}. \tag{86}$$

As the eigenvectors $\mathbf{v}_i$ are orthogonal, this equality holds if and only if $\xi_i \lambda_i = \beta_i$ for all i. Using that $\lambda_0 = 0$ is the only eigenvalue with value 0, this means that any solution $\mathbf{x}$ is defined by $\xi_i = \frac{\beta_i}{\lambda_i}$ for $i = 1, \ldots, N-1$, with arbitrary ξ_0, and that

$$\beta_0 = \frac{\langle \mathbf{1}, \mathbf{b} \rangle}{\langle \mathbf{1}, \mathbf{1} \rangle}\mathbf{1} = \frac{\langle \mathbf{1}, \mathbf{b} \rangle}{N}\mathbf{1} = 0 \Leftrightarrow \langle \mathbf{1}, \mathbf{b} \rangle = 0 \Leftrightarrow \sum_i b_i = 0.$$

The least squares solution of a system $L\mathbf{x} = \mathbf{b}$ is the solution of $L^T L\mathbf{x} = L^T \mathbf{b}$. Let $\mathbf{x}$ be a solution of the system $L\mathbf{x} = \mathbf{b} - \frac{\sum_i b_i}{N}\mathbf{1}$, which we know exists, then

$$L^T L\mathbf{x} = L^T (L\mathbf{x}) = L^T \left(\mathbf{b} - \frac{\sum_i \mathbf{b}_i}{N}\mathbf{1} \right) = L^T \mathbf{b}. \tag{87}$$

So $\mathbf{x}$ is indeed the least squares solution of $L\mathbf{x} = \mathbf{b}$.

B.4 Iterative Solution

Let L be the Laplacian matrix of a graph $G = (V, E)$, and let $\lambda_0 = 0$ be the only eigenvalue with value 0. Let $\mathbf{b}$ be such that $L\mathbf{x} = \mathbf{b}$ has a solution, i.e., $\langle \mathbf{b}, \mathbf{1} \rangle = 0$. Then there is always exactly one base solution $\tilde{\mathbf{x}}$ for which $\langle \tilde{\mathbf{x}}, \mathbf{1} \rangle = 0$, and the complete solution is given by $\mathbf{x} = \tilde{\mathbf{x}} + \alpha\mathbf{1}$, $\alpha \in \mathbb{R}$.

Let a splitting of L

$$L = M - N, \tag{88}$$

be given, where M is invertible, and M is such that systems of the form $M\mathbf{y} = \mathbf{q}$ are relatively easy to solve. Then we can write

$$L\mathbf{x} = \mathbf{b} \Leftrightarrow (M - N)\mathbf{x} = \mathbf{b} \Leftrightarrow M\mathbf{x} = N\mathbf{x} + \mathbf{b} \Leftrightarrow \mathbf{x} = M^{-1}N\mathbf{x} + M^{-1}\mathbf{b}.$$

Then the iteration scheme for the approximate solution $\mathbf{x}^k$, belonging to the splitting (88), is given by

$$\mathbf{x}^{k+1} = M^{-1}N\mathbf{x}^k + M^{-1}\mathbf{b} = \left(I - M^{-1}L\right)\mathbf{x}^k + M^{-1}\mathbf{b}. \tag{89}$$

Many properties of this iteration scheme, like convergence, depend fully on the choice of the matrix M. However we can still derive some properties for the general case. Note that it is possible to have a different splitting $L = M^k - N^k$ each iteration, but as this does not change any of our results, we are disregarding this for notational simplicity.

The residual $\mathbf{r}^k$ in iteration k, is defined by

$$\mathbf{r}^k = \mathbf{b} - L\mathbf{x}^k. \tag{90}$$

The residual is generally used as a measure for the error in each iteration.

Using the iteration formula (89) we can write

$$
\begin{aligned}
\mathbf{r}^{k+1} &= \mathbf{b} - L\mathbf{x}^{k+1} \\
&= \mathbf{b} - L\left(\left(I - M^{-1}L\right)\mathbf{x}^k + M^{-1}\mathbf{b}\right) \\
&= \mathbf{b} - L\left(I - M^{-1}L\right)\mathbf{x}^k - LM^{-1}\mathbf{b} \\
&= -\left(I - LM^{-1}\right)L\mathbf{x}^k + \left(I - LM^{-1}\right)\mathbf{b} \\
&= \left(I - LM^{-1}\right)\mathbf{r}^k.
\end{aligned}
$$

We call the matrix $\left(I - LM^{-1}\right) = NM^{-1}$ the residual propagation matrix.

As $\langle\mathbf{b}, \mathbf{1}\rangle = 0$ and $\langle L\mathbf{y}, \mathbf{1}\rangle = 0$ for all $\mathbf{y} \in \mathbb{R}^N$, from $\mathbf{r}^k = \mathbf{b} - L\mathbf{x}^k$ it is obvious that $\langle\mathbf{r}^k, \mathbf{1}\rangle = 0$, independent of $\mathbf{x}^k$, and that

$$\mathbf{r}^k = \mathbf{b} - L\mathbf{x}^k = \mathbf{b} - L\left(\mathbf{x}^k + \alpha\mathbf{1}\right). \tag{91}$$

The update for the approximate solution is defined as

$$\boldsymbol{\Delta}\mathbf{x}^{k+1} = \mathbf{x}^{k+1} - \mathbf{x}^k. \tag{92}$$

Using the iteration scheme (89) we can write

$$\boldsymbol{\Delta}\mathbf{x}^{k+1} = M^{-1}N\mathbf{x}^k + M^{-1}\mathbf{b} - M^{-1}N\mathbf{x}^{k-1} - M^{-1}\mathbf{b} = M^{-1}N\boldsymbol{\Delta}\mathbf{x}^k,$$

and

$$\boldsymbol{\Delta}\mathbf{x}^{k+1} = \left(I - M^{-1}L\right)\mathbf{x}^k + M^{-1}\mathbf{b} - \mathbf{x}^k = -M^{-1}L\mathbf{x}^k + M^{-1}\mathbf{b} = M^{-1}\mathbf{r}^k.$$

Summarizing, we have

$$M\boldsymbol{\Delta}\mathbf{x}^{k+1} = \mathbf{r}^k = N\boldsymbol{\Delta}\mathbf{x}^k. \tag{93}$$

The error $\mathbf{e}^k$ in iteration k, is generally defined by $\mathbf{e}^k = \mathbf{x} - \mathbf{x}^k$. However, in our case this definition is ambiguous, since the system $L\mathbf{x} = \mathbf{b}$ has an infinite number of solutions. Therefore, instead we will use the definition

$$\mathbf{e}^k = \left(\mathbf{x} - \frac{\langle\mathbf{x}, \mathbf{1}\rangle}{N}\mathbf{1}\right) - \left(\mathbf{x}^k - \frac{\langle\mathbf{x}^k, \mathbf{1}\rangle}{N}\mathbf{1}\right). \tag{94}$$

In words, the error is the difference of the base solution and the base approximate solution.

We can also write $\mathbf{e}^k = \mathbf{x}^* - \mathbf{x}^k$, where $\mathbf{x}^* = \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{1} \rangle - \langle \mathbf{x}^k, \mathbf{1} \rangle}{N} \mathbf{1}$. Note that $\mathbf{x}^*$ is indeed also a solution of the system $L\mathbf{x} = \mathbf{b}$. Then, again using the iteration scheme (89), we can write

$$
\begin{aligned}
\mathbf{e}^{k+1} &= \mathbf{x}^* - \left(I - M^{-1}L\right)\mathbf{x}^k - M^{-1}\mathbf{b} \\
&= \mathbf{x}^* - \left(I - M^{-1}L\right)\mathbf{x}^k - M^{-1}L\mathbf{x}^* \\
&= \left(I - M^{-1}L\right)\left(\mathbf{x}^* - \mathbf{x}^k\right) \\
&= \left(I - M^{-1}L\right)\mathbf{e}^k,
\end{aligned}
$$

where we call the matrix $\left(I - M^{-1}L\right) = M^{-1}N$ the error propagation matrix.

We also find

$$
\Delta\mathbf{e}^{k+1} = \mathbf{e}^{k+1} - \mathbf{e}^k = \left(I - M^{-1}L\right)\mathbf{e}^k - \mathbf{e}^k = -M^{-1}\mathbf{r}^k = -\boldsymbol{\Delta}\mathbf{x}^{k+1}.
$$

Further note that

$$
L\mathbf{e}^k = L\mathbf{x}^* - L\mathbf{x}^k = \mathbf{r}^k,
$$

which is the reason that the residual can be used as a measure for the error of an approximate solution, as obviuolsy the true error is not known.

From $\mathbf{x}^{k+1} = M^{-1}N\mathbf{x}^k + M^{-1}\mathbf{b}$, in the iteration scheme (89), and the fact that M is invertible, it is obvious that $\mathbf{x}^{k+1}$ is uniquely defined. However, for all $\alpha \in \mathbb{R}$ the vector $\mathbf{x}^{k+1} + \alpha\mathbf{1}$ would be an equally good approximation, in the sense that the error $\mathbf{e}^k$ is the same. From equation (91) it is clear that the residual also stays the same. Thus, in each iteration we are free to add $\alpha\mathbf{1}$ to the approximate solution $\mathbf{x}^k$, for any choice of $\alpha \in \mathbb{R}$. This will not influence convergence of the algorithm, though the solution it converges to is likely to be different. If we like, we could therefore set the approximate solution to be the base approximate solution in each iteration. But we can also leave the iterates as they are, and just set the final approximation of the algorithm to match the desired solution, be it the base solution or any other.

Let a base approximate solution $\mathbf{x}^{k*}$ be given. Using equation (93), the next iterate is given by $\mathbf{x}^{k+1} = \mathbf{x}^{k*} + M^{-1}\mathbf{r}^k$. Thus we find

$$
\left\langle \mathbf{x}^{k+1}, \mathbf{1} \right\rangle = \left\langle \mathbf{x}^{k*}, \mathbf{1} \right\rangle + \left\langle M^{-1}\mathbf{r}^k, \mathbf{1} \right\rangle = \left\langle M^{-1}\mathbf{r}^k, \mathbf{1} \right\rangle.
$$

Therefore, $\left\langle \mathbf{x}^{k+1}, \mathbf{1} \right\rangle = 0$ if and only if $\left\langle M^{-1}\mathbf{r}^k, \mathbf{1} \right\rangle = 0$.

Let $\mathbf{r}^k = (r_1, \ldots, r_N)$, and $M^{-1} = \begin{bmatrix} \mathbf{m}_1^{-1} & \ldots & \mathbf{m}_N^{-1} \end{bmatrix}$. Then

$$
\left\langle M^{-1}\mathbf{r}^k, \mathbf{1} \right\rangle = \mathbf{r}^T \left(M^{-1}\right)^T \mathbf{1} = \sum_i \left\langle \mathbf{m}_i^{-1}, \mathbf{1} \right\rangle r_i.
$$

Using the knowledge that $\sum_i r_i = 0$, we find that in general

$$
\left\langle \mathbf{x}^{k+1}, \mathbf{1} \right\rangle = \left\langle M^{-1}\mathbf{r}^k, \mathbf{1} \right\rangle = 0 \Leftrightarrow \left\langle \mathbf{m}_i^{-1}, \mathbf{1} \right\rangle = \mu \text{ for all } i.
$$

Therefore, having a base approximate solution $\mathbf{x}^{k*}$ does not guarantee the next iterate $\mathbf{x}^{k+1}$ to also be a base approximate solution, unless $\left\langle \mathbf{m}_i^{-1}, \mathbf{1} \right\rangle = \mu$ for all i. Similarly, a certain offset $\alpha\mathbf{1}$ from the base approximate solution is only guaranteed to be maintained through iterations if and only if $\left\langle \mathbf{m}_i^{-1}, \mathbf{1} \right\rangle = \mu$ for all i.

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