DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TEST SCIENTIFIC COMPUTING (wi4201)
Friday January 29 2016, 13:30-16:30

1. Below 5 statements are given. If the statement is true give a short proof. If the statement is wrong give a counter example or an explanation.

   (a) $A \in \mathbb{R}^{n\times n}$, $A = A^T \Rightarrow \|A\|_1 = \|A\|_\infty$. (2 pt.)

   (b) $A \in \mathbb{R}^{n\times n}$, A is SPD $\Rightarrow$ all elements of A are positive. (2 pt.)

   (c) $A \in \mathbb{R}^{n\times n}$, A is an M-matrix $\Rightarrow$ diag(A) is an M-matrix. (2 pt.)

   (d) $A = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \|A\|_2 = 3$. (2 pt.)

   (e) There exists a square real-valued matrix for which the condition number measured in 2-norm is smaller than 1. (2 pt.)

2. For a given function f we consider the following boundary value problem:

$$-\frac{d^2u(x)}{dx^2} + u(x) = f(x) \text{ for } 0 < x < 1\,, \tag{1}$$

   with boundary conditions

$$u(x=0) = 0 \text{ and } u(x=1) = 0\,. \tag{2}$$

   A finite difference method is used on a uniform mesh with N intervals and mesh width $h = 1/N$.

   (a) Give the finite difference stencil for internal grid points. (1 pt.)

   (b) Give the matrix A^h for $N = 3$ where the boundary conditions are eliminated. In the next questions one can use the fact that for general values of N the eigenvalues are given by: $\lambda_k^h(A^h) = 1 + \frac{2}{h^2} 2\sin^2(\frac{\pi\, h\, k}{2})$.

   (c) Prove that A^h is positive definite for every value of the mesh width h using either Gershgorin theorem or the eigenvalues of A^h. (1 pt.)

   (d) Show that for every value of the mesh width h that the inverse of A is elementwise positive. (2 pt.)

   (e) Compute the condition number of the matrix A^h in the 2-norm as a function of the mesh width h. (1 pt.)

(f) Give the error propagation matrix of the method of Jacobi B^h_{JAC} and weighted Jacobi $B^h_{JAC(w)}$ as a function of the mesh width h. (2 pt.)

(g) Compute the spectral radius of the Jacobi iteration matrix B^h_{JAC} as a function of the mesh width h. (2 pt.)

3. For a given function f we consider the following boundary value problem:

$$-\frac{d^2u(x)}{dx^2} = f(x) \text{ for } 0 < x < 1\,, \tag{3}$$

with boundary conditions

$$u(x=0) = 0 \text{ and } u(x=1) = 0\,. \tag{4}$$

A finite difference method is used on a uniform mesh with N intervals and mesh width $h = 1/N$. Let $A^h\mathbf{u}^h = \mathbf{f}^h$ denote the resulting linear system.

(a) Derive a basic iterative method for the iterand $\mathbf{u}^k$ using the defect-correction principle. (2.5 pt.)

(b) Consider an even number of intervals N and a red-black ordering of the grid nodes such that the boundary nodes correspond to the red nodes. Describe the 2×2 block structure of the matrix A^h induced by this ordering. (2.5 pt.)

(c) Show that after one red-black Gauss-Seidel sweep the residual vector $\mathbf{r}^k$ is zero in the components that corresponds to the black nodes. (2.5 pt.)

(d) Consider both a fine and coarse mesh in which the coarse mesh coincides with the red nodes of the fine mesh. Describe a two-grid iterative method that combines the red-black Gauss-Seidel sweep on the fine mesh and the defect correction scheme using a coarse mesh in a $V(1,0)$-cycle. (2.5 pt.)

4. In this exercise we have to solve a linear system $Au = b$, where A is an $n \times n$ non-singular matrix.

(a) Take $u_1 = \alpha b$. Derive an expression for α such that $\|b - Au_1\|_2$ is minimal. (2.5 pt.)

(b) Give the definition of a Krylov subspace of dimension k, matrix A and starting vector b. (2.5 pt.)

(c) Give the optimisation property of the GMRES method. Motivate why $u_n = u$ (without rounding errors). (2.5 pt.)

(d) Given the algorithm

GCR algorithm

Choose u^0, compute $r^0 = b - Au^0$

for $i = 1, 2, ...$ do
  $\mathsf{s}^i = \mathsf{r}^{i-1}$,
  $\mathsf{v}^i = \mathsf{A}\mathsf{s}^i$,
  for $j = 1, ..., i-1$ do
    $\alpha = (\mathsf{v}^j, \mathsf{v}^i)$,
    $\mathsf{s}^i := \mathsf{s}^i - \alpha \mathsf{s}^j$, $\mathsf{v}^i := \mathsf{v}^i - \alpha \mathsf{v}^j$,
  end for
  $\mathsf{s}^i := \mathsf{s}^i / \|\mathsf{v}^i\|_2$, $\mathsf{v}^i := \mathsf{v}^i / \|\mathsf{v}^i\|_2$
  $\beta = (\mathsf{v}^i, \mathsf{r}^{i-1})$;
  $\mathsf{u}^i := \mathsf{u}^{i-1} + \beta \mathsf{s}^i$;
  $\mathsf{r}^i := \mathsf{r}^{i-1} - \beta \mathsf{v}^i$;
end for

Determine the minimal amount of memory and flops for iteration i (+ motivation). (2.5 pt.)

5. (a) Give the outline of the LU-decomposition method (without pivotting) to solve $Au = b$, where $A \in \mathbb{R}^{n \times n}$ is a non-singular matrix. Give the amount of flops for a full matrix A. (2.5pt.)

   (b) Show that the inverse of the Gauss transformation $M_k = I - \alpha^{(k)} \mathbf{e}_k^T$ is the rank-one modification $M_k^{-1} = I + \alpha^{(k)} \mathbf{e}_k^T$. The k-th Gauss-vector $\alpha^{(k)} \in \mathbb{R}^n$ is defined as

$$\alpha^{(k)} = (\underbrace{0, \ldots, 0}_{k}, \underbrace{\mathbf{b}_k / a_{k,k}^{(k-1)}}_{n-k}) . \tag{5}$$

(2.5pt.)

   (c) Given the linear system $Au = b$ with $A \in \mathbb{R}^{n \times n}$ and the perturbed system $A(u + \Delta u) = b + \Delta b$. Derive an upperbound for $\frac{\|\Delta u\|}{\|u\|}$ where $\|.\|$ is an arbitrary vector norm, which has the multiplicative property. (2.5pt.)

   (d) Suppose we have a penta-diagonal matrix $A \in \mathbb{R}^{n \times n}$. For a given m, where $1 < m < n$, we know that the elements $a(i-m, i)$, $a(i-1, i)$, $a(i, i)$, $a(i, i+1)$, and $a(i, i+m)$, are nonzero. Give the non-zero pattern of the L and U matrix after the LU-decomposition without pivotting. (2.5pt.)