DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

TEST SCIENTIFIC COMPUTING (wi4201)
Friday February 2 2018, 13:30-16:30

1. Below 5 statements are given. If the statement is true give a short proof. If the statement is wrong give a counter example or an explanation.

   (a) $A \in \mathbb{R}^{n\times n}$, A is SPD $\Rightarrow \|A^{-1}\|_2 = \frac{1}{\lambda_{min}}$, and λ_{min} is the smallest eigenvalue of A. (2 pt.)

   (b) For any nonsingular $A, B \in \mathbb{R}^{n\times n}$, $\text{cond}_2(AB) \leq \text{cond}_2(A)\,\text{cond}_2(B)$; (2 pt.)

   (c) $R_1, R_2 \in \mathbb{R}^{n\times m} \Rightarrow R_1^T R_2$ is SPD. (2 pt.)

   (d) $u \in \mathbb{R}^n \Rightarrow \|u\|_1 \leq n\|u\|_\infty$. (2 pt.)

   (e) $A \in \mathbb{R}^{n\times n}$, and A is strictly row diagonal dominant and $a_{ii} \geq 0 \Rightarrow$ the real part of all eigenvalues is positive. (2 pt.)

2. For a given function f we consider the following boundary value problem:

$$-\frac{d^2u(x)}{dx^2} + 4u(x) = f(x) \text{ for } 0 < x < 1\,, \tag{1}$$

   with boundary conditions

$$u(0) = 0 \text{ and } u(1) = 0\,. \tag{2}$$

   A finite difference method is used on a uniform mesh with N intervals and mesh width $h = 1/N$.

   (a) Give the finite difference stencil for internal grid points and show that it is $O(h^2)$. (1 pt.)

   (b) Given the eigenvectors v^k of the matrix A^h with components

$$v_i^k = \sin(k\,\pi\,x_i) = \sin(k\,\pi\,(i-1)\,h) \text{ for } 1 \leq i \leq N+1 \tag{3}$$

   derive an expression for the corresponding eigenvalues λ_k as a function of the meshwidth h by computing the action of A^h on these eigenvectors. It suffices here to consider the matrix rows corresponding to the grid nodes not having any connections to the boundary nodes. (Hint: $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta)$). (3 pt.)

   (c) Prove that the eigenvalues λ_k of A^h are larger than or equal to 4, for every value of the meshwidth h using either the Gershgorin theorem or the eigenvalues of A^h; (2 pt.)

(d) Give an upperbound for $cond_2(A)$ as a function of the meshwidth h. (2 pt.)

(e) To solve the linear system, one can use a direct or an iterative method. Which method is preferred (motivate your answer)? (2 pt.)

3. Consider the linear system $Au = f$, where $A \in \mathbb{R}^{n\times n}$ is a full nonsingular matrix.

(a) Describe the LU method to solve such a system, both the decomposition and solution phase. Also derive the number of flops for the various steps. (2 pt.)

(b) We consider the perturbed system: $A(u + \Delta u) = f + \Delta f$. Give a proof of the following inequality:

$$\frac{\|\Delta u\|}{\|u\|} \leq \kappa(A)\frac{\|\Delta f\|}{\|f\|},$$

where $\kappa(A)$ is the condition number of A measured in the norm $\|\cdot\|$. (2 pt.)

(c) The k-th Gauss-vector $\alpha^{(k)} \in \mathbb{R}^n$ is defined as

$$\alpha^{(k)} = (\underbrace{0,\ldots,0}_{k}, \underbrace{b_k/a_{k,k}^{(k-1)}}_{n-k})^T . \tag{4}$$

The k-th Gaussian transformation $M_k \in \mathbb{R}^{n\times n}$ is defined as $M_k = I - \alpha^{(k)} e_k^T$. Show that

$$(M_{n-1}\cdot \ldots \cdot M_1)^{-1} = I + \sum_{k=1}^{n-1} \alpha^{(k)} e_k^T . \tag{5}$$

(2 pt.)

(d) Suppose that the pivot element is equal to zero. How can you prevent the LU method to break down? Is it always possible to finish the LU method with this adaptation (give a counter example or a proof)? (2 pt.)

(e) Do 1 step of the Gaussian elimination process on the following matrix:

$$\begin{pmatrix} 4 & -1 & 0 & 0 & -1 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ -1 & 0 & 0 & -1 & 4 \end{pmatrix}$$

What is the size of the fill in? (2 pt.)

4. Consider the linear system $Au = f$, where $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix.

(a) Given that a non-singular matrix M exists we can split A as follows: $A = M - N$. Show that $u^{k+1} = u^k + M^{-1}r^k$. (2 pt.)

(b) Derive the Jacobi iteration matrix B_{Jac}. (hint $e^{k+1} = B_{Jac}e^k$) (2 pt.)

(c) Consider a 2D Poisson equation with Dirichlet boundary conditions on a square domain, discretized by a 5-point stencil. Give the stencil notation of the Jacobi iteration matrix. (2 pt.)

(d) Derive the damped Jacobi method and give the damped Jacobi iteration matrix. (2 pt.)

(e) Do 1 iteration with the Gauss-Seidel method to the following linear system, where we start with the zero vector.

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

(2 pt.)

5. Consider the linear system $Au = f$, where $A \in \mathbb{R}^{n \times n}$ is a nonsingular matrix.

(a) We assume that $u^1 = \alpha_0 r^0$. Determine α_0 such that $\|u - u^1\|_2$ is minimal. (2 pt.)

(b) If A is SPD show that $< y, z >_A = y^T A z$ is an inner product. (an inner product has the following properties: $< u, v > = < v, u >$, $< c\,u, v > = c < u, v >$, $< u + v, w > = < u, w > + < v, w >$ and $< u, u > \geq 0$ with equality only for $u = 0$) (2 pt.)

(c) The matrix A is from the discretized Poisson operator. The eigenvalues are given by

$$\lambda_{k,\ell} = 4 - 2\cos\frac{\pi k}{31} - 2\cos\frac{\pi \ell}{31} \quad , \quad 1 \leq k, \ell \leq 30.$$

Determine the linear rate of convergence for the Conjugate Gradient method. (2 pt.)

(d) From the linear rate of convergence it appears that 280 iteration are necessary to have a reduction of a factor 10^{-12}. In practice only 120 iterations are necessary. Explain (in words and graphical) this behaviour. (2 pt.)

(e) Give for a general matrix A three classes of Krylov solvers and their properties. (2 pt.)