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TEST SCIENTIFIC COMPUTING (wi4201)
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1. Below 5 statements are given. If the statement is true give a short proof. If the statement is wrong give a counter example or an explanation.

   (a) $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix. The spectrum of A and Q^TAQ are the same. (2 pt.)

   (b) $A \in \mathbb{R}^{n\times n}$ is a tridiagonal matrix. The components are given by: $a_{i,i} = 4$, $a_{i,i-1} = -1$, and $a_{i,i+1} = 1$. All eigenvalues of A are elements of $\mathbb{R}$. (2 pt.)

   (c) $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix, where $d_{i,i} = \frac{i}{n}$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ the operator $(\mathbf{x}, \mathbf{y})_D$ is defined as: $(\mathbf{x}, \mathbf{y})_D = \mathbf{x}^T D\mathbf{y}$. $(\mathbf{x}, \mathbf{y})_D$ is an inner product. (2 pt.)

   (d) $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix. The Fröbenius norm of A and QA are the same. (2 pt.)

   (e) Matrix A is given by:

   $$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

   The spectral radius of A is equal to -3. (2 pt.)

2. (a) Matrix A^h is given by expression (3.48) of the lecture notes. Show that the eigenvalues as given in (3.53) are contained in the Gershgorin disks. (2.5 pt.)

   (b) We take $N = 3$. Show that the eigenvectors as given in (3.52) are orthogonal to each other. (2.5 pt.)

   (c) Give an expression for the eigenvalues of the coefficient matrix for the bi-Harmonic equation as given in (3.69). (2.5 pt.)

   (d) The two dimensional equation

   $$-u_{xx} + u_x - u_{yy} + 6u = x^2$$

   is discretized on an equidistant grid on the unit square where the step size is denoted by h. Give the stencil notation of the second order discretization (including the proof) in an internal grid point. (2.5 pt.)

3. (a) Give the inverse M_k^{-1} of Gauss transformation $M_k = I - \boldsymbol{\alpha}^{(k)}\mathbf{e}_k^T$ and show that the formula for M_k^{-1} is correct. (2.5 pt.)

   (b) Given matrix and vector

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \mathbf{f} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

   Show that standard Gaussian elimination is not possible in the second step. How can this problem solved? Compute the solution $\mathbf{u}$ of the linear system $A\mathbf{u} = \mathbf{f}$. (2.5 pt.)

   (c) We assume that A is invertible and strictly column diagonal dominant. Show that the first step of standard Gaussian elimination is always possible. (2.5 pt.)

   (d) Matrix L is a lower triangular matrix with $L_{i,i} = 1$. Show that L^{-1} is also lower triangular and $L_{i,i}^{-1} = 1$. (2.5 pt.)

4. (a) We consider the standard 2 dimensional model problem. Show that the stencil of the Jacobi iteration matrix is given by

$$B_{JAC}^h = \begin{bmatrix} 0 & 1/4 & 0 \\ 1/4 & 0 & 1/4 \\ 0 & 1/4 & 0 \end{bmatrix}.$$

   (2.5 pt.)

   (b) Give the iteration matrix of the Symmetric Gauss Seidel method if A can be written as in Figure 5.1 of the lecture notes. (2.5 pt.)

   (c) Matrix $A \in \mathbb{R}^{n \times n}$ has the following properties: $a_{i,i} = 1$, all eigenvalues are in $\mathbb{R}$ and $0 < \lambda_1 < .... < \lambda_n$. Give an expression of ω such that the convergence speed of damped Jacobi is optimal. (2.5 pt.)

   (d) For a model problem we know that $\rho(B_{JAC}) = 1 - \frac{\pi^2}{2}h^2$. Give an expression of the number of iterates such that $\frac{\|\mathbf{e}^k\|_2}{\|\mathbf{e}^0\|_2} \leq 10^{-4}$. (2.5 pt.)

5. In this exercise we consider a matrix A that is SPD

   (a) Given $\mathbf{u}^k = \mathbf{u}^{k-1} + \alpha_k \mathbf{p}^k$ compute α_k such that $\mathbf{e}^k$ is perpendicular to $\mathbf{p}^k$ in the A inner product. (2.5 pt.)

   (b) Show that if $\{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^n\}$ is a set of A-orthogonal vectors in $\mathbb{R}^n$ and $\mathbf{z}^T\mathbf{v}^i = 0$ for $i = 1, \ldots, n$ then $\mathbf{z} = 0$. (2.5 pt.)

   (c) Take $A = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{pmatrix}$ and $\mathbf{f} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$. We solve $A\mathbf{u} = \mathbf{f}$. Show that Conjugate Gradient method applied to this system should convergence in 1 or 2 iterations. (2.5 pt.)

(d) We use the example as given in Example 7.2.1.1 of the Lecture Notes. Show that expression (7.22) is correct for $i > m$. (2.5 pt.)