DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

ANSWERS OF THE TEST SCIENTIFIC COMPUTING (wi4201)
Wednesday January 6 2016, 13:30-16:30

1. (a) The 2-norm of a vector x is given by: $\|x\|_2 = \sqrt{x^T x} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$. The 2-norm of matrix A is given by

$$\|A\|_2 = sup \frac{\|Ax\|_2}{\|x\|_2}.$$

(b) A is an SPD matrix means that A is symmetric and Positive Definite ($x^T Ax > 0$ for all vectors $x \neq 0$). Since A is symmetric we know that the eigenvalues λ_j are real values and the eigenvectors span $\mathbb{R}^n$ and can be chosen as a orthonormal set. Let us denote the eigenvectors by v_j for $j = 1, \ldots, n$. From the definition of an eigenvector and the Positive Definiteness it follows that

$$v_j^T A v_j = v_j^T \lambda_j v_j = \lambda_j > 0,$$

so all eigenvalues are positive. Take an arbitrary vector $x = \alpha_1 v_1 + \ldots + \alpha_n v_n$. Substitute this vector in the definition of the 2-norm of matrix A and use the orthonormality of the eigenvectors which leads to:

$$\|A\|_2^2 = sup \frac{\|Ax\|_2^2}{\|x\|_2^2} = sup \frac{\alpha_1^2 \lambda_1^2 + \ldots + \alpha_n^2 \lambda_n^2}{\alpha_1^2 + \ldots + \alpha_n^2}$$

(c) Hint: Note that the matrix is symmetric. The 2-norm of a symmetric matrix is given by the maximal absolute eigenvalue. Using Gershgorin Leads to the upperbound $\|A\|_2 \leq 8$.

(d) Hint: see section 4.3.1 of the lecture notes.

2. (a) Hint: see the second half of page 100 of the lecture notes, where the answer is given for the A-norm

(b) Hint: see section 7.1.1

(c) Hint: Theorem 7.1.2

(d) Per iteration you need 2 innerproducts (store the norm of the residual) and 3 vector updates. This costs $10n$ flops. Furthermore a matrix vector product is needed. For a 2D Poisson equation, there are 5 nonzero elements per row, to this costs $9n$ flops. Memory: the matrix A has to be stored. Furthermore r, p, u has to be stored, which costs $3n$ memory positions.

3. (a) Hint, see Section 7.3.1
   (b) Hint, see Section 7.3.1 and properties mentioned during the lectures.
   (c) Hint, substitute A and simplify the expression
   (d) Hint, see Section 7.3.6.