All monotonically decreasing isotropic 2D filters can be decomposed into a stack of concentric pillboxes. Some filter shapes \( h \) can only be expressed analytically in the spatial domain, others in the Fourier domain. This appendix derives the spatial as well as Fourier representation of eq. (A.1) to allow error analysis to be done in both domains.

**Relative location error in the spatial domain**

The pillbox stack is a rotation symmetric 2D function \( h_{2D}(s_1,s_2) \) with cross section \( h(s) = h_{2D}(s,0) \) (defining \( h(-s) \equiv h(s) \) we can use this for \( s<0 \) too). The 2D Fourier transform \( H_{2D}(u,v) \) of \( h_{2D}(s_1,s_2) \) represents combined filters as simple products. It is rotation symmetric as well. Its cross section \( H(w) = H_{2D}(w,0) = H_{2D}(0,w) \) (the Hankel transform of \( h(s) \)) is not to be confused with the 1D Fourier transform of \( h(s) \) which equals \( \int_{-\infty}^{\infty} H_{2D}(u,v) \, dv \). In order to calculate eq. (A.1)

\[
\rho_0 -1 = \frac{1}{2} \int_0^{\infty} f h_s s^3 \, ds / 4 \int_0^{\infty} G h_s s^{-1} \, ds
\]  

(A.1)

with correction factors \( f \equiv (1 - s^2/4)^{-1/2} \) and \( G \equiv (1 - 3s^2/4 + 7s^4/16 - s^6/16) f^3 \), we first handle the factors \( f \) and \( G \).
For $0 < s < 1$, we have $1 < f < 1.15$ and $0.88 < G < 1$, hence the integrands keep their sign over the entire interval $(0, s_{\text{max}} < 1)$. Moreover, the integrands are zero outside that interval, such that for some $1 < f_0 < 1.15$ and $0.88 < G_0 < 1$, numerator and denominator can be written as

$$\frac{1}{2} \int_{0}^{\infty} fh_s s^3 ds = \frac{f_0}{2} \int_{0}^{\infty} h_s s^3 ds$$

and

$$4 \int_{0}^{\infty} Gh_s s^{-1} ds = 4G_0 \int_{0}^{\infty} h_s s^{-1} ds$$

With $M + f_0/G_0$, $1 < M < 1.15/0.88 = 1.3$ this yields eq. (A.2).

$$\rho_0 - 1 \approx \frac{M}{8} \int_{0}^{\infty} h_s s^3 ds / \int_{0}^{\infty} h_s s^{-1} ds$$

(A.2)

**Fourier expression for the numerator**

The numerator of eq. (1) can be expressed (note that $h(s)s^3 \to 0$ for $s \to 0$ and $s \to \infty$) in the Fourier transform

$$\frac{f_0}{2} \int_{0}^{\infty} h_s s^3 ds = \frac{-3f_0}{4} \int_{-\infty}^{\infty} h_{2D}(0,t) t^2 dt = \frac{-3f_0}{4} \int_{-\infty}^{\infty} \frac{\partial^2 H_{2D}(u,v)}{\partial (2\pi v)^2} du \bigg|_{v=0}$$

Because $H_{2D}(u,v)$ is rotation symmetric the integrand can be expressed in its cross section ($H(w)$)

$$\int_{0}^{\infty} H_{w,w^{-1}} dw = \frac{3f_0}{8\pi^2} \int_{0}^{\infty} \int_{0}^{\infty} \frac{\partial H_{2D}(u,0)}{\partial u} u^{-1} du$$

**Proof**

For $u=\text{constant}$, $v$ and $w$, are given by

$$v = \sqrt{w^2 - u^2}, \quad w_v = \frac{1}{v_w} = \frac{v}{w}$$

The derivatives with respect to $v$ can be expressed into derivatives to $w$

$$\frac{\partial H_{2D}(u,v)}{\partial v} = \frac{v}{w} \frac{\partial H_{2D}(u,v)}{\partial w}$$

$$\frac{\partial^2 H_{2D}(u,v)}{\partial v^2} = \frac{w^2 - v^2}{w^3} \frac{\partial H_{2D}(u,v)}{\partial w} + \frac{v}{w} \frac{\partial^2 H_{2D}(u,v)}{\partial w^2}$$

Due to the rotation invariance $u=w$ and $v=0$, therefore
Fourier expression for the denominator

A Fourier expression for the denominator of eq.(A.2) is more complicated to derive. An intuitive solution is similar to the result for the numerator in the sense that $H(w)$ and $h(s)$ change roles in the derivation, note that $H(w)w^3 \to 0$ for $w \to 0$ and $w \to \infty$.

$$4G_0 \int_0^\infty h_s s^{-1} ds = -16\pi^2 G_0 \int_0^\infty H(w) w^2 dw = -16\pi^2 G_0 \int_0^\infty H_{2D}(u, 0) u^2 du$$

**Proof**

$$4G_0 \int_0^\infty h_s s^{-1} ds = 4G_0 \left[ h_s \log(s) \right]_0^\infty - 4G_0 \int_0^\infty h_{ss} \log(s) ds$$

Here we assume that the bandlimiting component of $h$ near $s=0$ is $O(s^2)$ (which is already necessary to make the denominator converge) so $h_s$ as well as $h_s \log(s)$ are 0 for $s=0$. Since $h(s)=0$ for $s>s_{\text{max}}$, $h_s$ and $h_s \log(s)$ are 0 for $s \to \infty$.

$$\lim_{\varepsilon \to 0} 2G_0 \int_{-\infty}^\infty h_{ss} \log(s) ds = -G_0 \int_{-\infty}^\infty h_{ss} \left. N \to \infty \log \left( s^2 + \varepsilon^2 \right) - \log \left( 1 + s^2 N^{-2} \right) \right] ds$$

We can replace this scalar product of $h_{ss}$ and $\log(s^2+\varepsilon^2)-\log(1+s^2 N^{-2})$ by the scalar product of their Fourier transforms. The Fourier transform of $h_{ss}$ is $(2\pi i u)^2 H(u)$, with $H(u)$ the 1D Fourier transform of $h(s)$ which equals $\int_{-\infty}^\infty H_{2D}(u, v) dv$; the Fourier transform of $\log(s^2+\varepsilon^2)-\log(1+s^2 N^{-2})$ is found as follows.

As

$$\int_{-\infty}^\infty e^{2\pi i su} e^{-|u|} du = \frac{2}{1 + (2\pi s)^2}$$

we have

$$\int_{-\infty}^\infty e^{2\pi i su} \alpha \frac{\alpha}{\alpha^2 + s^2} ds = \pi e^{-2\pi|\alpha|}$$

Integrating both sides from $N$ to $\varepsilon$ with respect to $\alpha$ we get

$$\int_{-\infty}^\infty e^{2\pi i su} \left[ \log \left( s^2 + \varepsilon^2 \right) - \log \left( 1 + s^2 N^{-2} \right) - 2 \log \left( N \right) \right] ds = \frac{e^{-2\pi N|\alpha|} - e^{-2\pi|\alpha|}}{|\alpha|}$$

The scalar product of the Fourier transforms thus equals
\[ \lim_{\varepsilon \to 0} \left( e^{-2\pi N|u|} - e^{-2\pi \varepsilon |u|} \right) \cdot -2\log(N) \delta(u) \int_{-\infty}^{\infty} du \]

In the limit of \( N \to \infty \), \( e^{-N|u|} = 0 \) and in the limit of \( \varepsilon \to 0 \), \( e^{-\varepsilon |u|} = 1 \). The term containing \(-2\log(N)\delta(u)\) does not contribute, as \( u^2H(u) = 0 \) for \( u = 0 \) and \( \delta(u) = 0 \) for all \( u \neq 0 \).

\[ = -4\pi^2 G_0 \int_{-\infty}^{\infty} |u| H(u) du = -4\pi^2 G_0 \int_{-\infty}^{\infty} |u| H_{2D}(u, v) dudv \]

Because \( H_{2D}(u,v) \) is rotation symmetric the double integral can be reduced to a single integral in cylindrical coordinates.

\[ = -4\pi^2 G_0 \int_{0}^{2\pi} \cos(\phi) |d\phi| \int_{-\infty}^{\infty} u^2 H_{2D}(u, 0) du = -16\pi^2 G_0 \int_{-\infty}^{\infty} u^2 H_{2D}(u, 0) du \]

**Relative location error in the Fourier domain**

With \( M \equiv f_0 / G_0 \), \( 1 < M < 1.3 \) this yields eq. (A.3)

\[ \rho_0 - 1 \approx \frac{-3M}{128\pi^4} \int_{0}^{\infty} \frac{\partial H_{2D}(u, 0)}{\partial u} u^{-1} du \int_{0}^{\infty} H_{2D}(u, 0) u^2 du \quad (A.3) \]

Thus combining eqs.(A.2,A.3) gives

\[ \rho_0 - 1 \approx \frac{M}{8} \int_{0}^{\infty} \frac{\partial H_{2D}(u, 0)}{\partial u} u^{-1} du \int_{0}^{\infty} H_{2D}(u, 0) u^2 du \]
Appendix II

Gaussian Stack Shape

The relative location error of Gaussian filtered curved edges can be expressed in the spatial moments of the Gaussian stack shape. The moments depend on the truncation factor \( a \) (\( h(s) = 0 \) for \( s \geq a\sigma R \))

\[
h(s) = \left[ \exp\left(-\frac{1}{2}R^2s^2\sigma^{-2}\right) - \exp\left(-\frac{1}{2}a^2\right) \right] U(a\sigma - sR)
\]

Moments of the truncated Gaussian filter
For \( n=0 \) to 5 we shall evaluate eq. (A.4)

\[
I(n, a) = \left(\frac{R}{\sigma}\right)^{n+1} \int_0^{a\sigma/R} \exp\left(-\frac{1}{2}s^2R^2\sigma^{-2}\right)s^n ds = \int_0^a \exp\left(-\frac{1}{2}x^2\right)x^n dx
\]

(A.4)

According to the definition of the error function, \( \text{erf}(z) + (2/\sqrt{\pi}) \int_0^z \exp(-t^2) dt \), we have

\[
I(0, a) = \sqrt{\pi/2} \text{erf}\left(\frac{1}{2}a\right)
\]

For \( n=1 \) the defining integral can be directly evaluated

\[
I(1, a) = 1 - \exp\left(-\frac{1}{2}a^2\right)
\]

\( I(2, a) \) and \( I(4, a) \) are found taking \(-2\partial/\partial T\big|_{T=1}\) and \(4\partial^2/\partial T^2\big|_{T=1}\) on both sides of

\[
\frac{I(0, a)}{\sqrt{T}} = \int_0^{a/\sqrt{T}} \exp\left(-\frac{1}{2}x^2T\right) dx
\]
which gives
\[ I(0, a) = I(2, a) + a \exp \left( -\frac{1}{2} a^2 \right) \]

and
\[ 3I(0, a) = I(4, a) + \left( a^3 + 3a \right) \exp \left( -\frac{1}{2} a^2 \right) \]

\( I(3,a) \) and \( I(5,a) \) are found taking \(-2\partial^2/\partial T^2\big|_{T=1} \) and \( 4\partial^2/\partial T^2\big|_{T=1} \) on both sides of
\[ \frac{I(1,a)}{T} = \int_0^{a/\sqrt{T}} \exp \left( -\frac{1}{2} x^2 T \right) x dx \]

which gives
\[ 2I(1,a) = I(3,a) + a^2 \exp \left( -\frac{1}{2} a^2 \right) \]

and
\[ 8I(1,a) = I(5,a) + \left( a^4 + 4a^2 \right) \exp \left( -\frac{1}{2} a^2 \right) \]

Summarizing the expressions for \( I(0,a) \) to \( I(5,a) \) as function of the truncation factor \( a \).
\[
\begin{align*}
I(0,a) &= \sqrt{\pi/2} \operatorname{erf}(a/\sqrt{2}) \quad (A.5) \\
I(1,a) &= 1 - \exp(-a^2/2) \quad (A.6) \\
I(2,a) &= I(0,a) - a \exp(-a^2/2) \quad (A.7) \\
I(3,a) &= 2 - (a^2 + 2) \exp(-a^2/2) \quad (A.8) \\
I(4,a) &= 3I(0,a) - (a^3 + 3a) \exp(-a^2/2) \quad (A.9) \\
I(5,a) &= 8 - (a^4 + 4a^2 + 8) \exp(-a^2/2) \quad (A.10)
\end{align*}
\]

The sensitivity of the Gaussian moments to the choice of \( a \) is given in table A.1. Note that the bandlimitation improves with increasing \( a \).

**Table A.1**: Sensitivity of relative location error to the choice of truncation factor \( a \).

<table>
<thead>
<tr>
<th></th>
<th>( a = 1.8 )</th>
<th>( a = 2 )</th>
<th>( a = 3 )</th>
<th>( a = 4 )</th>
<th>( a = \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I(2,a) / I(0,a) )</td>
<td>1 – 37%</td>
<td>1 – 33%</td>
<td>1 – 03%</td>
<td>1 – 0.1%</td>
<td>1</td>
</tr>
<tr>
<td>( I(4,a) / I(0,a) )</td>
<td>3 – 77%</td>
<td>3 – 53%</td>
<td>3 – 11%</td>
<td>3 – 1.0%</td>
<td>3</td>
</tr>
<tr>
<td>( I(3,a) / I(1,a) )</td>
<td>2 – 40%</td>
<td>2 – 31%</td>
<td>2 – 05%</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( I(5,a) / I(1,a) )</td>
<td>8 – 72%</td>
<td>8 – 63%</td>
<td>8 – 16%</td>
<td>8 – 1.0%</td>
<td>8</td>
</tr>
</tbody>
</table>
Appendix III

Stack of Pillballs I

The edge position of Laplace \((p=1,q=1)\), SDGD \((p=1,q=0)\) and PLUS \((p=2,q=1)\) after applying a single pillball filter is

\[
2^{1-D} (D-1) V/(D-1) P(p, q, \rho^2, s, R)
\]

with \(s + l/R, \rho + r/R\), and

\[
P\left(p, q, \frac{r^2}{R^2}, \frac{l}{R}, R\right) \equiv -rc^{D-3}\left[p\frac{\partial (\rho^2)}{\partial (r^2)} + q \frac{c^2}{r^2}\right]
\]

For a stack of pillballs the edge position is given by an integral of single pillball results over radius \(l\). To estimate the zero-crossing position we expand the integrand of around \(r=R\).

\[
\int_{|r-R|}^{lmax} h_r P\left(p, q, \frac{r^2}{R^2}, \frac{l}{R}, R\right) dl \approx \int_{|\rho-|}^{smax} h_r \left[P\left(p, q, 1, s, R\right) + (\rho - 1) P_\rho \left(p, q, 1, s, R\right)\right] ds
\]

with Taylor coefficients \(P(p,q,1,s,R)\) and \(P_\rho(p,q,1,s,R)\) and filter shape \(h(s)+h(l)\).

\(P(p,q,1,s,R) \equiv -2^D R^{D-2} g_1 f^{3-D} s^{D-1}\) with correction factors \(f \equiv (1 - s^2/4)^{-1/2}\) and \(g_1 \equiv (2q-p)/4 + (p-q) s^2/8\). \(P_\rho(p,q,1,s,R)\) is stated in the lemma below.

**Lemma**

\[P_\rho(p,q,1,s,R) = 2^D R^{D-2} g_1 f^{3-D} s^{D-3}\]

with
Proof of lemma

For \( r = R \):

\[
g^2 = 4l^2 \left( 1 - \frac{l^2}{4R^2} \right) \equiv 4s^2R^2 \left( 1 - \frac{s^2}{4} \right) \equiv \frac{4l^2}{f^2}
\]

\[
\frac{\partial (g^2)}{\partial (r^2)} = -1 + \frac{(R^2 - l^2)^2}{R^4} \equiv -2s^2 \left( 1 - \frac{s^2}{2} \right)
\]

\[
\frac{\partial^2 (g^2)}{\partial (r^2)^2} = -\frac{2(R^2 - l^2)^2}{R^6} \equiv -\frac{2(1-s^2)^2}{R^2}
\]

Substitution into the following expression completes the proof

\[
P_p \left( p, q, \frac{r^2}{R^2}, \frac{l}{R}, R \right) = 2rR \frac{\partial P \left( p, q, \frac{r^2}{R^2}, \frac{l}{R}, R \right)}{\partial (r^2)}
\]

\[
= \left\{ -(D - 3) r^2 R c^{D-3} \frac{\partial (c^2)}{\partial (r^2)} - R c^{D-3} \left( p \frac{\partial (c^2)}{\partial (r^2)} + q \frac{c^2}{r^2} \right) \right\}
\]

\[
-2r^2 R c^{D-3} \frac{\partial}{\partial (r^2)} \left( p \frac{\partial (c^2)}{\partial (r^2)} + q \frac{c^2}{r^2} \right)
\]

In particular: for \( SDGD \) (\( p=1, q=0 \)),

\[
g = \frac{1}{2} \left( \frac{(D+4)s^2}{8} + \frac{(2D+3)s^4}{16} - \frac{Ds^6}{32} \right)
\]

for \( Laplace \) (\( p=1, q=1 \)),

\[
g = \frac{1}{2} \left( \frac{(D-2)s^2}{8} - \frac{(D-2)s^4}{16} \right)
\]

for \( PLUS \) (\( p=2, q=1 \)),

\[
g = 1 - \frac{6s^2}{8} + \frac{(D+5)s^4}{16} - \frac{Ds^6}{32}
\]
Appendix IV

Stack of Pillballs II

This appendix derives the spatial representation of eq. (A.11) to allow error prediction in the spatial domain.

In order to calculate eq. (A.11)

$$\rho_0 - 1 = C \int_0^\infty Fh_s s^{D-1+E} ds \int_0^\infty Gh_s s^{D-3} ds$$

we first handle the factors $F$ and $G$. We define bounds for $F$ and $G$ on the interval $0 < s < 1$, or $0 < l < R$: we have

$$F_{floor} < F < F_{ceiling} \quad \text{and} \quad G_{floor} < G < G_{ceiling} ;$$

the values of these bounds are given in table A.2.

The condition $l < R$ guarantees that the largest pillbox at the base properly intersects the object when the stack is on-edge.

If the stack shape function $h(s)$ is monotonous on $0 < s < s_{\text{max}}$ (so that multiple edge responses are excluded and $h_s$ does not change sign) and if $s_{\text{max}} < 1$ then there is a pair of values $F_0$, $G_0$ with $F_{floor} < F_0 < F_{ceiling}$ and $G_{floor} < G_0 < G_{ceiling}$ such that

$$\int_0^\infty Fh_s s^{D-1+E} ds = F_0 \int_0^\infty h_s s^{D-1+E} ds$$
and
\[ \int_{0}^{\infty} G h_s s^{D-3} ds = G_0 \int_{0}^{\infty} h_s s^{D-3} ds \]

With
\[ M_{\text{floor}} = \frac{F_{\text{floor}}}{G_{\text{ceiling}}} < M = \frac{F_0}{G_0} < M_{\text{ceiling}} = \frac{F_{\text{ceiling}}}{G_{\text{floor}}} \]

this yields eq. (A.12).
\[ \rho_0 - 1 \approx M \int_{0}^{s_{\text{max}}} h_s s^{D-1+E} ds \int_{0}^{s_{\text{max}}} h_s s^{D-3} ds \quad (A.12) \]

The bounds \( F_{\text{floor}}, F_{\text{ceiling}}, G_{\text{floor}}, G_{\text{ceiling}}, M_{\text{floor}} \) and \( M_{\text{ceiling}} \) are summarized in table A.2. The ratio \( M_{\text{ceiling}}/M_{\text{floor}} \) indicates that PLUS allows the most accurate estimate of location error, both in 2D and in 3D.

**Table A.2:** Bounds for \( F, G \) and \( M \) if \( 0 < s < 1 \)

<table>
<thead>
<tr>
<th></th>
<th>( D )</th>
<th>( F_{\text{floor}} )</th>
<th>( F_{\text{ceiling}} )</th>
<th>( G_{\text{floor}} )</th>
<th>( G_{\text{ceiling}} )</th>
<th>( M_{\text{floor}} )</th>
<th>( M_{\text{ceiling}} )</th>
<th>( M_{\text{ceiling}}/M_{\text{floor}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>SDGD</td>
<td>2</td>
<td>0.58</td>
<td>1</td>
<td>0.385</td>
<td>1</td>
<td>0.58</td>
<td>2.60</td>
<td>4.5</td>
</tr>
<tr>
<td>Laplace</td>
<td>2</td>
<td>1</td>
<td>1.15</td>
<td>1</td>
<td>1.55</td>
<td>0.65</td>
<td>1.15</td>
<td>1.8</td>
</tr>
<tr>
<td>PLUS</td>
<td>2</td>
<td>1</td>
<td>1.15</td>
<td>0.88</td>
<td>1</td>
<td>1</td>
<td>1.31</td>
<td>1.3</td>
</tr>
<tr>
<td>SDGD</td>
<td>3</td>
<td>0.50</td>
<td>1</td>
<td>0.25</td>
<td>1</td>
<td>0.50</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>Laplace</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1.50</td>
<td>0.67</td>
<td>1</td>
<td>1.5</td>
</tr>
<tr>
<td>PLUS</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0.83</td>
<td>1</td>
<td>1</td>
<td>1.20</td>
<td>1.2</td>
</tr>
</tbody>
</table>
This appendix derives the Fourier representation of eq. (A.12) of Appendix IV to allow error prediction in the Fourier domain.

The moments of $h_s$ are expressed in those of the $D$–dimensional Fourier transforms as follows. The volume of the $D$-dimensional hypersphere of radius $r$ is $V(D)r^D$ with $V(D) = \pi^{D/2}/\Gamma(\frac{1}{2}D + 1)$. The hyperarea of the $D$-dimensional hypersphere of radius 1 is $Q(D) = \partial/\partial r V(D)|_{r=1} = D V(D)$, e.g. $Q(3) = 4\pi$.

For $D \geq 2$, in the numerator for Laplace and SDGD

\[
\int_0^\infty h_s s^{D-1} ds = -(D-1) \int_0^\infty h(s) s^{D-2} ds
\]
\[
= \frac{-(D-1)}{Q(D-1)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_{DD}(0, s_2, \ldots, s_D) ds_2 \cdots ds_D
\]
\[
= \frac{-1}{V(D-1)} \int_{-\infty}^{\infty} H_{DD}(u_1, 0, 0) du_1 = \frac{-2}{V(D-1)} \int_0^\infty H(w) dw
\]

(A.13)

and in the numerator for PLUS

\[
\int_0^\infty h_s d^{D+1} ds = -(D+1) \int_0^\infty h(s) s^D ds
\]
\[
= \frac{-(D+1)}{Q(D-1)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(s_2^2 + \ldots + s_D^2\right) h_{DD}(0, s_2, \ldots, s_D) ds_2 \cdots ds_D
\]
\[
\frac{-(D+1)}{(2\pi i)^2 Q(D-1)} \int_{0}^{\infty} \left( \frac{\partial^2}{\partial u_2^2} + \ldots + \frac{\partial^2}{\partial u_D^2} \right) H_{DD}(u_1, u_2, \ldots, u_D) du_1 \bigg|_{u_2=\ldots=u_D=0} \\
= \frac{-2(D-1)(D+1)}{(2\pi i)^2 Q(D-1)} \int_{0}^{\infty} H_w w^{-1} dw \\
= \frac{(D+1)}{2\pi^2 V(D-1)} \int_{0}^{\infty} \frac{\partial H_{DD}(u, 0, 0)}{\partial u} u^{-1} du
\]

(A.14)

where use has been made of

\[
\frac{\partial^2}{\partial u_k^2} H_{DD}(u_1, 0, 0, u_k, 0) du_1 \bigg|_{u_1=w, u_k=0} = H_w w^{-1}
\]

which on its turn follows from \( \frac{\partial^2 H_{2D}(u, v)}{\partial v^2} \big|_{u=w, v=0} = H_w w^{-1} \)

(as proven in Appendix I: Stack of Pillboxes) identifying \( u_1=u, u_k=v, \)
\( H_{DD}(u_1, 0, 0, u_k, 0) = H_{2D}(u, v) \).

For \( D=3 \) (for \( D=2 \) see Appendix I: Stack of Pillboxes), in the denominator:

\[
\int_{0}^{\infty} h_s ds = h(\infty) - h(0) = -h(0) \\
= -\int \int \int \int H_{3D}(u_1, u_2, u_3) du_1 du_2 du_3 \\
= -Q(3) \int_{0}^{\infty} H(w) w^2 dw \\
= -4\pi \int_{0}^{\infty} H_{3D}(u, 0, 0) u^2 du
\]

(A.15)

For \( D \geq 4 \), in the denominator:

\[
\int_{0}^{\infty} h_s s^{D-3} ds = -(D-3) \int_{0}^{\infty} h(s) s^{D-4} ds \\
= \frac{-(D-3)}{Q(D-3)} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_{DD}(0, 0, 0, s_4, \ldots, s_D) ds_4 \ldots ds_D \\
= -\frac{1}{V(D-3)} \int \int \int \int \int \int H_{3D}(u_1, u_2, u_3, 0, 0) du_1 du_2 du_3 du_4 \ldots du_D \\
= \frac{-Q(3)}{V(D-3)} \int_{0}^{\infty} H(w) w^2 dw \\
= \frac{-4\pi}{V(D-3)} \int_{0}^{\infty} H_{DD}(u, 0, 0) u^2 du
\]

(A.16)
Appendix VI

Pillball FT

In this appendix we derive the Fourier transforms of pillballs in any dimension. The solutions for 1D, 2D and 3D are given explicitly.

A \(D\)-dimensional pillball filter of radius \(l_0\) is given by \((s_0 \equiv l_0/R, \nu \equiv s_0 u_1, \varsigma \equiv s_1/s_0)\)

\[
h^{\text{ball}}_{DD}(s_1, s_2, \ldots, s_D) \equiv \frac{U\left(s_0^2 - \left(s_1^2 + \ldots + s_D^2\right)\right)}{V(D) s_0^D} = \frac{U\left(s_0^2 - s_1^2\right)}{V(D) s_0^D} U\left(s_0^2 - \left(s_1^2 + \ldots + s_D^2\right)\right)
\]

Its Fourier Transform is rotation symmetric with cross section

\[
H^{\text{ball}}_{DD}(u_1, 0, \ldots, 0) \equiv H^{\text{ball}}_{DD}\left(\frac{\nu}{s_0}, 0, \ldots, 0\right) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h^{\text{ball}}_{DD}(s_1, s_2, \ldots, s_D) e^{-2\pi i s_1 u_1} ds_1 \ldots ds_D
\]

\[
= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \frac{U\left(s_0^2 - s_1^2\right)}{V(D) s_0^D} \left[\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} U\left(s_0^2 - \left(s_1^2 + \ldots + s_D^2\right)\right) ds_2 \ldots ds_D\right] e^{-2\pi i s_1 u_1} ds_1 \ldots ds_D
\]
As the expression between brackets is the volume of the \((D-1)\)-dimensional hypersphere of radius \(\sqrt{(s_0^2-s_1^2)}\), which equals \(V(D-1) \left(\sqrt{(s_0^2-s_1^2)}\right)^{D-1}\). The volume of a \(D\)-dimensional hyper-sphere of radius 1 is \(V(D) = \frac{\pi^{D/2}}{\Gamma\left(\frac{1}{2}D+1\right)}\) (A.17)

with \(\Gamma(\cdot)\) the Gamma function and \(V(1)=2, V(2)=\pi, V(3)=4\pi/3\).

\[H_{DD}^{ball}\left(\frac{v}{s_0},0,0\right) = \frac{V(D-1)}{V(D)} \int_{-1}^{1} \left(\sqrt{1-\zeta^2}\right)^{D-1} e^{-2\pi i \zeta v} d\zeta\]  \hspace{1cm} (A.187)

\[H_{1D}^{ball}\left(\frac{v}{s_0},0\right) = \frac{1}{\pi} \int_{-1}^{1} e^{-2\pi i \zeta v} d\zeta = \frac{\sin(2\pi v)}{\pi v}\]  \hspace{1cm} (A.19)

\[H_{2D}^{ball}\left(\frac{v}{s_0},0,0\right) = \frac{2}{\pi} \int_{-1}^{1} \sqrt{1-\zeta^2} e^{-2\pi i \zeta v} d\zeta = \frac{J_1(2\pi v)}{\pi v}\]  \hspace{1cm} (A.20)

\[H_{3D}^{ball}\left(\frac{v}{s_0},0,0\right) = \frac{3}{4} \int_{-1}^{1} \left(1-\zeta^2\right) e^{-2\pi i \zeta v} d\zeta = \frac{-6\pi v \cos(2\pi v) + 3\sin(2\pi v)}{4\pi^3 v^3}\]  \hspace{1cm} (A.21)

As for any function \(f(\zeta)\)

\[\frac{\partial^2}{\partial(2\pi i v)^2} \int_{-1}^{1} f(\zeta) e^{-2\pi i \zeta v} d\zeta = \int_{-1}^{1} f(\zeta) \zeta^2 e^{-2\pi i \zeta v} d\zeta\]

we can express the cross section of the Fourier transform for odd \(D\) as

\[H_{DD}^{ball}\left(\frac{v}{s_0},0,0\right) = \frac{2V(D-1)}{V(D)} \left(1-\frac{\partial^2}{\partial(2\pi i v)^2}\right)^{D-1/2} \frac{\sin(2\pi v)}{\pi v}\]  \hspace{1cm} (A.22)

and that for even \(D\) as

\[H_{DD}^{ball}\left(\frac{v}{s_0},0,0\right) = \frac{\pi V(D-1)}{2V(D)} \left(1-\frac{\partial^2}{\partial(2\pi i v)^2}\right)^{D/2} \frac{J_1(2\pi v)}{\pi v}\]  \hspace{1cm} (A.23)

As \(\partial/\partial x (x^n J_n) = x^n J_{n+1}\) and \(J_{n+1} J_{n+2} = 2nx^{-1} J_{n+1}\), we have

\((1+\partial^2/\partial x^2)x^n J_n = (2n-1)x^{(n+1)} J_{n+1}\) and hence

\((1+\partial^2/\partial x^2)^m x^J \{1(x) = [(2m+1)!/(2m)!] x^{(m+1)}J_{m+1}(x)\} such that for even \(D\)

\[H_{DD}^{ball}\left(\frac{v}{s_0},0,0\right) = \frac{\pi V(D-1)(D-1)}{V(D)2^{(D-2)/2}(D-2)! (2\pi v)^{D/2}} J_{D/2}(2\pi v)\]  \hspace{1cm} (A.24)
From the normalization

\[ 1 = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} h_{DD}^{ball}(s_1, \ldots, s_D) \, ds_1 \ldots ds_D \]

\[ = H_{DD}^{ball}(0, 0, 0) = \frac{V(D-1)}{V(D)} \int_{-1}^{1} \left( \sqrt{1 - \zeta^2} \right)^{D-1} \, d\zeta \]

we get

\[ \frac{V(2m+1)}{V(2m)} = 2 \int_{0}^{\pi/2} \sin^{2m+1}(\theta) \, d\theta = 2 \frac{2 \cdot 4 \ldots 2m}{1 \cdot 3 \ldots (2m+1)} \]

and

\[ \frac{V(2m)}{V(2m-1)} = 2 \int_{0}^{\pi/2} \sin^{2m}(\theta) \, d\theta = 2 \frac{1 \cdot 3 \ldots (2m-1) \pi}{1 \cdot 3 \ldots 2m \cdot 2} \]

which justifies the definition for \( V(D) \) of eq. (A.17).

Inserting these definitions in eq. (A.24) we get

\[ H_{DD}^{ball} \left( \frac{v}{s_0}, 0, 0, 0 \right) = \left( \frac{1}{2} D \right)! \frac{J_{D/2}(2\pi v)}{(\pi v)^{D/2}} \]

(A.25)

The asymptotic behavior of eqs. (A.22 – A.25) for large \( u = v/s_0 \) \((u > 1/s_0)\) follows from

\[ \left( 1 + \frac{\partial^2}{\partial (2\pi v)^2} \right) \frac{\sin(2\pi v)}{2\pi v} = -2n \frac{\cos(2\pi v)}{(2\pi v)^{n+1}} + n(n+1) \frac{\sin(2\pi v)}{(2\pi v)^{n+2}} \]

\[ \left( 1 + \frac{\partial^2}{\partial (2\pi v)^2} \right) \frac{\cos(2\pi v)}{2\pi v} = 2n \sin(2\pi v) \frac{\cos(2\pi v)}{(2\pi v)^{n+1}} + n(n+1) \cos(2\pi v) \frac{\cos(2\pi v)}{(2\pi v)^{n+2}} \]

and

\[ J_{D/2}(2\pi v) \approx \frac{\cos(2\pi v - (D+1)\pi/4)}{\pi \sqrt{v}} \]

Both for odd and even \( D \) we get

\[ H_{DD}^{ball}(u, 0, 0, 0) = O \left( \frac{\cos(2\pi s_0 \mu - (D+1)\pi/4)}{u^{(D+1)/2}} \right) \]

(A.26)
To obtain the influence of $R$ on the slope length we study the maximum gradient. The gradient and therefore the edge slope is maximal at the zero-crossing $r_0$ of the $SDGD$.

The gradient at $r=r_0$ is given by the Taylor series expansion

$$ B_g \bigg|_{r=r_0} = B_g \bigg|_{r=R} + (r_0 - R) B_{gg} \bigg|_{r=R} $$

At $r=R$, $c=2lf$ with $f=(1-s^2/4)^{-0.5}$ and $w=(R/\sigma)^2$

$$ B_g = \int_0^\infty h_t c dl = \int_0^\infty h_t c dl = -w \int_0^{a/\sqrt{w}} s \exp \left( -\frac{1}{2} s^2 w \right) ds $$

$$ \Rightarrow B_g \bigg|_{r=R} = -2Rw \int_0^{a/\sqrt{w}} \frac{s^2}{f} \exp \left( -\frac{1}{2} s^2 w \right) ds $$

with $1<f<1.15$.

$$ \left| B_g \right|_{r=R} = \frac{2R\sqrt{w}^3}{f_0 \sqrt{w}} \int_0^{a/\sqrt{w}} s^2 \exp \left( -\frac{1}{2} s^2 w \right) ds = \frac{2R}{f_0 \sqrt{w}} I(2) $$

The $SDGD$ or $B_{gg}$ is given by
The maximum gradient is thus

\[
B_{g_2} = \int_0^\infty h_l \left( -1 + \frac{s^2}{2} \right) f s d l = -w f_0 \int_0^{a/\sqrt{w}} \left( -1 + \frac{s^2}{2} \right) s \exp\left( -\frac{1}{2} s^2 w \right) ds
\]

\[
= w f_0 \int_0^{a/\sqrt{w}} s \exp\left( -\frac{1}{2} s^2 w \right) ds - f_0 \frac{w^2}{2w} \int_0^{a/\sqrt{w}} s^2 \exp\left( -\frac{1}{2} s^2 w \right) ds
\]

\[
= f_0 I(1,a) - \frac{f_0}{2w} I(3,a)
\]

The relative correction term due to finite \( R \) can for \( R=2\sigma \) be approximated by

\[
\frac{\sigma}{2R} I(1,a) \left( 0.25 \text{ to } 1.0 \right) \approx 0.06 \text{ to } 0.25
\]
Appendix VIII

Bending Energy of Parametric Curves/Surfaces using Differential Geometry

The circumference of a 2D ellipse is given by an elliptical integral. In order to evaluate the proposed measures for bending energy we need to calculate the true bending energy of ellipses in 2D and ellipsoids in 3D. These values are obtained using numerical integration in Mathematica™ (Wolfram 1988) based upon the theory of Differential Geometry. The function EllipseLengthBE computes the perimeter and bending energy of 2D ellipses. The function EllipsoidAreaBE computes the surface area and bending energy (note that we have set Poisson’s ratio equal to zero) of 3D ellipsoids. For readability purposes we have adopted the exact notation used by (Boehm 1990 a; Boehm 1990 b).
(*
SYNOPSIS:
<<Calculus'VectorAnalysis';
EllipseLengthBE[a,b]

AUTHOR: Lucas J. van Vliet
*)

EllipseLengthBE[a_, b_] :=
(* a, b: principal axes of 2D ellipse *)
Module[{t, (* curve parameter *)
  x, (* position vector x(t) *)
  xt, (* 1st derivatives of x to t *)
  xtt, (* 2nd derivatives of x to t *)
  d, (* dummy vector *)
  ds, (* length element *)
  K, (* curvature *)
  L, (* total length *)
  BE (* 2D bending energy *)
},
  x  = {a*Cos[t], b*Sin[t], 0};
  xt = D[x,t];
  xtt = D[x,{t,2}];
  ds = Sqrt[xt.xt];
  d = CrossProduct[xt,xtt];
  K=Sqrt[d.d]/(xt.xt)^(3/2);
  L  = 4 * NIntegrate[ds, {t,0,Pi/2}];
  BE = 4 * NIntegrate[K^2 * ds, {t,0,Pi/2}];
  {L,BE}]

(*)

SYNOPSIS:

<<Calculus'VectorAnalysis';

EllipsoidAreaBE[a,b]

AUTHORS: James C. Mullikin and Lucas J. van Vliet

*)

EllipsoidAreaBE[a_, b_, c_] := (* a, b, c: principal axes of 3D ellipsoid *)

Module[{%
    u, (* surface parameter 1 *)
    v, (* surface parameter 2 *)
    x, (* position vector x(u,v) *)
    xu, xv, (* 1st derivatives of x to u & v *)
    xuu, xuv, xv, (* 2nd derivatives of x to u, uv, & v *)
    E, F, G, (* used in the first fundamental form *)
    dA, (* area element *)
    n, (* (normalized) surface normal vector *)
    L, M, N, (* used in the second fundamental form *)
    Km, (* mean curvature *)
    Kg, (* Gaussian curvature *)
    A, (* total area *)
    BE, (* 3D bending energy *)
    },

    x = {a*Sin[v]*Cos[u], b*Sin[v]*Sin[u], c*Cos[v]};

    xu = D[x, u];
    xv = D[x, v];

    xuu = D[x, {u, 2}];
    xuv = D[x, u, v];
    xv = D[x, {v, 2}];

    E = xu.xu;
    F = xu.xv;
    G = xv.xv;

    dA = Sqrt[E*G - F*F];
    n = CrossProduct[xu, xv]/dA;

    L = xuu.n;
    M = xuv.n;
    N = xv.n;

    Km = (E*N - 2*F*M + G*L)/(2(E*G - F*F));
    Kg = (L*N - M*M)/(E*G - F*F);

    A = 8*NIntegrate[dA, {u, 0, Pi/2}, {v, 0, Pi/2}];
    BE = 8*NIntegrate[(4*Km^2 - 2*Kg)*dA, {u, 0, Pi/2}, {v, 0, Pi/2}];

}
}   (* End of Module EllipsoidAreaBE *)
Appendix IX

Second Order
Rotation Invariance

The point measurements described in part II of this thesis are rotation invariant operators of the second order. This appendix gives an overview of rotation invariant operators expressed in first and second order derivatives along the axes of a Cartesian coordinate system. The operators can be implemented as convolutions with derivatives-of-Gaussians. The “pure mathematical” derivatives are applied to image $B$, a Gaussian filtered version of image $A$. The resulting rotation invariant operators, expressed in derivatives of image $B$, produce estimates of the underlying properties of image $A$. The gradient operator $\nabla$ determines the lowest order rotation invariance. Applying this operator to an image $B$ yields a gradient vector $g$ whose modulus is called “gradient strength” $G$. The product of the gradient operator with itself $(\nabla \nabla^T)$ applied to an image $B$ results in the Hessian matrix $H$. The “physical” interpretation of the Hessian is that for any normalized vector $(\cos \phi, \sin \phi) \equiv a_\phi$, the second derivative in that direction is given by $a_\phi^T H a_\phi$. Starting from a Hessian matrix we calculate some rotation invariant scalars.
Second Order Rotation Invariance in Two Dimensions

Let \( H \) be the Hessian matrix of image \( B \), \( g \) the gradient vector and \( c \) the isophote contour vector.

\[
H = \begin{pmatrix} B_{xx} & B_{xy} \\ B_{xy} & B_{yy} \end{pmatrix}, \quad g = \begin{pmatrix} B_x \\ B_y \end{pmatrix}, \quad c = \begin{pmatrix} -B_y \\ B_x \end{pmatrix}
\]

**Principal second derivatives**

The principal second derivatives can be found by setting \( \det(H - \lambda I) = 0 \).

\[
SD_{\text{max}} = \frac{1}{2} \left( B_{xx} + B_{yy} \right) + \frac{1}{2} \sqrt{\left( B_{xx} - B_{yy} \right)^2 + 4B_{xy}^2} \quad \text{(A.27)}
\]

\[
SD_{\text{min}} = \frac{1}{2} \left( B_{xx} + B_{yy} \right) - \frac{1}{2} \sqrt{\left( B_{xx} - B_{yy} \right)^2 + 4B_{xy}^2} \quad \text{(A.28)}
\]

**Mean second derivative**

The mean second derivative is defined as the average of the two principal second derivatives \( SD_{\text{max}} \) and \( SD_{\text{min}} \)

\[
SD_{\text{mean}} = \frac{1}{2} \left( SD_{\text{max}} + SD_{\text{min}} \right) = \frac{1}{2} \left( B_{xx} + B_{yy} \right)
\equiv \frac{1}{2} \text{Laplace}(B) \equiv \frac{1}{2} L = \frac{1}{2} \text{trace}(H) \quad \text{(A.29)}
\]

**Quadratic variation**

We define the quadratic variation \( QV \) as

\[
QV = (SD_{\text{max}} - SD_{\text{mean}})^2 + (SD_{\text{min}} - SD_{\text{mean}})^2 \quad \text{(A.30)}
\]

Using the principal second derivatives as computed above we get

\[
QV = \frac{1}{2} \left( \left( B_{xx} - B_{yy} \right)^2 + 4B_{xy}^2 \right) \quad \text{(A.31)}
\]

**Second-derivative magnitude**

We define the second-derivative magnitude as

\[
M = 2(\text{SD}_{\text{mean}})^2 + QV = \frac{1}{2} L^2 + QV
\equiv \frac{1}{2} \left( B_{xx} + B_{yy} \right)^2 + \frac{1}{2} \left( B_{xx} - B_{yy} \right)^2 + 4B_{xy}^2 \quad \text{(A.32)}
\]

\[
= B_{xx}^2 + B_{yy}^2 + 2B_{xy}^2
\]
“Gaussian” second derivative
The “Gaussian” second derivative is defined as the product of the two principal second derivatives $SD_{\text{max}}$ and $SD_{\text{min}}$

$$SD_{\text{Gaussian}} = SD_{\text{max}} SD_{\text{min}} = \det(\mathbf{H}) = B_{xx} B_{yy} - B_{xy}^2$$  \hspace{1cm} (A.33)

Second derivative in gradient direction, SDGD
The second derivative in gradient direction is given by $(a_\varphi = g/|g|)$

$$SDGD = \frac{\mathbf{g}^T \cdot \mathbf{H} \cdot \mathbf{g}}{|\mathbf{g}|^2} = \frac{B_{xx} B_x^2 + 2B_{xy} B_x B_y + B_{yy} B_y^2}{B_x^2 + B_y^2} \equiv B_{gg}$$  \hspace{1cm} (A.34)

Second derivative in contour/isophote direction, SDCD
The second derivative in contour/isophote direction is given by $(a_\varphi = c/|c|)$

$$SDCD = \frac{\mathbf{c}^T \cdot \mathbf{H} \cdot \mathbf{c}}{|\mathbf{c}|^2} = \frac{B_{xx} B_y^2 - 2B_{xy} B_x B_y + B_{yy} B_x^2}{B_x^2 + B_y^2} \equiv B_{cc}$$  \hspace{1cm} (A.35)

Change in isophote direction (Curvature)
The contour/isophote direction $\theta$ is defined as

$$\theta = \arccos \left( -\frac{B_y}{G} \right) = \arcsin \left( \frac{B_x}{G} \right) = \arctan \left( -\frac{B_x}{B_y} \right)$$

with

$$G = \sqrt{B_x^2 + B_y^2}$$

To differentiate along the curve with respect to the arc length $s$ we use the operator

$$\frac{d}{ds} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} = -\frac{B_y}{G} \frac{\partial}{\partial x} + \frac{B_x}{G} \frac{\partial}{\partial y}$$

The change in isophote direction is also called the isophote curvature

$$\kappa = \frac{d\theta}{ds} = -\frac{B_{xx} B_y^2 - 2B_{xy} B_x B_y + B_{yy} B_x^2}{\left( B_x^2 + B_y^2 \right)^{3/2}} \equiv -\frac{B_{cc}}{G} = -SDCD \frac{G}{G}$$  \hspace{1cm} (A.36)

Change in gradient direction
The gradient direction $\varphi$ is defined as

$$\varphi = \arccos \left( \frac{B_x}{G} \right) = \arcsin \left( \frac{B_y}{G} \right) = \arctan \left( \frac{B_y}{B_x} \right)$$
with
\[ G = \sqrt{B_x^2 + B_y^2} \]

To differentiate along the gradient with respect to the arc length \( s \) we use the operator
\[
\frac{d}{dg} = \cos \varphi \frac{\partial}{\partial x} + \sin \varphi \frac{\partial}{\partial y} = \frac{B_x}{G} \frac{\partial}{\partial x} + \frac{B_y}{G} \frac{\partial}{\partial y}
\]

The change in gradient direction is given by
\[
\frac{d\varphi}{dg} = \frac{B_{xy} \left( B_x^2 - B_y^2 \right) - B_{xx} B_{yy} \left( B_{xx} - B_{yy} \right)}{\left( B_x^2 + B_y^2 \right)^{3/2}} \equiv -\frac{B_{sc}}{G}
\] \hspace{1cm} (A.37)

If we consider the gradient as drain direction, then \( d\varphi/dg \) denotes the change in drain direction.

**Second Order Rotation Invariance in Three Dimensions**

Let \( \mathbf{H} \) be the Hessian matrix of image \( \mathbf{B} \) and \( \mathbf{g} \) the gradient vector.

\[
\mathbf{H} = \begin{pmatrix}
B_{xx} & B_{xy} & B_{xz} \\
B_{xy} & B_{yy} & B_{yz} \\
B_{xz} & B_{yz} & B_{zz}
\end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix}
B_x \\
B_y \\
B_z
\end{pmatrix}
\]

The extension of the contour vector \( \mathbf{c} \) is a tangent plane \( \mathbf{T} \) spanned by \( \mathbf{c}_1 \) and \( \mathbf{c}_2 \).

**Principal second derivatives**

The principal second derivatives can be found by setting \( \det(\mathbf{H} - \lambda \mathbf{I}) = 0 \). This results in a cubic equation of the form
\[ x^3 + ax^2 + bx + c = 0 \]

with real coefficients \( a, b, \) and \( c \).

\[
a = -\left( B_{xx} + B_{yy} + B_{zz} \right)
\]

\[
b = B_{xx} B_{yy} + B_{xx} B_{zz} + B_{yy} B_{zz} - B_{xy}^2 - B_{xz}^2 - B_{yz}^2
\]

\[
c = B_{xx} B_{yz}^2 + B_{yy} B_{xz}^2 + B_{zz} B_{xy}^2 - 2B_{xy} B_{xz} B_{yz}
\]

To find the roots we use the variables \( Q, R, \) and \( \theta \) (Press et al. 1990)
The cubic equation has three real roots if $Q^3 - R^2 \geq 0$. The roots are

$$SD_1 = -2\sqrt{Q}\cos\left(\theta/3\right) - \frac{1}{3}a$$
$$SD_2 = -2\sqrt{Q}\cos\left((\theta + 2\pi)/3\right) - \frac{1}{3}a$$
$$SD_3 = -2\sqrt{Q}\cos\left((\theta + 4\pi)/3\right) - \frac{1}{3}a$$

(A.38)

Note that $0 \leq \theta \leq \pi/3$ and thus $SD_2 \geq SD_3 \geq SD_1$.

Mean second derivative

The mean second derivative is defined as the average of the three principal second derivatives

$$SD_{mean} = \frac{1}{3} (SD_1 + SD_2 + SD_3) = \frac{1}{3} (B_{xx} + B_{xy} + B_{zz})$$
$$\equiv \frac{1}{3} \text{Laplace}(B) \equiv \frac{1}{3} L = \frac{1}{3} \text{trace}(H)$$

(A.39)

Quadratic variation

Using the notation from above, the quadratic variation is given by

$$QV = (SD_1 - SD_{mean})^2 + (SD_2 - SD_{mean})^2 + (SD_3 - SD_{mean})^2$$
$$= \frac{3}{2} (-2\sqrt{Q})^2 = 6Q$$

(A.40)

Second-derivative magnitude

We define the second-derivative magnitude as

$$M = 3(SD_{mean})^2 + QV = \frac{1}{3} L^2 + 6Q$$
$$= \frac{1}{3} L^2 + \left(\frac{2}{3} L^2 - 2\left(B_{xx}B_{yy} + B_{xx}B_{zz} + B_{yy}B_{zz}\right) + 2\left(B_{xy}^2 + B_{xz}^2 + B_{yz}^2\right)\right)$$
$$= B_{xx}^2 + B_{yy}^2 + B_{zz}^2 + 2\left(B_{xy}^2 + B_{xz}^2 + B_{yz}^2\right)$$

(A.41)

“Gaussian” second derivative

The “Gaussian” second derivative is given by the determinant of the Hessian matrix

$$SD_{Gaussian} = SD_1 SD_2 SD_3 = \det(H)$$
$$= B_{xx}B_{yy}B_{zz} + 2B_{xy}B_{xz}B_{yz} - B_{xx}B_{yy}^2 - B_{xx}B_{zz}^2 - B_{yy}B_{zz}^2 - B_{xy}^2$$

(A.42)
Second derivative in gradient direction, SDGD

The second derivative in gradient direction is given by \((a_\phi = \mathbf{g}/|\mathbf{g}|)\)

\[
SDGD = \frac{\mathbf{g}^\top \mathbf{H} \cdot \mathbf{g}}{|\mathbf{g}|^2} = \frac{B_{xx} B_x^2 + 2 B_{xy} B_x B_y + B_{yy} B_y^2 + 2 B_{xz} B_x B_z + 2 B_{yz} B_y B_z + B_{zz} B_z^2}{B_z^2 + B_z^2 + B_z^2} \tag{A.43}
\]

### Derivatives in Gradient Direction

This section shows how to construct higher derivatives in the gradient direction in \(D\)-dimensional image space.

**Two-dimensional space**

The first derivative in gradient direction (2D) is obtained by applying the operator

\[
\frac{\partial}{\partial g} = \cos \varphi \frac{\partial}{\partial x} + \sin \varphi \frac{\partial}{\partial y} \tag{A.44}
\]

with

\[
\cos \varphi = \frac{\mathbf{g} \cdot \mathbf{x}}{|\mathbf{g}|} \quad \text{and} \quad \sin \varphi = \frac{\mathbf{g} \cdot \mathbf{y}}{|\mathbf{g}|}
\]

where \(\mathbf{g}\) denotes the gradient vector, \(\mathbf{x}\) and \(\mathbf{y}\) are the unit vectors along the \(x\) and \(y\) axes of the Cartesian grid. The \(D\)th derivative operator in gradient direction (2D) is defined as

\[
\frac{\partial^D}{\partial g^D} = \left(\cos \varphi \frac{\partial}{\partial x} + \sin \varphi \frac{\partial}{\partial y}\right)^D \tag{A.45}
\]

**D-dimensional space**

In \(D\)-dimensional space the first derivative in gradient direction is given by

\[
\frac{\partial}{\partial g}(D) = \sum_{i=1}^{D} \alpha_i \frac{\partial}{\partial r_i} \tag{A.46}
\]

with \(r_i\) the axes of a Cartesian coordinate system and the weights \(\alpha_i\) defined as

\[
\alpha_i = \frac{\mathbf{g} \cdot \mathbf{r}_i}{|\mathbf{g}|}
\]

For example, in 3D the third derivatives operator in gradient direction (TDGD) is
\[ TDGD_{3D} = \left( \sum_{i=1}^{D} \alpha_i \frac{\partial}{\partial r_i} \right)^3 \]  

(A.47)

Applied to a Gaussian filtered image \( B \) we get

\[
TDGD_{3D} (B) = B_{ggs} = \left( B_{xxx} B_x^3 + B_{yyy} B_y^3 + B_{zzz} B_z^3 + 6B_{xxy} B_x B_y B_z + 3B_{xxy} B_x^2 B_y + 3B_{xxy} B_y B_z \right) \\
+ 3B_{xxz} B_x^2 B_z + 3B_{xyc} B_y^2 B_z + 3B_{yyc} B_y^2 B_z + 3B_{zzz} B_z^2 B_x + 3B_{zzz} B_z^2 B_y \right)^{\frac{1}{2}} 
\]

(A.48)

**PLUS Operator**

The PLUS operator is defined as the sum of Laplace and SDGD (c.f. Chapter 3).

**Lemma**

Applying the PLUS operator to a \( D \)-dimensional image \( B \) can be written as

\[
PLUS (B) = \frac{1}{G} \left( \nabla \cdot (G \nabla B) \right) 
\]

(A.49)

with \( G \) the gradient magnitude and \( \nabla \) the gradient operator.

**Proof of lemma**

In \( D \)-dimensional space, \( G \) is given by

\[
G = \sqrt{\sum_{r=1}^{D} \frac{B_r^2}{G}} 
\]

(A.50)

where the subscript \( r \) denotes the derivative along the \( r \)th Cartesian coordinate.

The gradient vector \( \mathbf{g} = B \). Calculation of the inner product yields

\[
PLUS (B) = \frac{1}{G} \sum_{r=1}^{D} \frac{\partial}{\partial r} (GB_r) \\
= \frac{1}{G} \sum_{r=1}^{D} \left( GB_{rr} + \frac{B_r}{G} \left( B_r B_{rr} + \sum_{q=1, q \neq r}^{D} B_q B_{rq} \right) \right) \\
= \frac{1}{G} \sum_{r=1}^{D} GB_r + \frac{1}{G^2} \sum_{r=1}^{D} \sum_{q=1}^{D} B_r B_q B_{rq} \\
= \text{Laplace} + \text{SDGD}
\]

□