Chapter 8

Euler Number and 3D Length

Existing estimators for curve length in 3D are applied to a binary representation of a curve. In this paper we estimate the 3D curve length through grey-volume measurements. Line like objects of constant intensity are transformed into volumes proportional to their length by applying the second derivative in the gradient direction (SDGD). To guarantee a constant contrast along the curve we first apply a “soft-clipping” operation to the linear region of the slope around the line. This technique measures curve length in 3D and \((D-2)\)-dimensional hyper-length in \(D\)-dimensional hyper-space.

A slight modification of the method allows the enumeration of simply-connected closed objects in two dimensions. For arbitrary objects (various topologies) in two dimensions the measure yields the 2D Euler number. An alternative method to estimate Euler numbers of 2D and 3D surface models is based upon results from differential geometry. The contour/surface integral of the Gaussian curvature is invariant under translation, rotation, scaling, and local stretching of the object. After normalization it produces the Euler numbers.
8.1. Introduction

Length measurement of space curves in 3D is a difficult task. For length measurements in 3D there exist two different approaches.

The first method is an extension of the 2D chain-code based length estimators. A digital straight line consisting of \( N \) string elements is decomposed into three types of transitions: grid parallel, square diagonal, and cube diagonal. The number of elements in each class (\( n_{\text{grid parallel}} \), \( n_{\text{square diagonal}} \), \( n_{\text{cube diagonal}} \)) are scaled using predetermined weight coefficients (\( \alpha, \beta, \gamma \)).

\[
L = \alpha n_{\text{grid parallel}} + \beta n_{\text{square diagonal}} + \gamma n_{\text{cube diagonal}}
\]  

(8.1)

Various authors have published optimal weights derived using different optimization criteria. Beckers & Smeulders (Beckers and Smeulders 1992) have minimized the mean square error for a string of length \( N \) randomly oriented lines in three-dimensional space. Verwer (Verwer 1991) published recipes and results for minimizing the mean square error or the maximum absolute error for randomly oriented line segments of the same Euclidean length. Kiryati and Kübler (Kiryati and Kübler 1992) calculated the optimal weights starting from a digitized line. Using conditional probabilities they came up with a mean square error slightly larger than Verwer’s. Applying a string substitution to the “chain-code” elements that replaces two elements (a grid parallel and a square diagonal) by a single element (cube diagonal) yields the result published by Verwer.

None of the above techniques were designed for measuring the length of an arbitrary space curve. It is beyond the scope of this thesis to test which set of weights comes out best.

The second technique is based upon techniques from stereology. For these techniques to work, one often needs a minimum number of curves. The length is measured by counting the intersections between the projection of the original curve on a plane and some equi-distantly spaced straight lines in that plane (Gokhale 1986; Cruz-Orive and Howard 1990).

Our new method shows some similarities with earlier work on estimating 2D edge length and 3D surface area (Verbeek and van Vliet 1992 a; Verbeek and van Vliet 1993 a). This method can be extended to measure hyper-length in hyper-space. To allow comparison with binary length estimators we will also measure the length contribution of randomly oriented straight lines per unit length. In contrast with the other techniques we do not apply our operator on a binary representation of the curve. We believe that the thresholding, which normally produces the binary object, destroys valuable information.
Applying this method to 2D images containing closed objects (not necessarily convex) we obtain the number of objects present in the image. For arbitrarily shaped objects we find the global Euler number (total number of objects minus the total number of holes inside these objects) of the input image.

8.2. Length Estimation through Grey-Volume Measurement

A global sampling-error free measure is the volume under the grey-scale landscape, called grey-volume. In theorem 2.1 (Chapter 2) we have shown that the sum of the samples (grey-volume) is directly proportional to the volume under the grey-scale landscape for all bandlimited signals sampled at higher than half the Nyquist rate.

To transform a 3D object into a grey-volume proportional to its length, the object must be of constant grey level. This can be accomplished by applying erf-clipping (c.f. Chapter 2, section 2.6) to the linear region of the slope around the line. To solve the isophote selection problem as well as problems associated with shading, the erf-clipping can be applied around the zero-crossing of a second derivative image.

We wish to measure the length of a space curve through a grey-volume measurement. Assume that we have a 3D object of constant grey value \( H \) and whose thickness does not change rapidly such as a cylinder of radius \( R \) and length \( L \). The grey-volume of the cylinder equals \( H\pi R^2 L \). To transform this volume into another volume proportional to \( L \) requires a second derivative perpendicular to the centerline of the cylinder, thus in the radial direction. The second derivative in the gradient direction (SDGD) accomplishes this.

\[
\text{grey-volume} \left( \text{SDGD} \left( B_{\text{cylinder}} \right) \right) = \iiint_{\text{image}} \text{SDGD} \left( B_{\text{cylinder}} \right) dx dy dz = H2\pi L \quad (8.2)
\]

According to the equation above, this length measure is independent of the diameter \((2R)\) of the cylinder.

**Lemma 8.1**
The length of a cylinder with diameter \( R \) and length \( L \) is given by

\[
L = \frac{1}{2\pi H} \iiint_{\text{image}} \text{SDGD} \left( B \right) dx dy dz \quad (8.3)
\]
with $B$ the grey-value image containing the cylinder.

**Proof of lemma 8.1**

Assume an edge profile in cylindrical coordinates $B(r, \varphi)$

$$B(r, \varphi) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \sigma}} \exp\left(-\left(\frac{\xi - R}{2\sigma^2}\right)^2\right) d\xi$$  \hspace{1cm} (8.4)

where $H=1$. The SDGD of image $B$ is the second derivative in the radial direction.

$$SDGD(B) = \frac{\partial^2 B}{\partial r^2} = \frac{r-R}{\sqrt{2\pi \sigma^3}} \exp\left(-\left(\frac{r-R}{2\sigma^2}\right)^2\right)$$  \hspace{1cm} (8.5)

Using eqs. (8.3, 8.4, 8.5) the length of the cylinder is

$$L = \frac{1}{2\pi} \iiint_{\text{image}} SDGD(B) dxdydz = \frac{1}{2\pi} \int_0^L 2\pi \int_0^\infty SDGD(B) r dr dz$$

$$= \int_0^L \left[ \int_0^\infty \exp\left(-\left(\frac{r-R}{2\sigma^2}\right)^2\left(\frac{r}{\sigma}\right)^2\right) d\left(\frac{r}{\sigma}\right) \right] dz$$

$$= \int_0^L dz = L$$

The integral between square brackets is independent of the diameter $R$ and the edge-slope parameter $\sigma$. This result remains valid for any linear combination of “Gaussian” edge slopes of different $\sigma$ and $R$. \hfill \Box

The lemma shows that we can measure the length of spaghetti (not necessarily of constant width). The length of macaroni, however, remains an unsolved problem. The inside tube gives a negative contribution producing a total length of macaroni equal to zero. This is explained by the theory on Euler numbers in the next section.

For a sphere of radius $R$, the method results in

$$L_{\text{sphere}} = \frac{1}{2\pi H} \iiint_{\text{image}} SDGD(B_{\text{sphere}}) dxdydz = 4R$$  \hspace{1cm} (8.6)

The obtained length estimate is exactly twice the diameter of the sphere. This is due to the fact that, at all positions on the surface, the principal curvatures $\kappa_1$ and $\kappa_2$ contribute to the final result. This also means that the type of end-points alters the length estimate of strongly elongated objects of finite length. For a cylinder of radius $R$ and length $L$ topped at both ends by half a sphere this method produces
as length estimate $L+4R$. In other words the length estimate adds up the total length $L+2R$ and the diameter $2R$.

Extension to higher dimensions yields the $(D-2)$-dimensional hyper-length of a $D$-dimensional cylinder in $D$-dimensional space. The $(D-2)$-dimensional hyper-length is given by

$$L_{(D-2)}(I) = \frac{1}{H} \left( \frac{\partial^2 \left( V(D) r^D \right)}{\partial r^2} \right)_{r=1}^{-1} \int \cdots \int_{\text{image}} \frac{\partial^2 B}{\partial y^2} \, dx_1 dx_2 \cdots dx_D \quad (8.7)$$

where $V(D) r^D$ is the volume of a $D$-dimensional hyper-sphere of radius $r$ (c.f. Appendix VI: Pillball FT) and $V(D)$ is given by

$$V(D) = \frac{\pi^{D/2}}{\Gamma\left(\frac{D}{2}+1\right)}$$

Note that the factor before the integral of eq. (8.7) merely takes care of normalization.

### 8.3. Object Enumeration and the Euler Number

The previous section introduces a method for measuring the length of space curves (or any other object) in 3D. It also shows that it is trivial to extend the method to higher dimensions. This section deals with the extension to a lower dimension.

#### 8.3.1. Euler number in two-dimensional space

Extension of the method to 2D will result in a $2\pi$ contribution for each closed object. This allows us to count the number of objects in the image. For arbitrarily shaped objects we get the global Euler number (the total number of objects minus the total number of holes inside these objects)

$$n_{Euler}^{2D}(B) = \frac{1}{2\pi H} \int_{\text{image}} SDGD(B) \, dx dy \quad (8.8)$$

which characterizes the topology of the 2D object. The integrated second derivative in the gradient direction yields a $2\pi$ contribution for every SCC object. A hole yields a contribution of $-2\pi$.

The two-dimensional Euler number is defined as the number of connected components ($c$) minus the number of holes ($h$) (Hilbert and Cohn-Vossen 1932). For polygonal networks there exists a relationship, called the Euler formula.
\[ n_{\text{Euler} \ 2D} = c - h = v - e + f \]  
(8.9)

with \( v \) the number of vertices, \( e \) the number of edges, and \( f \) the number of faces of the polygonal description of the object. In this chapter \( v - e + f \) is called the Euler number.

An alternative way to estimate the Euler number from a grey-scale image is based upon theory from Differential Geometry (Stoker 1969). Hopf (Hopf 1935) proved that the total curvature of a positively oriented regular Jordan curve is \( 2\pi \).

\[ \int \kappa \, dl = 2\pi \]  
(8.10)

This measure can also be applied to properly sampled grey-scale images. When the curvature \( \kappa \) in eq. (8.10) is set to 1, the formula reduces to the 2D edge length estimator as described in chapter 7. Multiplication of the grey-volume proportional to the edge length (GCL) by the curvature information yields the 2D Euler number

\[ n_{\text{Euler} \ 2D} = \frac{1}{2\pi H} \int \kappa(x, y) \, GCL(x, y) \, dx \, dy \]  
(8.11)

Comparing eqs. (8.8) and (8.11) we conclude that the first method of eq. (8.8) is easier to compute than the one of eq. (8.11).

### 8.3.2. Euler number in three-dimensional space

In the literature, the 3D Euler number is used as a topology test in 3D skeletonization algorithms (Lobregt et al. 1980; Toriwaki et al. 1982). For a polygonal subdivision of the surface of a single simply-connected closed (SCC) 3D object, Euler’s formula (Hilbert and Cohn-Vossen 1932) states

\[ n_{\text{Euler} \ 3D} = v - e + f = 2 \]  
(8.12)

This means that \( n \) SCC objects produce a total Euler number equal to \( 2n \). A similar result exist for more general surfaces

\[ n_{\text{Euler} \ 3D} = v - e + f = 2c - 2t \]  
(8.13)

where \( t \) denotes the number of tunnels (also called handles). In general, the Euler number \( v - e + f \) characterizes the topology of a surface. Unfortunately, the Euler relations do not distinguish between surfaces from distinct objects and surfaces originating from the same object (such as an object with a hole \( h \)).
Similar to the two-dimensional Euler numbers we would like to extend our methods (eqs. (8.8,8.11) to three dimensions. A natural extension of eq. 8.8 to $D$-dimensional space would be

$$n_{Euler}^{D} (B) = \frac{1}{H} \left( \frac{\partial^D (V (D) r^D)}{\partial r^D} \right) \int_{r=1}^{...} \int_{\text{image}} \frac{\partial^D B}{\partial g^D} dx_1 dx_2 ... dx_D$$

(8.14)

which is indeed dimensionless. However, the above formula is not independent of the edge slope of the object and therefore useless as the $D$-dimensional “Euler” number.

The second method of eq. (8.8) can be extended easily to three-dimensional space. The Gauss-Bonnet formula applied to closed surfaces is related to the 3D Euler number (Stoker 1969) as defined in eq. (8.13) as follows

$$\int \int_{\text{surface}} \kappa_{\text{Gaussian}} dA = 2\pi (v - e + f) \equiv 2\pi n_{Euler}^{3D}$$

(8.15)

with $\kappa_{\text{Gaussian}} = \kappa_1 \kappa_2$. Again, using the principal curvature images (c.f. Chapter 5) and the grey-volume proportional to the surface area (c.f. Chapter 7) we estimate the 3D Euler number as follows

$$n_{Euler}^{3D} = \frac{1}{2\pi H} \int \int \int_{\text{image}} \kappa_1 (x, y, z) \kappa_2 (x, y, z) GCL (x, y, z) dx dy dz$$

(8.16)

**Handles versus holes**

In two-dimensions, the topology of an object is determined by the number of holes inside the object. In three dimensions, the Euler number corresponds to the topology of a closed surface rather than the topology of an object. Whereas in surface models the topology is determined by the number of tunnels (such as the handle of a coffee cup), in volume based models we also encounter holes (such as the air-filled cavities in cheese before slicing). Although the outer and inner surfaces do not intersect, they are connected to each other through the bulk of the object. Unfortunately, this distinction cannot be made by the 3D Euler number as defined in eqs. (8.13, 8.15, 8.16).
8.4. Effects of Low-pass Filtering

Low-pass filtering of an image displaces the object boundary towards smaller object radii (c.f. Chapter 4). Fortunately, the length measurements on a space cylinder are independent of the diameter of the cylinder and therefore independent of the size of the low-pass filter. More generally, all surface patches in 3D images with only one non-zero principal curvature produce an unbiased length contribution. At the spherical ends the contributions do depend on the edge radius $R$. For spheres we may expect an under-estimate due to these effects. The relative bias will be exactly the same as the relative shift of the edge position as described in part II of this thesis (c.f. Chapter 3 and 4).

8.5. Experiments

The section describes some experiments to test the presented theory and compare the performance of our length estimator with an existing method.

8.5.1. Straight lines in three dimensions

Although the method is designed to estimate the length of arbitrary space curves (cooked spaghetti) we have tested it on 1000 randomly oriented cylinders (uncooked spaghetti). The test images contained bandlimited cylinders of radius 20 sampled at the Nyquist rate. The SDGD used a built-in $\sigma$ of 1.35. The length contributions per unit length can be compared with the Verwer’s (Verwer 1991) 3x3x3 chamfer method (c.f. table 8.1). Both methods produce unbiased length measurements. The other errors (coefficient-of-variation, maximum error, and minimum error) are very different. As expected, the orientation dependency is negligible compared to the binary chamfer method.

<table>
<thead>
<tr>
<th>Error</th>
<th>3D chamfer method</th>
<th>integrated SDGD</th>
</tr>
</thead>
<tbody>
<tr>
<td>bias</td>
<td>0.00 %</td>
<td>0.00 %</td>
</tr>
<tr>
<td>CV</td>
<td>2.30 %</td>
<td>2.1 10^{-3} %</td>
</tr>
<tr>
<td>maximum error</td>
<td>2.95 %</td>
<td>4.7 10^{-3} %</td>
</tr>
<tr>
<td>minimum error</td>
<td>-10.60 %</td>
<td>-3.2 10^{-3} %</td>
</tr>
</tbody>
</table>

Variances in length contribution due to the width and shape of a cross section perpendicular to the line are similar to the size and shape dependency of Euler numbers in two dimensions.
8.5.2. Length of a sphere
According to the theory, the length of a sphere in three dimensions should be four times the radius $4R$. Due to the edge displacement, we will expect an underestimate proportional to the zero-crossing error of the 3D SDGD. The length estimator applied to 100 randomly positioned spheres of various radii ($10 \leq R \leq 45$) showed that the bias indeed behaves as predicted. The length is underestimated by $\sigma^2/R^2 \times 100\%$. The CV is on the order of $10^{-2}\%$.

8.5.3. Euler Numbers in two dimensions
We have distinguished two methods for estimating 2D Euler numbers. To test the theory for the Euler number in two dimensions, we apply the methods to objects of different topology:
- simply-connected closed (SCC) objects (convex as well as concave boundaries),
- objects containing holes.

For the method to work, the objects need to fulfill two requirements:
- constant grey level,
- the smallest strip of either object or background needs to be larger than the support of the filter (built-in $\sigma$ of the derivative-of-Gaussian filters).

The first requirement can be satisfied by erf-clipping as described in section 2.6 of chapter 2. The resulting Euler numbers will be related to the images after rescaling and not so much to an analog property of the underlying analog object. To produce a sampling-error-free estimate of the Euler number, the zero-frequency should stay free of aliasing. The SDGD filter that transforms the grey-scale landscape into a grey-volume proportional to the Euler number needs Gaussian derivatives of size $\sigma \approx \frac{1}{2} \times 2.7 = 1.35$.

The second requirement applies to the images after erf-clipping. A small Gaussian $\sigma$ in the SDGD filter is preferred. To satisfy the second requirement, the sampling density should be high enough.

Method 1, eq. (8.8): Integrated SDGD
We have tested the integrated SDGD method on a wide range of randomly oriented convex and concave test images (10 realizations for each object). All objects have a smallest diameter larger than 10 pixels. The SDGD is constructed using Gaussian derivatives of size $\sigma = 1.5$. We measured a positive bias of 0.01 per SCC object ($\approx 1\%$) and of coarse an equal negative bias of 0.01 per hole ($\approx -1\%$). The coefficient-of-variation (CV) over all test objects is very small $10^{-2}\%$.
Method 2, eq. (8.11): Integrated curvature
For the same set of objects we measured a bias smaller than 0.1% and a CV smaller than 10^{-2}%. Here the curvature estimation uses Gaussian derivatives of size $\sigma = 2.7$.

Application: object counting
Applying the SDGD with $\sigma_{SDGD} = 1.0$ to an image of gold particles embedded in glass. After elimination of all objects connected to the image borders and all objects smaller than 100 pixels in area we measured their Euler numbers after erf-clipping. First, we measured the Euler number of individual objects (one object per image). The statistics are listed in table 8.2. Secondly we measured the global Euler of all objects in a single image. The small discrepancy between the sum of “individual” Euler numbers and the global Euler number is due to the spatial arrangement of objects in the image. The distance between some objects may be smaller than the support of the SDGD filter, producing a slightly lower Euler number.

Table 8.2: Statistics on the Euler numbers of an image containing 41 SCC objects with a minimum area of 100 pixels.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>minimum Euler number</td>
<td>0.991</td>
</tr>
<tr>
<td>maximum Euler number</td>
<td>1.004</td>
</tr>
<tr>
<td>average Euler number</td>
<td>0.9985</td>
</tr>
<tr>
<td>standard deviation</td>
<td>0.00245</td>
</tr>
<tr>
<td>sum of Euler numbers</td>
<td>40.9385</td>
</tr>
<tr>
<td>global Euler number</td>
<td>40.900</td>
</tr>
</tbody>
</table>

8.5.4. Euler Numbers in three dimensions
To test the theory on 3D Euler numbers we applied them to a set of objects such as spheres, ellipsoids, halters, and a donut. The test for objects with holes can be derived from the single object cases if there is enough distance between the outer surface of the object and the inner surface (around the hole). We have applied the method with a $\sigma_{\kappa} = 3.8$ (c.f. Chapter 9) to objects with a Gaussian smoothed edge profile of $\sigma_{\text{slope}} = 1.0$. The GCL method to construct the grey-volume proportional to the surface area uses a $\sigma_{\text{Laplace}} = 1.5$. For 10 realizations (random placement) of the same type of object the coefficient-of-variation (CV) is around $10^{-2}\%$ to $10^{-3}\%$. The bias terms are given in table 8.3. The bias of the measures depends on the smallest edge radius for each object. For edge radii larger than 20 pixels, the bias is around $10^{-2}\%$. For smaller edge radii (down to $3 \times 3.8 = 11.4$ pixels) the bias increases to around 1%, depending on the shape and topology of the object.
In this chapter we estimate the length of 3D grey-value lines through grey-volume measurements. The method requires constant line intensity along the curve. This can be accomplished by applying erf-clipping to the slopes along the lines. The proposed method is independent of the line diameter and hence insensitive to the clipping level. The length contribution is virtually independent of the orientation of the line. After bias correction the resulting CV is $2.1 \times 10^{-3}$ %, the maximum error is $4.7 \times 10^{-3}$ % and minimum error is $-3.1 \times 10^{-3}$ %. All errors are three orders of a magnitude smaller than the ones reported by Verwer (Verwer 1991) for the 3x3x3 chamfer method.

We have proposed and tested two methods for estimating Euler numbers in two dimensions. The first is the extension of the 3D length estimator to two-dimensions. The integrated second derivative in the gradient direction yields a $2\pi$ contribution for every SCC object. A hole yields a contribution of $-2\pi$. The bias is around 1%. The second method is derived from the field of differential geometry. The integrated curvature of SCC objects always produce $2\pi$ per revolution. Around holes the curvature has an opposite sign and results in an integrated curvature of $-2\pi$. This method has a bias of 0.1%. A smaller bias in exchange for a more complex algorithm.

In three dimensions we have only one method. From the field of differential geometry we know that the surface integral of the Gaussian curvature yields $4\pi$ for SCC objects. Each tunnel (handle) through the object contributes a factor of $-4\pi$. Unfortunately, we cannot differentiate between surfaces around objects and holes. The bias is again smaller than 0.1%.