Sudden collapse
Shells are very efficient in carrying load. However, this efficiency comes at a price.
If a shell buckles, it collapses with a bang. There will be no warning and it will collapse faster than we can run.
Truss, frame and plate structures do not have this problem. Usually, they slowly deform a lot before collapsing and therefore they give clear warnings to evacuate the area.
Consequently, shells need to be extra safe. In other words, for shells we often use larger load factors and material factors than for most structures. In the eurocode this is organised in consequence classes. Often, the highest consequence class is appropriate.

Tucker High School
On September 14, 1970, the gymnasium of The Tucker High School in, Henrico County, Virginia, collapsed completely [1]. Some school children were injured but fortunately there was no loss of life. The structure was a four element **hypar** with a plan of 47.2 m by 49.4 m (fig. 1). It had a **sagitta** of about 4.6 m, large inclined supporting ribs and centre ribs that were essentially concentric with the shell. The shell was 90 mm thick for the most part. Therefore, it had a ratio $\frac{a}{t} = \frac{47.2/2 \times 49.4/2}{4.6 \times 0.090}$ = 1400.
The failure was due to progressive deflection. The lightweight concrete showed much creep [1]. Three other similar structures were subsequently demolished. One of these had a deflection of 460 mm at the centre. Research showed that the collapse could have been simply prevented by cambering upward the centre point of the shell [1].

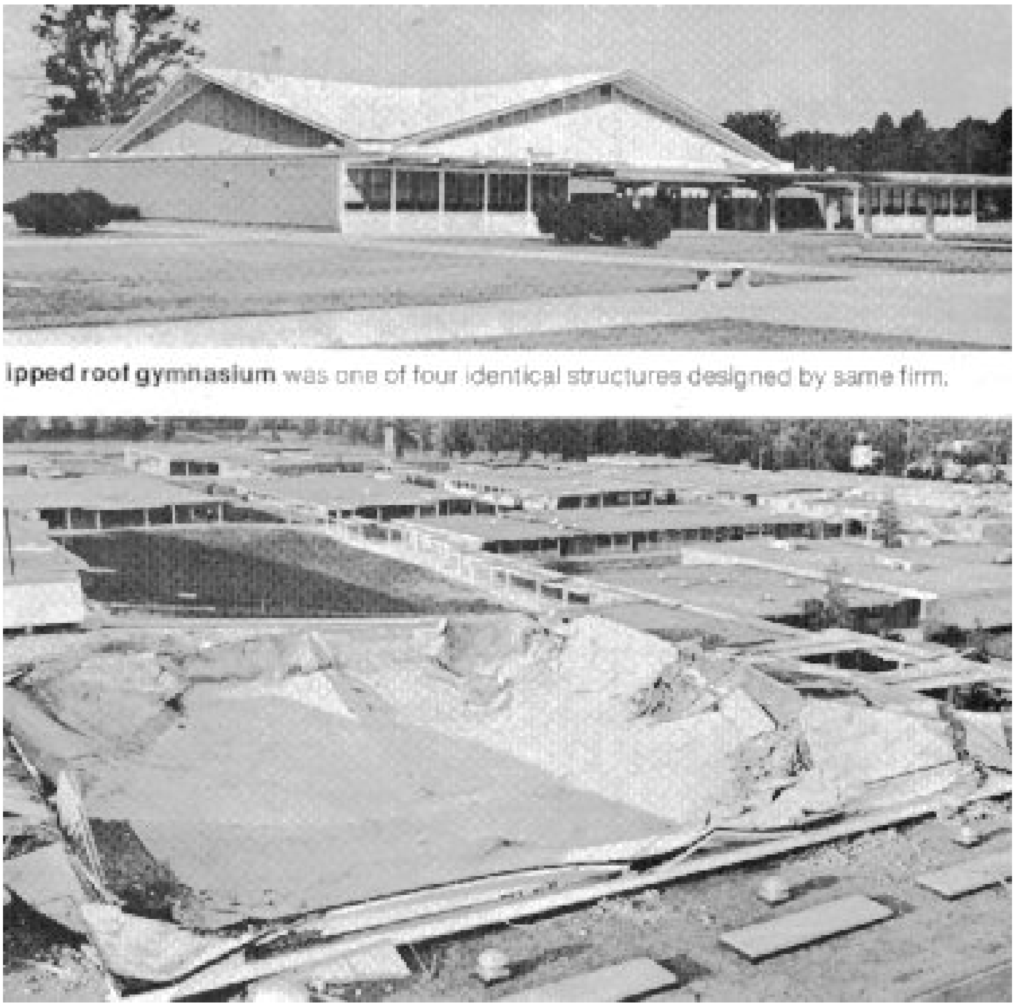
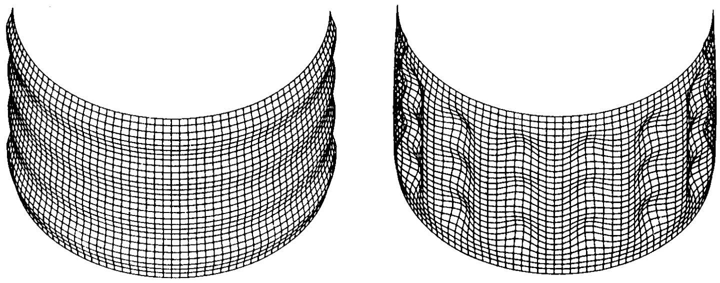


Figure 1. Newspaper photograph of the collapsed hypar shell [1]

Cylinder buckling shapes

The buckling shape of an axially loaded cylinder can be a ring pattern or a chess board pattern (fig. 2). Which one occurs depends on the shell thickness and its radius. When buckling progresses the ring pattern can transform into the chess board pattern. These deformations are very small and usually not visible. When the material starts to deform plastically the ring pattern develops into an elephant foot (fig. 3); the chess board pattern develops into a Yoshimura[1] pattern (fig. 4), which are clearly visible.



Ring pattern *Chess board pattern*

Figure 2. Buckling modes of axially compressed cylinders computed by the finite element method (The deformation is enlarged to make it visible.)

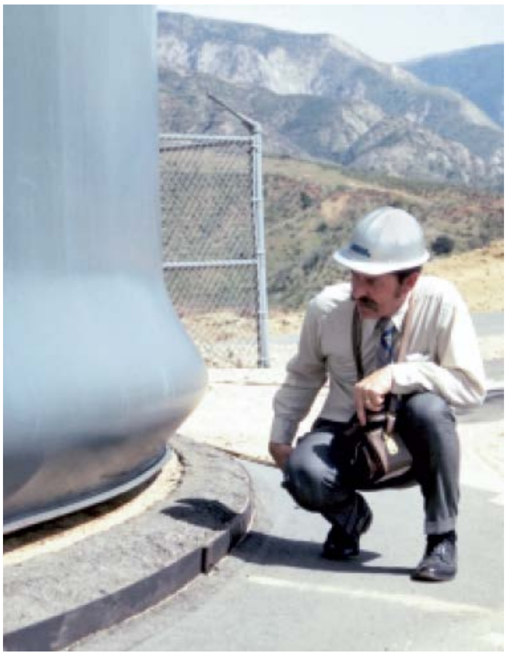


Figure 3. Elephant foot buckling of a tank wall [5]

Figure 4. Yoshimura buckling of an aluminium cylinder

Exercise: The Yoshimura pattern can be obtained as an origami exercise. You can do this exercise. Take a sheet of paper and draw the lines of figure 5 on it. Fold all horizontal lines towards you and all diagonal lines away from you. When all folds are made the sheet tends to curve away from you. Curve the sheet further and close it with sticky tape.

---

[1] Yoshimura Yoshimaru (吉村 慶丸) (19..-19..) was a professor of applied mechanics at Tokyo University of Technology. Nine years after the second world war, he was invited to the USA and work on shell structures. There, he wrote a report [6] which explained the buckling shape that was often observed in cylinder experiments. Unfortunately for many of us, most of his other publications are in Japanese.

Remarkable about the Yoshimura pattern is that it is in-extensional (see **in-extensional deformation**). Fortunately, large extensions are needed to transform a cylinder directly into a Yoshimura pattern [3]. You can try this too: Take a sheet of paper, curve it into a cylinder shape and close it with sticky tape. Then load the cylinder axially until it buckles. If the cylinder and the load are nearly perfect then the cylinder deforms into a Yoshimura pattern. Clearly, reality is not perfect. Nevertheless, several Yoshimura buckles can be recognised in the overloaded cylinder.

Sheet of paper *Yoshimura pattern* *Buckled paper cylinder*

Figure 5. Origami exercise

Buckling of a beam supported by springs

A beam supported by evenly distributed springs (fig. 6) can be a good way to understand the behaviour of shell structures. The bending stiffness of the beam is *EI* [Nm²]. The stiffness of the distributed springs is *k* [N/m²]. The beam is loaded by an axial force *P* [kN]. The differential equation that describes this beam is

$$EI\frac{d^4w}{dx^4}+P\frac{d^2w}{dx^2}+k\,w=0\,.$$

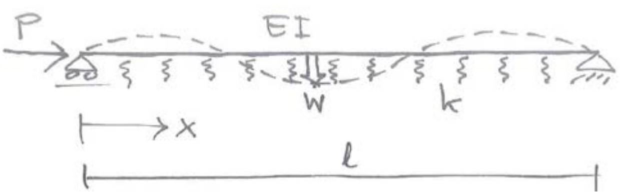


Figure 6. Elastic beam supported by distributed springs

The following buckling shape is proposed

$$w=b\sin\frac{n\pi x}{l},$$

where *n* is the number of half waves of the buckled shape. Substitution of the buckling shape into the differential equation gives the following solution.

```
> w:=b*sin(n*Pi*x/l):
```

```
> eq:=EI*diff(w,x,x,x,x)+P*diff(w,x,x)+k*w=0:
> Pcr:=expand(solve(eq,P));
```

$$P_{cr} = \frac{n^2\pi^2 EI}{l^2} + \frac{kl^2}{n^2\pi^2}$$

This solution is plotted in figure 7 in dimension less quantities. It shows that for long beams a good approximation is

$$P_{cr} = 2\sqrt{k\,EI}\ .$$

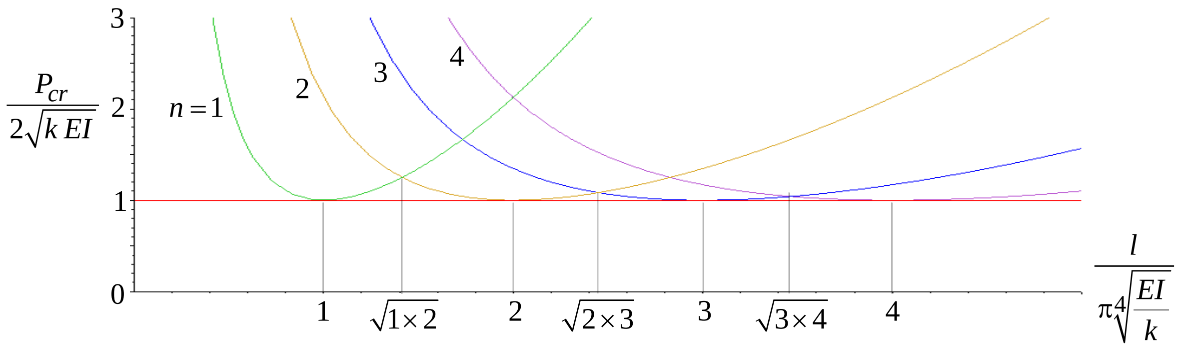


Figure 7. Buckling load as a function of the beam length

Ring buckling of an axially compressed cylinder
Consider an circular cylinder (fig. 8).

$$k_{xx} = 0, \quad k_{yy} = \frac{1}{a}, \quad k_{xy} = 0, \quad \alpha_x = \alpha_y = 1, \quad 0 \le u \le l, \quad 0 \le v \le 2\pi a$$

Somebody proposes the following deformation.

$$u_x = \frac{\nu}{a}\int w(u)\,du, \quad u_y = 0, \quad u_z = w(u)\,.$$

This deformation is axial symmetric and depends on an unknown function *w*. Please note the difference between ν (Poisson's ratio) and *v* (curvilinear coordinate).

Substitution in the 21 **Sanders-Koiter equations** gives

$$\frac{E\,t^3}{12(1-\nu^2)}\frac{d^4 w}{dx^4} + \frac{E\,t}{a^2}w = n_{xx}\frac{d^2 w}{dx^2}\,.$$

This is the same differential equation as that of **buckling of a beam supported by springs**. Apparently we can make the following interpretations.

$$\frac{Et^3}{12(1-v^2)} = EI\,, \quad \frac{Et}{a^2} = k, \quad n_{xx} = -P$$

Using this analogy, the buckling load of a not short cylinder is calculated as

$$n_{cr} = -2\sqrt{k\,EI} = \frac{-1}{\sqrt{3(1-\nu^2)}}\frac{Et^2}{a} \approx -0.6\frac{Et^2}{a}$$

and the buckling length is

$$l_{cr} = \pi\sqrt[4]{\frac{EI}{k}} = \frac{\pi\sqrt{at}}{\sqrt[4]{12(1-\nu^2)}} \approx 1.7\sqrt{a\,t}\ .$$

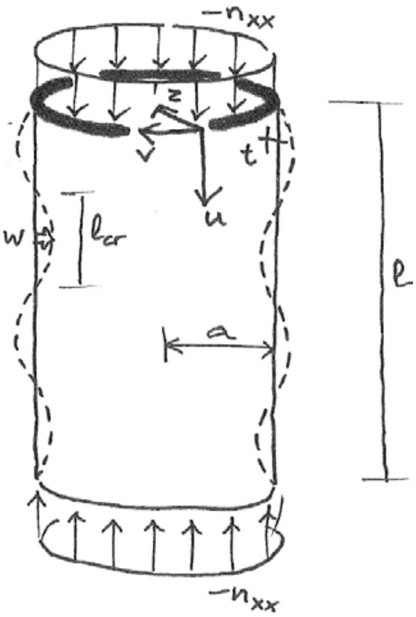


Figure 8. Cylinder coordinate system

Differential equation for shell buckling
The structural behaviour of shells including large displacements is described by an eight order differential equation [2] (see **Reduced differential equation**)

$$\frac{E\,t^3}{12(1-\nu^2)}\nabla^2\nabla^2\nabla^2\nabla^2 u_z + E\,t\,\Gamma\Gamma u_z = \nabla^2\nabla^2(\,p_z + n_{xx}\frac{\partial^2 u_z}{\partial x^2} + 2n_{xy}\frac{\partial^2 u_z}{\partial x \partial y} + n_{yy}\frac{\partial^2 u_z}{\partial y^2})\ .$$

The differential equation can be solved analytically for elementary shell shapes and elementary loading. The buckling loads thus obtained are called critical loads. In Table 1, *E* is Young's modulus, ν is Poisson's ratio, *t* is the shell thickness and *a* is the radius of the middle surface of the shell. The scientists who made significant contributions are Rudolf Lorenz [3], Stephen Timoshenko [4], Richard Southwell [5], Richard von Mises […], Wilhelm Flügge [1932], Lloyd Donnell [1934] [2].

[2] Rudolf Lourenz (18..-19..) was a civil engineer in Dortmund, Germany [..].
Richard Southwell (1888-1970) was a mathematician and engineer. He taught at the University of Cambridge, Oxford and Imperial College London [wikipedia]
Richard von Mises (1883-1953) was a professor of applied mathematics in Straßburg, Dresden, Berlin, Istanbul and Cambridge (Harvard) [wikipedia]
Stephen Timoshenko (1878-1972) was a professor at Kyiv Polytechnic Institute, Ukraine, University of Michigan and Stanford University, USA [wikipedia]
Wilhelm Flügge (1904-1990) was a German engineer. After the second world war he moved to the USA and became professor at Stanford University [German wikipedia]

Critical membrane force
In table 1 we observe that most shell shapes buckle at a membrane force of

$$n_{cr} = \frac{-1}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a}$$

The exceptions occur due to in-extensional deformation and due to shear stresses. It is not a big step to assume that this formula is valid for shells of any shape. For realistic values of ν the formula can be approximated to

$$\boxed{n_{cr} = -0.6 \frac{Et^2}{a}}$$

Clearly, the principal membrane forces need to be larger than the critical membrane force.

$$n_2 \geq n_{cr}$$

The radius *a* needs to be measured in a section perpendicular to the direction of n_2.

Table 1. Critical loading and critical membrane forces of elementary shells

	Critical loading p_{cr} [N/m²]	Critical membrane force n_{cr} [N/m]	Imperfection sensitive
Open cylinder, radially loaded (in-extensional deformation)	$\frac{1}{4(1-\nu^2)} \frac{Et^3}{a^3}$	$\frac{-1}{4(1-\nu^2)} \frac{Et^3}{a^2}$	no
Open cylinder, axially loaded		$\frac{-1}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a}$	yes
Open cylinder, torsion loading		$\frac{1}{3\sqrt{2}(1-\nu^2)^{\frac{3}{4}}} E\sqrt{\frac{t^5}{a^3}}$ shear force	no
Hyperboloid, axially loaded (cooling tower)		$\frac{-1}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a}$	no
Closed cylinder, loaded in all directions	$\frac{2}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a^2}$	$\frac{-1}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a}$ hoop direction	yes
Sphere	$\frac{2}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a^2}$	$\frac{-1}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a}$	yes
Dome base radius $> 3.8\sqrt{at}$	$\frac{2}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a^2}$	$\frac{-1}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a}$	yes
Hypar	…	…	no

Lloyd Donnell (1895-1997) was an American engineer, professor at Illinois Institute of Technology and Stanford University [wikipedia]

Imperfection sensitivity
Before 1930, airplanes consisted of frames covered with a fabric which was painted. However, engineers wanted to build airplanes from aluminium plates that were joint to form a cylindrical shape. Therefore, scientists started to do experiments on cylinders [Andrew Robertson 1928]. Figure 9 shows the ultimate loads of axially compressed aluminium cylinders. They are much smaller than the critical load. This is caused by invisible shape imperfections. At first sight, imperfection sensitivity is hard to believe because the experiments were performed very carefully. The aluminium cylinders had perfectly cut edges and were beautifully polished. The cylinders were perfectly centred in the testing machines. The testing machines were modern and very accurate measuring instruments were used. Nonetheless, the ultimate loads were much smaller than the critical loads. Not only compressed cylinders but also bend cylinders and radially compressed domes are very sensitive to imperfections.

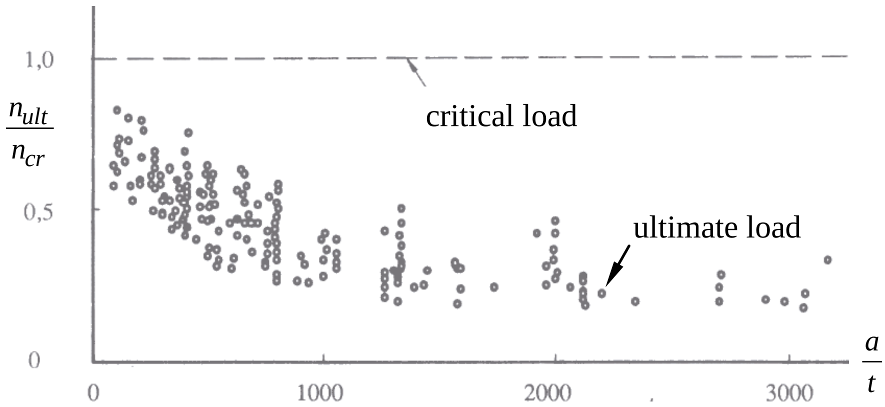
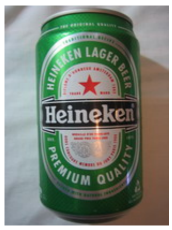


Figure 9. Experimental ultimate loads of 172 axially loaded aluminium cylinders [6]

Experiment
What is the ultimate load of an axially loaded empty beer can? We model the can as an open cylinder. The wall thickness is 0.08 mm the radius is 32.8 mm, Yong's modulus is 2.1e5 N/mm² and Poisson's ratio is 0.35. According to Table 1 the critical loading is



$$n_{cr} = -0.6 \frac{Et^2}{a} = -0.6 \frac{2.1 \times 10^5 \times 0.08^2}{32.8} = -25.3 \text{ N/mm}$$

$$F_{cr} = 2\pi a \, n_{cr} = 2 \times 3.14 \times 32.8 \times (-25.3) = -5200 \text{ N}$$

Therefore, it should be able to carry a mass of 520 kg. Carefully stand on the can and it will – probably – carry your weight. Subsequently, use your thumbs to push a dimple in the can and push it out again. Doing so makes typical clicking sounds. Notice that the imperfections you made are hardly visible. Now, try standing on the can again. It will collapse abruptly. The explanation is imperfection sensitivity.

Puzzle
The large difference between the theoretical buckling load (critical load) and the experimental buckling load (ultimate load) puzzled scientists for approximately 10 years. Is the differential equation wrong? Are the solutions to the differential equation wrong? Are there more solutions that we have not found? Is there some mistake the experimental set up? Has thin aluminium less strength than solid aluminium?

The solution was discovered by Theodore von Kármán and Qian Xuesen (钱 学 森 pronounce tsien? sue? sen) in 1940 [7, 7b].[3] They calculated the load-displacement curve after buckling. Figure 10a shows the result of their calculation; n_{xx} is the membrane force in a cylinder and w is the shortening of the cylinder. Note that load on a perfect cylinder can be increased until the critical load after which the strength will drop strongly. This behaviour is typical for shell structures and very different from other structures. Figure 10b shows that very small shape imperfections cause the ultimate load to be much smaller than the critical load.

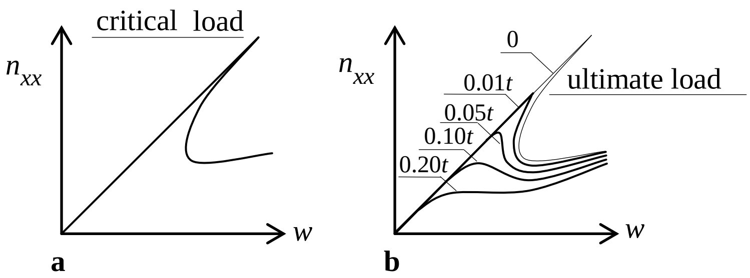


Figure 10. Buckling of cylinders for different shape imperfection amplitudes [6]

Exceptions to imperfection sensitivity
Some shells are not sensitive to imperfections (table 1). Radially loaded open cylinders are not because they buckle in-extensionally. (They look like a shell but they behave like an arch.) Cylinders with torsion loading and hypar roofs are not sensitive to imperfections because they are tensioned in the other direction. This tension reduces shape imperfections and stabilises the compressed direction. For this reason hypar roofs can be much thinner than other shells.

Koiter's law [4]

The equilibrium of a perfect system can be described by

$$\lambda = \lambda_{cr}\left(1 - c_1 w - c_2 w^2\right),$$

Where λ is the load factor, λ_{cr} is the critical load factor, w is the amplitude of the deflection, c_1 and c_2 are constants characterising the given structure. There are three types of post critical behaviour (Fig. 11). Type I behaviour occurs when $c_1 = 0$ and $c_2 < 0$. The structure is not sensitive to imperfections. Type II behaviour occurs when $c_1 = 0$ and $c_2 > 0$. The structure is sensitive to imperfections. Koiter showed that the ultimate load factor is equal to

[3] Von Kármán (1881-1963) and Qian (1911-2009) worked at Caltech (California Institute of Technology) as rocket scientists. They developed the knowledge that later showed necessary for the Apollo program (1961-1972), in which USA astronauts walked on the moon. Von Karman was Hungarian and he immigrated to the USA in 1930, Qian was Chinese. He immigrated to the USA in 1935 and back to China in 1955 in not friendly circumstances. This discovery was just a footnote in their lives. More on Von Kármán and on Qian can be found in Wikipedia (Qian's name is often spelled as H.S. Tsien).

[4] Warner Koiter (1914-1997) was professor at Delft University of Technology at the faculties of Mechanical Engineering and Aerospace Engineering (1949-1979). He wrote his dissertation during the Second World War and published it in 1945 just after the war [8]. The English translation appeared in 1967 [9]. It became famous because it quantifies the imperfection sensitivity of thin shells.

$$\lambda_{ult} = \lambda_{cr}\left(1 - 3\left(w_0 \tfrac{1}{2}\rho\sqrt{c_2}\right)^{\frac{2}{3}}\right),$$

Where ρ is a coefficient depending on the imperfection shape and w_0 is the imperfection amplitude. Type III behaviour occurs when $c_1 > 0$. The structure is very sensitive to imperfections. The ultimate load factor is equal to

$$\lambda_{ult} = \lambda_{cr}\left(1 - 2\left(w_0\,\rho\, c_1\right)^{\frac{1}{2}}\right).$$

This is called Koiter's law. Koiter obtained this result by a mathematical method called perturbation analysis. This method is much simpler than that of Von Karman and Qian, however, it still involves a lot of advanced mathematics.

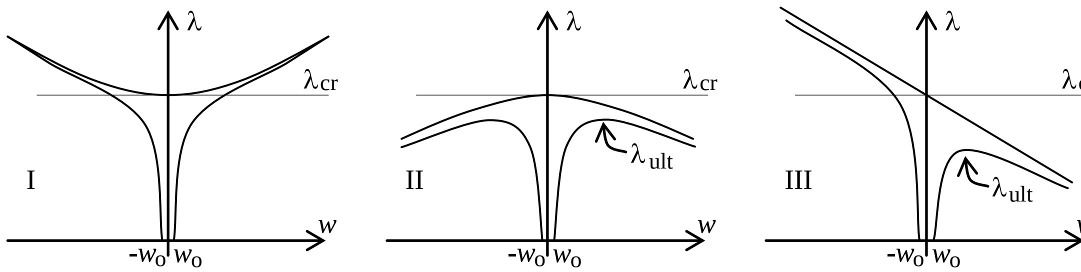


Figure 11. Basic types of post buckling behaviour according to Koiter

Buckling of flat plates
Flat plates buckle at small normal forces. Fortunately, buckling of a flat plate does not mean that it fails. After buckling the load can be increased substantially. This is type I behaviour (Fig. 11). Most shells display type III behaviour, which is totally different.

Knock down factor
In shell design often the following procedure is used. First the critical loading is computed by using the formula or a finite element program. Than this loading is reduced by a factor *C* that accounts for imperfection sensitivity. This factor is called "knock down factor". The result needs to be larger than the design loading. Often it is determined experimentally. For example for reinforced concrete cylindrical shells loaded in bending the following knock down factor is used.

$$C = 1 - 0.73(1 - e^{-\frac{1}{16}\sqrt{\frac{a}{t}}}).$$

The range in which it is valid is $0.5 < \frac{l}{a} < 5$ and $100 < \frac{a}{t} < 3000$ where *l* is the cylinder length [10].

If little information is available the following knock down factor can be used.

$$\boxed{C = \frac{1}{6}}$$

This is based on figure 5 in which all of the tests show an ultimate load more than 0.166 times the critical load.

Linear buckling analysis

Finite element programs can compute critical load factors and the associated normal modes. This is called a linear buckling analysis. A finite element model has as many critical load factors as the number of degrees of freedom. The real critical load is represented by the smallest critical load factor because a shell will buckle at the first opportunity it gets. We can specify how many of the smallest critical load factors the software will compute. If the second smallest buckling load is very close (say within 2%) to the smallest buckling load we can expect the structure to be highly sensitive to imperfections.

Typically, the loading on a finite element model is a load combination that includes partial safety factors for the ultimate limit state. The program multiplies this load combination with a load factor λ. This means that every imposed force, imposed displacement and imposed temperature is multiplied by λ. The program computes the critical load factors λ_{cr} at which buckling occurs. Clearly, the critical load factors need to be multiplied by the knockdown factor. The results need to be larger than 1. Consequently, if all critical load factors are larger than 6 the structure is safe for buckling. A linear buckling analysis can be performed by any well-educated structural engineer.

Ship design

A steel ship consists of plates strengthened by stiffeners. A linear buckling analysis of the ship model produces critical load factors for each plate that buckles. However, we are not interested in **buckling of flat plates** because that does not cause failure. We are interested in buckling of the curved ship as a whole because this will cause failure. A computer cannot tell the difference between plate buckling and shell buckling. The only thing we can do is go through the load factors from small to large, look at each buckling mode and continue until we see buckling that involves more than one plate. This can take much time because a large ship consists of hundreds of plates and has many load combinations.

Nonlinear finite element analysis

When a shell design is ready it is sensible to check its performance by nonlinear finite element analyses. In these analyses the loading is applied in small increments for which the displacements are computed. Figure 12 shows the results of different finite element analyses of a simply supported shallow dome.

The ultimate load is mainly affected by shape imperfections, support stiffness imperfections and inelastic effects. When these are measured and included in the finite element model then the predicted ultimate load has a deviation less than 10% of the experimental ultimate load [11].

Clearly, before a shell has been build we cannot measure the imperfections. Instead these are estimated. For example, the amplitude of the geometric imperfections is estimated by the designer and the contractor. Often, the analyst will assume that the shape of the geometric imperfections is the first buckling mode. He (or she) will add this imperfection to the finite element model.

It seems logical that an imperfection shape equal to the buckling shape gives the smallest ultimate load. For columns this is true. However, for shells there exists no mathematical proof of this. Therefore, another imperfection shape might give an even smaller ultimate load [12]. Of course, the analyst can consider only a limited number of imperfection shapes.

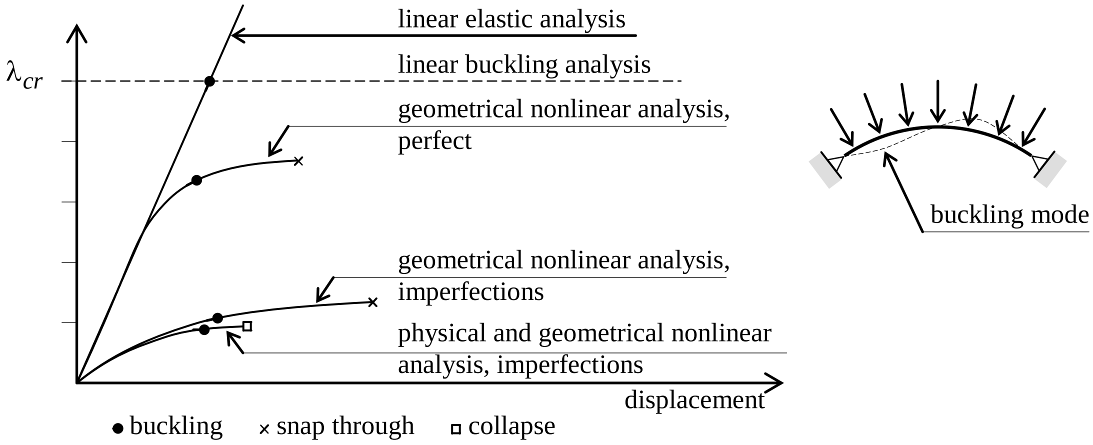


Figure 12. Shell finite element analyses of a steel spherical dome [12]

Knock down factor example
A particular shell has a critical load factor of 8.76. This had been obtained by a finite element linear buckling analysis of the perfect geometry. The shell was also analysed with a nonlinear finite element program including a shape imperfection. This imperfection resembled the buckling shape and its amplitude was 30 mm. This resulted in a ultimate load factor of 4.39. Using Koiters law we find

$$\lambda_{\max} = \lambda_{cr}\left(1 - 2\left(w_0\,\rho\,c_1\right)^{\frac{1}{2}}\right)$$

$$4.39 = 8.76\left(1 - 2\left(30\,\rho\,c_1\right)^{\frac{1}{2}}\right) \Rightarrow \rho\,c_1 = 0.00207\,.$$

This gives the following knock down factor for this shell.

$$C = \left(1 - 2\left(w_o\,0.00207\right)^{\frac{1}{2}}\right) = 1 - 0.091\sqrt{w_o}$$

Mystery solved
The critical and ultimate load of shell structures can be determined by both analytical and numerical analysis. However, these analyses are complicated and many engineers and scientists feel that we still do not understand imperfection sensitivity […]. Here it is argued that shell buckling is not a mystery at all.

In nonlinear finite element analysis we see that when a small load is applied the shell deforms in a buckling mode. The buckling mode increases the shape imperfections of the shell. The deformation is very small and invisible to the naked eye. Nonetheless, the deformation changes the curvature, in some locations the curvature has become larger and in other locations the curvature has become smaller. It also changes the membrane forces. Inwards buckles have extra compression and outward buckles have extra tension. The load is increased and the curvatures and membrane forces change further. At some location the curvature becomes small and the compression membrane force becomes large. At this location a local buckle starts, which has a larger length than the earlier buckling mode. This local buckle grows quickly and other buckles occur next to it and this spreads through the shell in a second. The shell collapses.

Stick model for shell buckling

Consider a cylinder shell with a radius a and a wall thickness t. The shell has an invisible imperfection that has the same shape as the ring bucking pattern. The amplitude of the imperfection is d and its length is l. The shell is axially loaded in compression. This increases the imperfection by an amplitude w (fig. 13).
The cylinder can be seen as a stack of rings. A ring that has become larger has a tension membrane force n_{yy} in the lateral direction. A ring that has become smaller has a compression membrane force in the lateral direction. The material in the latter ring is compressed in two directions, by $-n_{xx}$ and by $-n_{yy}$. This is the location were the shell will buckle.

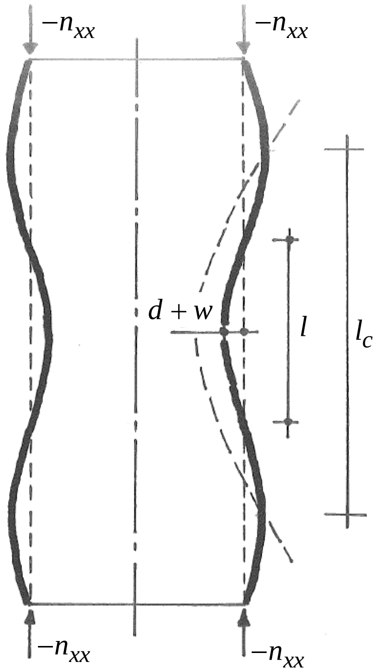


Figure 13. Cross-section of the cylinder shell

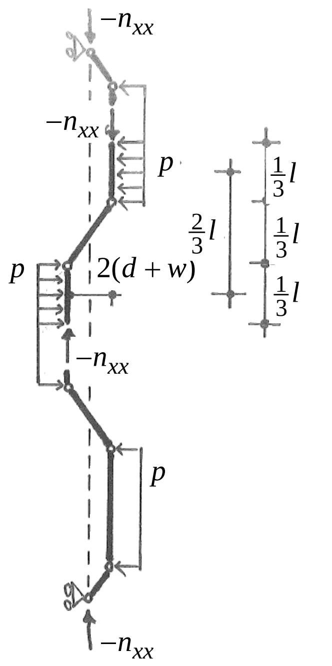


Figure 14. Stick model of the shell wall

The curvatures of the deformed shell at the location of the compressed ring are

$$k_{xx} = -8\frac{d+w}{l^2} \qquad \text{(see } \textbf{sagitta,} \text{ see } \textbf{line curvature}\text{)} \qquad (1)$$

$$k_{yy} = \frac{1}{a-d-w} \qquad \text{(see } \textbf{line curvature}\text{)} \qquad (2)$$

These curvatures have different signs so the local mean curvature is small (in absolute value). A small mean curvature has a small buckling load. This is another cause for the shell to buckle at the compressed ring. A shell with zero mean curvature buckles like a flat plate. The buckling load of a square flat plate is.

$$n_{xx} + n_{yy} = -\frac{4\pi^2 D}{l_c^2} \qquad \text{(see } \textbf{plate buckling}\text{)} \qquad (3)$$

where $D = \dfrac{Et^3}{12(1-\nu^2)}$. (4)

Finite element analyses show that the final buckling length is twice the initial buckling length.

$$l_c = 2l \qquad (5)$$

A stick model is made of the situation (fig. 14). The effect of the rings is replaced by a distributed load p. The following equations quantify the stick model.

$l = 1.7\sqrt{at}$	(see **Ring buckling of an axially compressed cylinder)**	(6)
$-n_{xx} 2(d+w) = p\frac{1}{3}l\frac{2}{3}l$	(fig. 14)	(7)
$\varepsilon_{yy} = -\frac{w}{a}$	(derive yourself)	(8)
$n_{yy} = Et\,\varepsilon_{yy}$		(9)
$-n_{yy} = pa$	(Barlow's formula)	(10)

Equations 1 to 10 can be solved by Maple. The result is shown in figure 15 and 16. Note that the local membrane force $|n_{yy}|$ in the hoop direction is larger than the local membrane

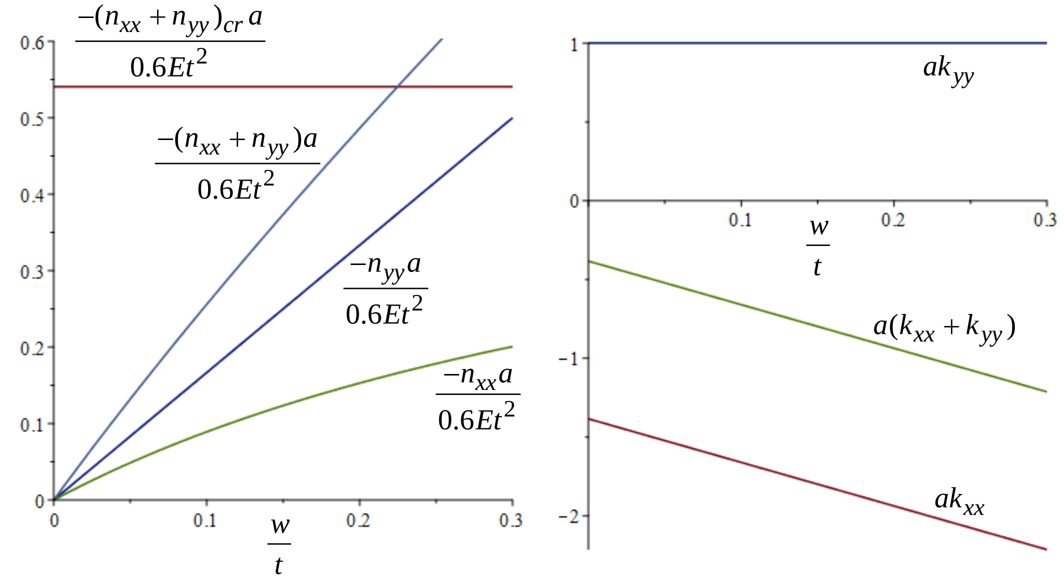


Figure 15. Membrane forces n_{xx}, n_{yy} *and curvature* k_{xx}, k_{yy} *at the buckling location as a function of the deformation w. (d =* $\frac{1}{2}t$, $\nu = 0.35$*)*

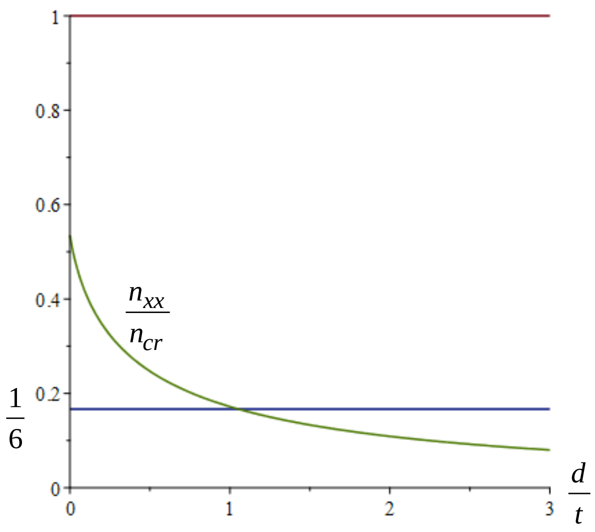


Figure 16. Koiter's law as produced by the stick model

force $|n_{xx}|$ in the axial direction. This is due to the invisible imperfections. Note that the local curvature $|k_{xx}|$ in the axial direction is larger than the local curvature $|k_{yy}|$ in the hoop direction. The imperfections are invisible but have large consequences.

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