

Stability of Shells

Shells are very efficient in carrying load. However, this efficiency comes at a price. If a shell fails it fails with a bang. There will be no warning and it will collapse faster than we can run.

Tucker High School

On September 14, 1970, the gymnasium of The Tucker High School in, Henrico County, Virginia, collapsed completely [1]. Some school children were injured but fortunately there was no loss of life. The structure was a four element hyperboloid shell with a plan of 47.2 m by 49.4 m (Fig. 1). It had a rise of about 4.6 m, large inclined supporting ribs and centre ribs that were essentially concentric with the shell. The shell was 90 mm thick for the most part. Therefore,

$$\frac{a}{t} = \frac{47.2/2 \times 49.4/2}{4.6 \times 0.090} = 1400.$$

The failure was due to progressive deflection. The lightweight concrete showed much creep and construction had taken place in cold weather [1]. Three other similar structures were subsequently demolished. One of these had a deflection of 460 mm at the centre. Research showed that the collapse could have been simply prevented by cambering upward the centre point of the shell [1].



ipped roof gymnasium was one of four identical structures designed by same firm.



Figure 1. Newspaper photograph of the collapsed hyperboloid shell [1]

Differential equation for shell buckling

The structural behaviour of shells including large displacements is described by an eight order differential equation [2] (see **Reduced differential equation**)

$$\frac{Et^3}{12(1-\nu^2)} \nabla^2 \nabla^2 \nabla^2 \nabla^2 u_z + Et \Gamma \Gamma u_z = \nabla^2 \nabla^2 (p_z + n_{xx} u_{z,xx} + 2n_{xy} u_{z,xy} + n_{yy} u_{z,yy}).$$

The differential equation can be solved analytically for elementary shell shapes and elementary loading (Table 1). The buckling loads thus obtained are called critical loads. In Table 1, E is Young's modulus, ν is Poisson's ratio, t is the shell thickness and a is the radius of the middle surface of the shell. The scientists who made significant contributions are R. Lorentz [1908], Southwell [], Richard von Mises [], Stephen Timoshenko [1910], W. Flügge [1934], L. Donnell [].

Yoshimura pattern

The buckling shape of an axially loaded cylinder can be a ring pattern or a column pattern (Fig. 2). Which one occurs depends on the shell thickness and its radius. When buckling progresses the ring pattern can transform into the column pattern. However, these deformations are very small and usually not visible. When the material starts to deform plastically the shape adopts a ring pattern (Fig. 3) or a Yoshimura pattern (Fig. 2 and 4) which are clearly visible. Remarkable about the Yoshimura pattern is that it is **in-extensional deformation**. Fortunately, large extensions are needed to transform a cylinder into a Yoshimura pattern [3].

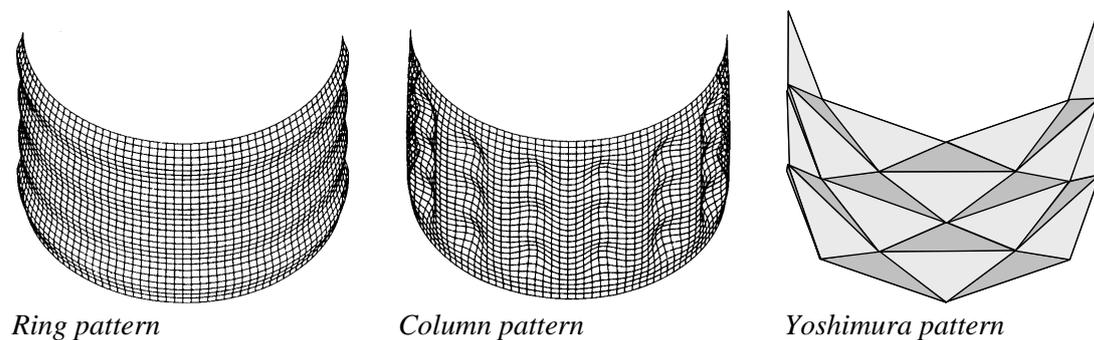


Figure 2. Buckling modes of an axially compressed cylinder [4] (only half the cylinder is shown)

Critical membrane force

In Table 1 we observe that most elementary shells buckle at a membrane force of

$$n_{cr} = \frac{-1}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a}.$$

The exceptions occur due to in-extensional deformation and due to shear stresses. It is not a big step to assume that this formula is valid for shells of any shape. For realistic values of ν the formula can be approximated to

$$n_{cr} = -0.6 \frac{Et^2}{a}$$

which is suitable for hand calculations. Clearly, the principal membrane forces need to be smaller than the critical membrane force (in absolute value).

$$n_1 \geq n_{cr}, \quad n_2 \geq n_{cr}$$

The radius a needs to be measured in a section perpendicular to the considered principal direction.

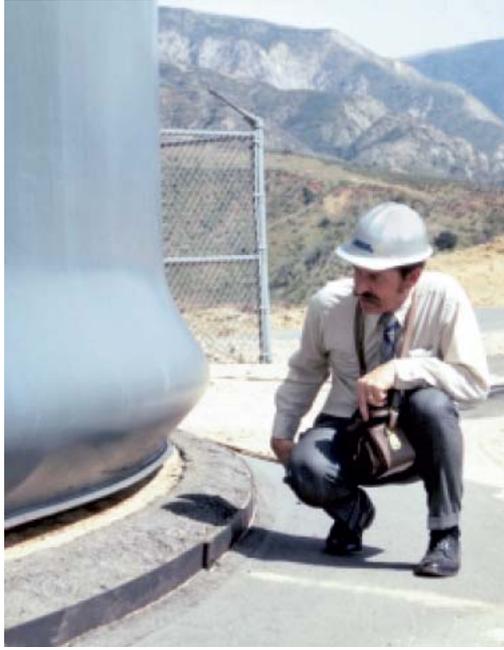


Figure 3. Elephant foot buckling of a tank wall [5]



Figure 4. Yoshimura buckling of an aluminium cylinder

Table 1. Critical loading and critical membrane forces of elementary shells

	Critical loading p_{cr} [N/m ²]	Critical membrane force n_{cr} [N/m]	Imperfection sensitive
Open cylinder, radially loaded (in-extensional deformation)	$\frac{1}{4(1-\nu^2)} \frac{Et^3}{a^3}$	$\frac{-1}{4(1-\nu^2)} \frac{Et^3}{a^2}$	no
Open cylinder, axially loaded		$\frac{-1}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a}$	yes
Open cylinder, torsion loading		$\frac{1}{3\sqrt{2}(1-\nu^2)^{\frac{3}{4}}} E \sqrt{\frac{t^5}{a^3}}$ shear force	no
Hyperboloid, axially loaded (cooling tower)		$\frac{-1}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a}$	no
Closed cylinder, loaded in all directions	$\frac{2}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a^2}$	$\frac{-1}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a}$ hoop direction	yes

Sphere	$\frac{2}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a^2}$	$\frac{-1}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a}$	yes
Dome base radius $> 3.8\sqrt{at}$	$\frac{2}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a^2}$	$\frac{-1}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a}$	yes
Hypar	$\frac{2}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a^2}$	$\frac{-1}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a}$	no

Imperfection sensitivity

Before 1930 airplanes consisted of frames covered with a fabric which was painted. However, engineers wanted to build airplanes from aluminium plates that were joint to form a cylindrical shape. Therefore, scientists started to do experiments on cylinders [Andrew Robertson 1928]. Figure 5 shows the ultimate loads of axially compressed cylinders. They are much smaller than the critical load. This is caused by imperfections such as shape deviations, small dimples, residual stresses, temperature stresses, inhomogeneities, creep, shrinkage, eccentricity of loading, first order deformations and non-uniform support stiffness. At first sight, imperfection sensitivity is hard to believe because the experiments were performed very carefully. The aluminium cylinders had perfectly cut edges and were beautifully polished. The cylinders were perfectly centred in the testing machines. The testing machines were modern and very accurate instruments were used. Nonetheless, the ultimate loads were much smaller than the critical loads, which is caused by imperfections that are often not visible with a naked eye. Not only compressed cylinders but also bend cylinders and radially compressed domes are very sensitive to imperfections. Hypars are not sensitive to imperfections.

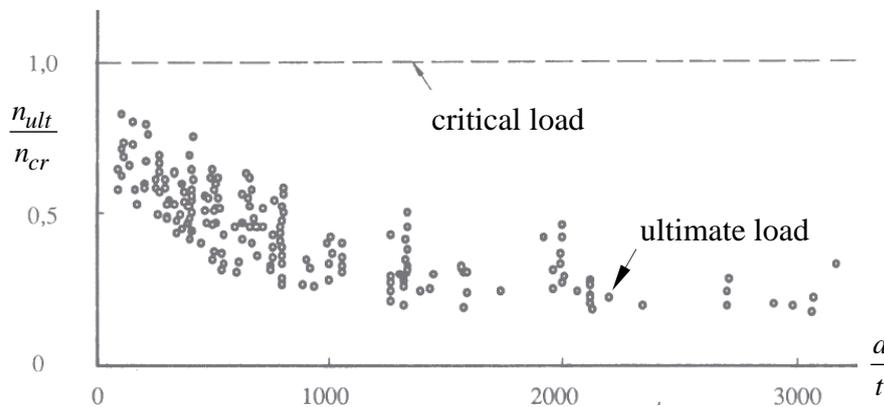


Figure 5. Experimental ultimate loads of 172 axially loaded cylinders [6]

Experiment

What is the ultimate load of an axially loaded empty beer can? We model the can as an open cylinder. The wall thickness is 0.08 mm the radius is 32.8 mm, Yong's modulus is 2.1×10^5 N/mm² and Poisson's ratio is 0.35. According to Table 1 the critical loading is



$$n_{cr} = \frac{-1}{\sqrt{3(1-\nu^2)}} \frac{Et^2}{a} = \frac{-1}{\sqrt{3(1-0.35^2)}} \frac{2.1 \times 10^5 \times 0.08^2}{32.8} = -25.3 \text{ N/mm}$$

$$F_{cr} = 2\pi a n_{cr} = 2 \times 3.14 \times 32.8 \times (-25.3) = -5200 \text{ N}$$

Therefore, it should be able to carry a mass of 520 kg. Carefully stand on the can and it will – probably – carry your weight. Subsequently, use your thumbs to push a dimple in the can and push it out again. Doing so makes typical clicking sounds. Notice that the imperfections you made are hardly visible. Now, try standing on the can again. It will collapse abruptly. The explanation is imperfection sensitivity.

Puzzle

The large difference between the theoretical buckling load (critical load) and the experimental buckling load (ultimate load) puzzled scientists for many years. The solution of imperfection sensitivity was discovered by Theodore von Kármán and Qian Xuesen (钱学森 pronounce tsien? sue? sen) in 1940 [7, 7b]. At that time they were working at Caltech (California Institute of Technology) as rocket scientists. They developed the knowledge that later showed necessary for the Apollo program (1961-1972), in which USA astronauts walked on the moon. Von Karman was Hungarian and he immigrated to the USA in 1930, Qian was Chinese. He immigrated to the USA in 1935 and back to China in 1955. This discovery was just a footnote in their lives. More on Von Kármán and on Qian can be found in Wikipedia (Qian's name is often spelled as H.S. Tsien).

These scientists solved the puzzle by calculating the load-displacement curve after buckling. Figure 6a shows the result of their calculation; n_{xx} is the membrane force in a cylinder and w is a displacement perpendicular to the cylinder. Note that load on a perfect cylinder can be increased until the critical load after which the strength will drop strongly. This behaviour is typical for shell structures and very different from other structures; compressed columns and compressed plates keep much of their strength after buckling. Figure 5b shows that very small shape imperfections cause the ultimate load to be much smaller than the critical load.

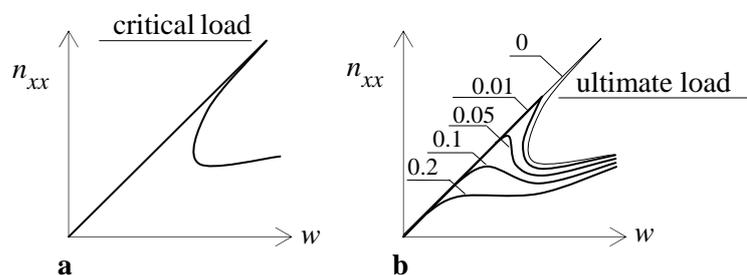


Figure 6. Buckling of cylinders for different shape imperfection amplitudes [6]

Exceptions to imperfection sensitivity

Some shells are not sensitive to imperfections (Table 1). Radially loaded open cylinders are not because they buckle in-extensionally. (They look like a shell but they behave like an arch.) Cylinders with torsion loading and hypars are not sensitive to imperfections because they are tensioned in the other direction. This tension removes shape imperfections and stabilises the compressed direction. For this reason hypars can be much thinner than other shells (see **Thickness/radius**).

Prof. Koiter

Warner Koiter (1914-1997) was professor at Delft University of Technology at the faculties of Mechanical Engineering and Aerospace Engineering (1949-1979). He wrote his dissertation during the Second World War and published it just after the war [8]. The English translation appeared in 1967 [9]. It became famous because it quantifies the imperfection sensitivity of thin shells.

Koiter's law

The equilibrium of a perfect system can be described by

$$\lambda = \lambda_{cr} (1 - c_1 w - c_2 w^2),$$

Where λ is the load factor, λ_{cr} is the critical load factor, w is the amplitude of the deflection, c_1 and c_2 are constants characterising the given structure. There are three types of post critical behaviour (Fig. 7). Type I behaviour occurs when $c_1 = 0$ and $c_2 < 0$. The structure is not sensitive to imperfections. Type II behaviour occurs when $c_1 = 0$ and $c_2 > 0$. The structure is sensitive to imperfections. Koiter showed that the ultimate load factor is equal to

$$\lambda_{ult} = \lambda_{cr} \left(1 - 3 \left(w_0 \frac{1}{2} \rho \sqrt{c_2} \right)^{\frac{2}{3}} \right),$$

Where ρ is a coefficient depending on the imperfection shape and w_0 is the imperfection amplitude. Type III behaviour occurs when $c_1 > 0$. The structure is very sensitive to imperfections. The ultimate load factor is equal to

$$\lambda_{ult} = \lambda_{cr} \left(1 - 2 \left(w_0 \rho c_1 \right)^{\frac{1}{2}} \right).$$

This is called Koiter's law. Koiter obtained this result by a mathematical method called perturbation analysis. This method is much simpler than that of Von Karman and Qian, however, it does involve a lot of advanced mathematics.

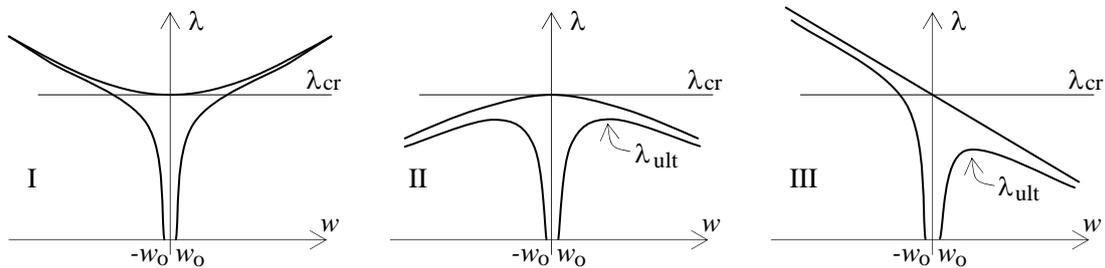


Figure 7. Basic types of post buckling behaviour

Buckling of flat plates

Flat plates buckle at small normal forces. Fortunately, buckling of a flat plate does not mean that it fails. After buckling the load can be increased substantially. This is type I behaviour (Fig. 7). Most shells display type III behaviour, which is totally different.

Knock down factor

In shell design often the following procedure is used. First the critical loading is computed by using the formula or a finite element program. Then this loading is reduced by a factor that accounts for imperfection sensitivity. The result needs to be larger than the design loading. This factor is called "knock down factor". It is often experimentally determined. For example for reinforced concrete cylindrical shells loaded in bending the following knock down factor C is used.

$$C = 1 - 0.73 \left(1 - e^{-\frac{1}{16} \sqrt{\frac{a}{t}}} \right).$$

The range in which it is valid is $0.5 < \frac{l}{a} < 5$ and $100 < \frac{a}{t} < 3000$ where l is the cylinder length [10].

If little information is available the following knock down factor can be used.

$$C = \frac{1}{6}$$

This is based on Figure 5 in which all of the tests show an ultimate load more than 0.166 times the critical load.

Linear buckling analysis

Finite element programs can compute critical load factors and the associated normal modes. This is called a linear buckling analysis. A finite element model has as many critical load factors as the number of degrees of freedom. The real critical load is represented by the smallest critical load factor because a shell will buckle at the first opportunity it gets. We can specify how many of the smallest critical load factors the software will compute. If the second smallest buckling load is very close (say within 2%) to the smallest buckling load we can expect the structure to be highly sensitive to imperfections. This is because the interaction of buckling modes gives a strong softening response after the critical state.

Typically, the loading on a finite element model is a load combination that includes partial safety factors for the ultimate limit state. The program multiplies this load combination with a load factor λ . This means that every imposed force, imposed displacement and imposed temperature is multiplied with λ . The program computes the critical load factors λ_{cr} at which buckling occurs. Clearly, the critical load factors need to be multiplied by the knockdown factor. The results need to be larger than 1. Consequently, if all critical load factors are larger than 6 the structure is safe for buckling. A linear buckling analysis can be performed by any well-educated structural engineer.

Ship design

A steel ship consists of plates strengthened by stiffeners. A linear buckling analysis of the ship model produces critical load factors for each plate. However, we are not interested in **buckling of flat plates** because this does not cause failure. We are interested in buckling of the curved ship as a whole because that will cause failure. A computer cannot tell the difference between plate buckling and shell buckling. The only thing we can do is go through the load factors from small to large, look at each buckling mode and continue until we see buckling that involves more than one plate. This can take much time because a large ship consists of hundreds of plates and has many load combinations.

Nonlinear finite element analysis

When a shell design is finalised it can be sensible to check its performance by nonlinear finite element analyses. In these analyses the loading is applied in small increments for which the displacements are computed. Such analyses should be performed by experts only (TNO, femto). It involves equilibrium iterations, path following methods and termination criteria. (See the course CT5142 Computational methods in nonlinear solid mechanics). Figure 8 shows the results of different finite element computations of a simply supported shallow dome.

The ultimate load is mainly affected by shape imperfections, support stiffness imperfections and inelastic effects. When these are measured and included in the finite element model then the predicted ultimate load has a deviation less than 10% of the experimental ultimate load [11].

Clearly, before a shell has been build we cannot measure the imperfections. Instead these are estimated. For example, the amplitude of the geometric imperfections is estimated by the designer and the contractor. Often, the analyst will assume that the shape of the geometric imperfections is the first buckling mode. He (or she) will add this imperfection to the finite element model.

It seems logical that an imperfection shape equal to the buckling shape gives the smallest ultimate load. For columns this is true. However, for shells there exists no mathematical proof of this. Therefore, another imperfection shape might give an even smaller ultimate load [12]. Of course, the analyst can consider only a limited number of imperfection shapes.

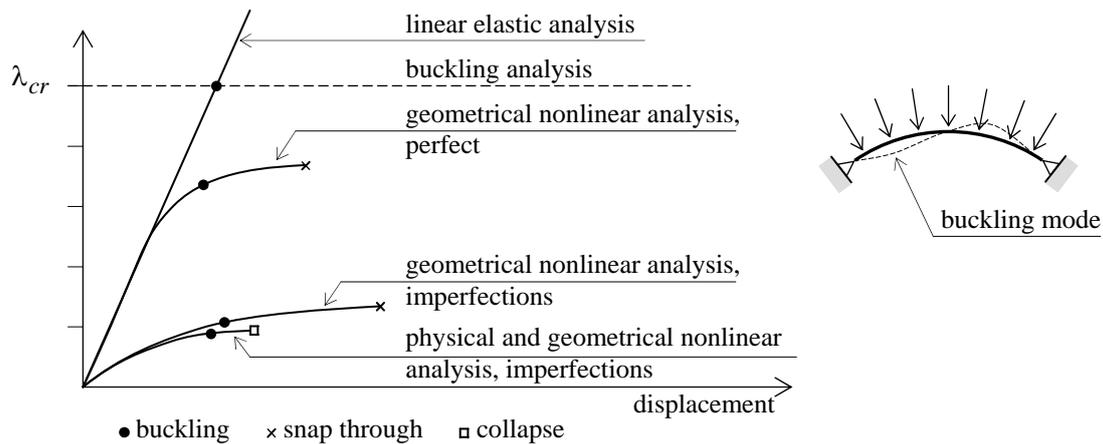


Figure 8. Shell finite element analyses of a shallow spherical dome [12]

Knock down factor example

A particular shell has a critical load factor of 8.76. This had been obtained by a finite element linear buckling analysis of the perfect geometry. The shell was also analysed with a nonlinear finite element program including a shape imperfection. This imperfection resembled the buckling shape and its amplitude was 30 mm. This resulted in a ultimate load factor of 4.39. Using Koiters half-power law we find

$$\lambda_{\max} = \lambda_{cr} \left(1 - 2(w_0 \rho c_1)^{\frac{1}{2}} \right)$$

$$4.39 = 8.76 \left(1 - 2(30 \rho c_1)^{\frac{1}{2}} \right) \Rightarrow \rho c_1 = 0.00207.$$

This gives the following knock down factor for this shell.

$$C = \left(1 - 2(w_0 0.00207)^{\frac{1}{2}} \right) = 1 - 0.091\sqrt{w_0}$$

Mystery solved

The critical and ultimate load of shell structures can be determined by both analytical and numerical analysis. However, these analyses can be complicated and many engineers and scientists felt that we still do not understand imperfection sensitivity. In 1983 Chris Calladine wrote a book on shell structures in which he gave a simple explanation [13]. Consider a thin shell with a very small inward dimple. When the shell is compressed the dimple becomes larger (second order effect). Due to the dimple the local radius of curvature is larger than that

of the perfect shell. The shell buckles at a membrane force (including residual stresses) of $-0.6 E t^2/a$ where a is the local radius of curvature.

This explanation can be written in equations. The theoretical buckling load is

$$n_{cr} = -0.6 \frac{Et^2}{a}.$$

According to Calladine, the real buckling load is

$$n_{ult} = -0.6 \frac{Et^2}{a_2},$$

where a_2 is the local radius of the deformed shell (Fig. 9). The radius of curvature can be accurately approximated by (see **Sagitta**)

$$a = \frac{1}{8} \frac{l^2}{s},$$

where l is the buckling length. Including an imperfection w the radius of curvature is

$$a_2 = \frac{1}{8} \frac{l^2}{s - w}.$$

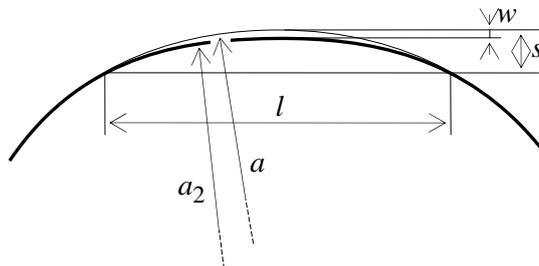


Figure 9. Cross-section of a shell with an imperfection

The imperfection w can be calculated from the initial imperfection w_o by

$$w = w_o \frac{n_{cr}}{n_{cr} - n_{ult}}.$$

This equation is used for columns and frames and is a good approximation for shells too. The latter 5 equations can be evaluated to

$$n_{ult} = n_{cr} \left(1 - 2\sqrt{2} \frac{\sqrt{a w_o}}{l} \right),$$

which is Koiter's law.

Calladine's explanation produces Koiter's law in a much simpler way than Koiter's perturbation analysis. A scientific principle is that the simplest solution is the right solution (Occam's razor). So, after more than 40 years of research of hundreds of scientists the mystery of imperfection sensitivity was solved. It is strange that the solution is so simple.

Yielding, crushing or buckling?

Relative slenderness is defined as

$$\lambda = \sqrt{\frac{n_{cr}}{n_p}} = \sqrt{\frac{-0.6Et^2}{a}} = \sqrt{0.6 \frac{E t}{f a}} .$$

If $\lambda \gg 1$ than buckling occurs before yielding or crushing.

If $\lambda \ll 1$ than plastic failure or crushing occurs before buckling.

If $\lambda \approx 1$ than interaction occurs between yielding or crushing and buckling.

Table 2 shows that a shell made of plastic is more likely to buckle than a shell made of glass.

Figure 8 shows buckling curves for steel columns based on hundreds of experiments [...].

Similar curves could be made for shell structures, however, these curves do not exist.

Table 2. Properties of materials

Material	Young's modulus E	Compressive strength f	$\frac{E}{f}$
Glass	70000 N/mm ²	50 N/mm ²	1400
Concrete	35000	40	875
Aluminium	70000	110	636
Steel	210000	350	600
Wood (Pine)	13000	40	325
Plastic (Acrylic)	2300	70	33

Figure 8. Eurocode buckling curves

Design formula

Using the formulas in Table 1 and the knock down factor in Figure 4 we can derive a formula for the required thickness of a shell

$$t = \sqrt{10 \frac{-n_2 a}{E}} ,$$

Where E is Young's modulus, a is the shell radius in the direction of n_1 and n_2 is the smallest principal membrane force. Note that n_2 needs to have a negative value because shells buckle only due to compression. For hypars the factor 10 in the formula needs to be replaced by 1.7. Small shells of reinforced concrete often will be thicker than this formula predicts because there needs to be sufficient cover on the reinforcing bars.