New ray-tracing method for radial gradient-index lenses

Florian Bociort* and Jürgen Kross
Optisches Institut, TU Berlin
Straße des 17. Juni 135, D-1000 Berlin 12, FRG

ABSTRACT

A new method to derive algebraic formulae for tracing skew rays in radial gradient-index media is described. We obtained power series for the ray position and direction, optical path length and coordinates of the intersection point of the curved ray with the spherical end surface of the GRIN lens. All these formulae include the effects of terms up to the eighth power of the radius in the refractive index distribution.

1. INTRODUCTION

The use of gradient-index lenses enables a considerable reduction of the number of elements in optical systems due to the additional degrees of freedom for the correction of aberrations provided by these lenses. As a consequence the lens design with gradients alone or together with homogeneous lenses is receiving increased attention.

In radial gradient-index (RGRIN) lenses the refractive index varies as function of distance from the optical axis. An example of a curved skew-ray path inside a thick RGRIN medium is shown in Fig. 1.

Fig. 1: Helical ray path in positive RGRIN media

At the present time the optical design with RGRIN lenses is based mainly on the iterative solution of the differential ray equations inside the lens. An alternative ray-tracing method, however, is to find approximate analytic solutions of these equations. In this case the ray position and direction and the optical path length are

* Permanent address: Institute of Atomic Physics, Laser Dept., R-76900 Bucharest, Romania
given as functions of the initial ray data and gradient profile parameters. Such formulae have been first obtained by Streifer and Paxton and Marchand.

The analytic formulae can be faster than iterative ray-tracing methods and are also useful for other purposes e.g. for derivation of aberration formulae and determination of the refractive index profile from measurements. However, if accurate results are required, these formulae become lengthy and the computation by hand of higher order corrections in these formulae is cumbersome. At the present time accurate analytic formulae are known only for RGRIN lenses with plane faces (Wood lenses).

In this work a new method for the derivation of analytic skew ray tracing formulae in RGRIN media is presented. Instead of attempting to solve the coupled ray equations directly, first a differential equation for an intermediate variable is solved. By inserting this solution in the ray equations the complexity of the skew-ray calculations is considerably reduced.

The resulting formulae for the ray position and direction and optical path length are power series where each coefficient can be calculated from the coefficients of preceding orders. The concise statement of the derivation steps makes it possible to perform the last stage of the lengthy computations of the formulae automatically on a computer using widespread symbolic mathematics software. This method also enables the derivation of analytic expressions for the coordinates of the intersection point of the curved ray with the spherical end surface of the RGRIN lens. Thus the range of applicability of the class of analytic methods is extended, the method being now suitable for arbitrary RGRIN lenses. We have considered in all these formulae the effects of terms up to the eighth power of the radius in the refractive index distribution for both positive and negative RGRIN media. Formulae of the same order of approximation but only for ray position and direction in positive RGRIN media have been previously described in Ref. 5.

### 2. RAY EQUATIONS

In RGRIN media the optical direction cosine with respect to the optical axis is invariant along any ray, \( l = l_0 \). With the z-axis as the axis of symmetry, the differential ray equations can be written as

\[
\frac{d^2x}{dz^2} = \frac{1}{2l_0^2} \frac{\partial}{\partial x} n^2(r^2)
\]

\[
\frac{d^2y}{dz^2} = \frac{1}{2l_0^2} \frac{\partial}{\partial y} n^2(r^2)
\]

and the optical direction cosines with respect to the x and y-axis

\[
p = l_0 \frac{dx}{dz}, \quad q = l_0 \frac{dy}{dz}
\]

satisfy the condition

\[
p^2 + q^2 = n^2(r^2) - l_0^2
\]
We assume a parabolic-like radial refractive index distribution of the form

\[ n^2(r^2) = n_0^2 \left( 1 - kr^2 + h_4 k^2 r^4 + h_6 k^3 r^6 + h_8 k^4 r^8 + \ldots \right) \]  \hspace{1cm} (4)

where \( r^2 = x^2 + y^2 \). In the paraxial approximation the terms of order higher than two in (4) can be neglected. Thus, the index on the optical axis \( n_0 \) and the coefficient of the quadratic term \( k \) determine the paraxial properties of the medium. For example the power of a thin Wood lens with thickness \( d \) is given by

\[ \varphi_w = n_0 k d \]  \hspace{1cm} (5)

The higher order coefficients \( h_4, h_6, h_8, \ldots \) are additional parameters that influence the aberrations of the system.

We first consider the case \( k > 0 \) (positive gradient). It is convenient to denote \( g = k^{1/2} \) and to introduce as independent variable

\[ t = \frac{n_0 g}{l_0} z \]  \hspace{1cm} (6)

In this case Eqs. (1) for a skew ray take the form

\[ \ddot{x} + x = x \left[ 2h_4 (gr)^2 + 3h_6 (gr)^4 + 4h_8 (gr)^6 + \ldots \right] \]

\[ \ddot{y} + y = y \left[ 2h_4 (gr)^2 + 3h_6 (gr)^4 + 4h_8 (gr)^6 + \ldots \right] \]  \hspace{1cm} (7)

where a dot indicates \( d/dt \).

In thick positive RGRIN media the solution of these ray equations is periodic. (Eqs.(7) are identical to those for a two-dimensional conservative anharmonic oscillator.) In the paraxial approximation the right side of (7) vanishes and the ray path is described by a simple sinusoidal solution. From this solution the paraxial features of the RGRIN lens (e.g. the cardinal points) can be obtained. For an accurate ray-tracing Eqs (7) must be solved in the general case, as shown in the next section.

### 3. NEW ANALYTIC METHOD

Equations (7) are coupled, second-order nonlinear ordinary differential equations. Streifer and Paxton\(^5\) and Marchand\(^7\) developed approximate analytical methods to solve Eqs. (7) directly. We found it more convenient to derive a differential equation for an intermediate variable \( \chi \) and to divide the computation of the solution of Eqs. (7) into two steps. As shown in this section, first the equation for \( \chi \) is solved. By introducing this solution in (7) these equations are reduced to two independent linear differential equations with variable coefficients and can then be solved much more comfortably. Moreover, the optical path length can be computed directly from the solution for \( \chi \).

By denoting \( \chi = r^2 \) it can be shown from (1-3) that for any radial refractive index distribution \( \chi \) satisfies the equation
We now define

\[ \chi = g^2 \chi = (gr)^2 \]  

(9)

For the index distribution (4) the differential equation for \( \chi \) can be written

\[ \ddot{\chi} + 4\chi = 4\mu + 6h_1\chi^2 + 8h_2\chi^3 + 10h_3\chi^4 + \ldots \]  

(10)

with

\[ \mu = \frac{1}{2} \left[ 1 - \frac{n^2}{n_0^2} \right] . \]  

(11)

In positive RGRIN media the parameter \( \mu \) indicates how close the ray path to the optical axis is. For example, for parabolic index profiles, i.e., \( h_4 = h_6 = h_8 = \ldots = 0 \), it follows from (3) and (4) that

\[ \mu_p = \frac{1}{2} \left[ g^2 (x_0^2 + y_0^2) + \frac{1}{n_0^2} (p_0^2 + q_0^2) \right] \]  

(12)

where the index 0 denotes the initial values of the corresponding ray data. It can be observed that \( \mu_p = 0 \) for a ray along the optical axis and that \( \mu_p \) increases with increasing initial ray height and inclination. For more general parabolic-like profiles (4) \( \mu \) exhibits the same features.

We first find an approximate solution of Eq. (10). Since \( \mu \) can be regarded as a small quantity this solution will be sought as the series

\[ \chi(t) = \mu \chi_0(t) + \mu^2 \chi_1(t) + \mu^3 \chi_2(t) + \mu^4 \chi_3(t) + \ldots \]  

(13)

Substituting (13) into (10) and equating the terms with equal powers of \( \mu \) we arrive at a set of equations for the coefficients \( \chi_i \).

The first equation in this set

\[ \ddot{\chi}_0 + 4\chi_0 - 4 = 0 \]  

(14)

has the solution

\[ \chi_0(t) = a \cos 2t + b \sin 2t + 1 \]  

(15)

where \( a \) and \( b \) are found from the initial conditions

\[ a = \frac{1}{\mu} g^2 (x_0^2 + y_0^2) - 1, \quad b = \frac{g}{n_0 \mu} (x_0 p_0 + y_0 q_0) \]  

(16)
For the higher order coefficients $\chi_i, i=1,2,3,\ldots$, it is convenient to define

$$
\begin{align*}
\chi_1(t) &= h\chi_{11}(t) \\
\chi_2(t) &= h^2\chi_{21}(t) + h\chi_{22}(t) \\
\chi_3(t) &= h^2\chi_{31}(t) + h^2\chi_{32}(t) + h\chi_{33}(t) \\
&\quad \quad \vdots
\end{align*}
$$

In this work we shall consider in all formulae the effects of terms up to the eighth power of the radius in the refractive index distribution (4) i.e. $i=1,2,3$. The set of equations for $\chi_{ij}$ is given by

$$
\begin{align*}
\chi_{ij} + 4\chi_{ij} &= f_{rij}(t) \\
f_{r11} &= 6\chi_0^2 \\
f_{r21} &= 12\chi_0\chi_{11} \\
f_{r22} &= 8\chi_0^3 \\
f_{r31} &= 6\chi_0^2 + 12\chi_0\chi_{21} \\
f_{r32} &= 24\chi_0^2\chi_{11} + 12\chi_0\chi_{22} \\
f_{r33} &= 10\chi_0^4 \\
\end{align*}
$$

It can be seen that for each order $i$ the coefficients $\chi_{ij}$ can be successively computed from the coefficients of preceding orders. The equations (18) are second order linear differential equations with an $t$-dependent inhomogeneous term on the right side. Following Marchand\textsuperscript{6}, we find the solution of this type of equations in the form

$$
\chi_i(t) = \frac{\sin 2t}{2} \int_0^t f_{rij}(t') \cos 2t' dt' - \frac{\cos 2t}{2} \int_0^t f_{rij}(t') \sin 2t' dt'.
$$

For $i=1$ we obtain

$$
\begin{align*}
\chi_{11}(t) &= -\frac{1}{4}(a^2 - b^2) \cos 4t - \frac{1}{2}absin 4t - 3bt \cos 2t - \frac{1}{2}(a^2 + 2b^2 + 3) \cos 2t + \\
&\quad + 3atsin 2t + \frac{1}{2}(2ab + 3b) \sin 2t + \frac{3}{4}(a^2 + b^2 + 2)
\end{align*}
$$

Since for higher order coefficients the length of the formulae increases rapidly with $i$ we do not reproduce them here. However, applying recently developed computer algebra software (e.g. DERIVE\textsuperscript{TM} or MATHEMATICA\textsuperscript{TM}) makes the evaluation of these coefficients using (19) an easy matter. The results have the following structure

$$
\chi_{ij}(t) = T_{rij} + \sum_{k=1}^{i+1} \sum_{l=1}^k (C_{ijkl} t^{k-l} \cos 2lt + S_{ijkl} t^{k-l} \sin 2lt)
$$

where $T_{rij}, C_{ijkl}$ and $S_{ijkl}$ are polynomials of degree $i+1$ in $a$ and $b$ with rational coefficients.
After inserting the solution \( \chi(t) \) into Eqs. (7) these equations are reduced to two similar linear differential equations with variable coefficients:

\[
\begin{align*}
\ddot{x} + x &= f(t)x \\
\ddot{y} + y &= f(t)y \\
f(t) &= 2h_{4}\chi(t) + 3h_{6}\chi^{2}(t) + 4h_{8}\chi^{3}(t) + \ldots
\end{align*}
\]  
(22)

The solution for both \( x \) and \( y \) can be written as a linear combination of the two fundamental solutions \( \phi_{c}(t) \) and \( \phi_{s}(t) \) of Eqs. (22). We seek both \( \phi_{c}(t) \) and \( \phi_{s}(t) \) as series expansions of the form

\[
\phi(t) = \phi_{0}(t) + \mu\phi_{1}(t) + \mu^{2}\phi_{2}(t) + \mu^{3}\phi_{3}(t) + \ldots
\]
(23)

As in the case of \( \chi(t) \) we define

\[
\begin{align*}
\phi_{1}(t) &= h_{4}\phi_{11}(t) \\
\phi_{2}(t) &= h_{6}\phi_{21}(t) + h_{6}\phi_{22}(t) \\
\phi_{3}(t) &= h_{8}\phi_{31}(t) + h_{8}\phi_{32}(t) + h_{8}\phi_{33}(t)
\end{align*}
\]  
(24)

and insert (23) and (24) into (22). Equating the terms with equal powers of \( \mu \) we obtain the set of equations

\[
\begin{align*}
\ddot{\phi}_{0}(t) + \phi_{0}(t) &= 0 \\
\ddot{\phi}_{\mu}(t) + \phi_{\mu}(t) &= f_{\mu\mu}(t)
\end{align*}
\]  
(24)

and

\[
\begin{align*}
f_{s11} &= 2\chi_{0}\phi_{0} \\
f_{s21} &= 2\chi_{1}\phi_{0} + 2\chi_{0}\phi_{1} \\
f_{s22} &= 3\chi_{0}^{2}\phi_{0} \\
f_{s21} &= 2\chi_{21}\phi_{0} + 2\chi_{11}\phi_{11} + 2\chi_{0}\phi_{21} \\
f_{s32} &= 6\chi_{0}\chi_{1}\phi_{0} + 2\chi_{22}\phi_{0} + 3\chi_{0}^{2}\phi_{11} + 2\chi_{0}\phi_{22} \\
f_{s33} &= 4\chi_{0}^{3}\phi_{0}
\end{align*}
\]  
(25)

The two fundamental solutions \( \phi_{c}(t) \) and \( \phi_{s}(t) \) are then obtained by inserting the two linearly independent solutions of Eq.(24)

\[
\begin{align*}
\phi_{c0}(t) &= \cos t \\
\phi_{s0}(t) &= \sin t
\end{align*}
\]  
(26)

as \( \phi_{0} \) in (25) and using

\[
\phi_{\mu}(t) = \sin t \int_{0}^{t} f_{\mu\mu}(t') \cos t' dt' - \cos t \int_{0}^{t} f_{\mu\mu}(t') \sin t' dt' 
\]  
(27)
If these calculations are symbolically performed on the computer the lengthy final formulae can also be automatically translated into various programming languages.

Finally, the ray path inside the RGRIN lens is given by

\[ x(t) = x_0 \phi_e(t) + \frac{p_0}{n_0 g} \phi_s(t) \]
\[ y(t) = y_0 \phi_e(t) + \frac{q_0}{n_0 g} \phi_s(t) \]
\[ p(t) = n_0 g x_0 \phi_e(t) + p_0 \phi_s(t) \]
\[ q(t) = n_0 g y_0 \phi_e(t) + q_0 \phi_s(t) \]

The optical path length can be computed directly from \( x(t) \)

\[ L = \int n ds = \frac{1}{L_0} \int_0^z n^2 dz = \frac{n_0}{g} \phi(t) \]
\[ \phi(t) = \int_0^t \left(1 - \chi(t') + h_0 \chi^2(t') + h_2 \chi^3(t') + h_4 \chi^4(t') + \ldots \right) dt' \]  

For positive RGRIN media the present method yields an accurate solution of Eqs (7) as long as the thickness of the medium is not greater than a few periods. In this case the convergence of the series expansions for \( \chi, \phi_e, \phi_s \), and \( \phi \) is determined only by \( \mu \). The two quantities \( a \) and \( b \) appearing in these formulae are of the order of magnitude of one.

For negative RGRIN media, i.e. \( k < 0 \), a similar formalism can be developed. However, since the series expansions for \( \chi, \phi_e, \phi_s \), and \( \phi \) can be regarded as analytical functions, we found it more convenient to allow \( g = k^{1/2} \) to become imaginary for \( k < 0 \) and to use the same ray-tracing formulae for both positive and negative RGRIN media. In the latter case several intermediate results have complex values but the final results are always real. For \( k < 0 \) and for certain values of the initial conditions, \( \mu \) can be zero and according to (16) \( a \) and \( b \) become infinite. This problem can be avoided by taking advantage of the fact that each term of \( \chi, \phi_e, \phi_s \), and \( \phi \) is a polynomial in \( a \) and \( b \) of the same degree as the corresponding power of \( \mu \), and by defining \( \bar{a} = \mu a, \quad \bar{b} = \mu b \). Thus \( \chi, \phi_e, \phi_s \), and \( \phi \) become power series expansions in all three rotational invariants \( \bar{a}, \bar{b}, \mu \).

4. CURVED END FACES

At the present time the accurate computation of the ray-surface intersection point in the case of RGRIN lenses with curved end faces is achieved only with methods relying on iterative solutions of the ray equations\(^2\). In this section we show how this computation can be performed starting from the analytic formulae of the previous section.

The coordinates of the intersection point of the ray with a rotationally symmetric end surface satisfy an equation of the form

\[ \bar{z} = F(r) \]  

(30)
where \( \bar{z} \) is measured from the vertex of the end face. For a spherical surface of radius \( R \) we have

\[
F(r^2) = -R \left( 1 - \sqrt{1 - \frac{r^2}{R^2}} \right) = -R \left[ \frac{1}{2} \left( \frac{r^2}{K^2} \right) + \frac{1}{2 \cdot 4} \left( \frac{r^2}{K^2} \right)^2 + \frac{3}{2 \cdot 4 \cdot 6} \left( \frac{r^2}{K^2} \right)^3 + \frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \left( \frac{r^2}{K^2} \right)^4 + \ldots \right]
\]  
(31)

We first compute the ray position and direction in the tangent plane at the vertex using (28) and (6) i.e., as for a plane end face and regard these values as new initial conditions. The \( z \)-coordinate of the intersection point is given by the solution of the system of equations

\[
\begin{align*}
\bar{r}^2 &= \frac{1}{\varepsilon^2} \chi(\bar{r}) \\
\bar{t} &= \frac{n_0 g}{l_0} F(r^2)
\end{align*}
\]  
(32)

with

\[
\bar{z} = \frac{l_0}{n_0 g} \bar{t}
\]  
(33)

Since \( \bar{r} \) is small we can use Taylor series expansions for the trigonometric functions in (15) and (21) for the evaluation of \( \chi \). We now seek the solution \( \bar{r} \) as the series

\[
\bar{t} = \mu^0 \bar{t}_1 + \mu^1 \bar{t}_2 + \mu^3 \bar{t}_3 + \mu^4 \bar{t}_4 + \mu^5 \bar{t}_5 + \mu^6 \bar{t}_6 + \ldots
\]  
(34)

Introducing (34) into (32) and equating (on the computer) the terms with equal powers of \( \mu \) we arrive at a set of equations for \( t_i \) where, as in the case of \( \chi \) and \( \phi \), each coefficient \( t_i \) can be calculated from the coefficients of preceding orders. We found that, in order to consider the effects of terms up to the eighth power of the radius in the refractive index distribution (4) we had to compute six terms in (34). We note at this point that the procedure presented above works only due to the fact that the series expansion (13) for \( \chi \) does not contain a zeroth power of the expansion variable \( \mu \). Finally, using \( \bar{r} \) in (28) the ray position and direction (before the refraction) at the intersection point can be determined.

5. TEST OF ACCURACY

To verify the validity of the present method we have compared the results generated by our analytic formulae with the corresponding iterative solutions generated by CODE V and by a code developed earlier in our research group. For both positive and negative RGRIN media the agreement was excellent.

We also compared our results with earlier analytic results of Marchand on the basis of a numerical example described in Ref. 7. For the same order of approximation as used there i.e., sixth power in (4) for ray position and direction and fourth power for optical path length our results have nearly the same accuracy as those of Ref. 7. Our eighth order results are, however, in full agreement with a very accurate iterative solution which was used there by the author to test his formulae.
From these comparisons we concluded that for parabolic-like refractive index profiles as given by (4) the accuracy of these various analytic and iterative methods is nearly the same and is given mainly by the number of terms considered in (4).

In order to illustrate the accuracy improvement with increasing order of approximation a simple example of axial imaging by a quarter-pitch rod lens will be presented. These lenses have a positive RGRIN index profile and flat end faces and the thickness \( d \) is chosen such that the ray travels one quarter of a period inside the lens. It is known that in this case the collimation of an axial point source situated directly in front of the lens is aberration-free for the refractive index profile

\[ n^2(r) = n_0^2 \text{sech}^2(\mu r) = n_0^2 \left[ 1 - (\mu r)^2 + \frac{2}{3} (\mu r)^4 - \frac{17}{45} (\mu r)^6 + \frac{62}{315} (\mu r)^8 + \ldots \right] \quad (35) \]

where \( \text{sech} x = \frac{2}{e^x + e^{-x}} \). The exact value of the optical path length inside the lens is then independent of the initial ray direction and is equal to its value along the optical axis \( n_0 d \). In this example (Fig.2) the parameter \( \mu \) which determines the convergence of the solution series is a function only of the numerical aperture. We have computed the optical path length with our method in four different orders of approximation by successively considering the effects of terms up to the 2-nd, 4-th, 6-th and 8-th power in (35).

![Fig. 2 Axial imaging by a quarter-pitch rod lens](image)

By comparing these results with the exact value we determined the computation error of the wave aberration in the four cases as function of the numerical aperture as shown in Fig.3.

It can be seen that the number of terms that have to be computed in order to achieve a prescribed accuracy increases with the numerical aperture. For example, with eighth order terms the error at N.A.=0.5 is less than 1/100 of a wavelength. For not too high values of the N.A. the accuracy improves with each additional order by a factor between 10 and 100, faster at low N.A. and slower at higher N.A.

We have found the same features in the general skew-ray case for all series described in this work. In this case the convergence is determined by the size of both aperture and field.

**5. CONCLUSIONS**

A new systematic method to derive approximate analytic formulae for skew-ray tracing in both positive and negative RGRIN lenses has been developed. Since all formulae include the effects of the eighth power of the radius in the refractive index distribution these formulae are in all more accurate than previously known analytic formulae. For the first time the analytic method is used also for accurate computation of the ray-surface intersection point in the case of curved end faces. These formulae are useful not only for ray tracing but also in various...
cases if the analytic dependence of the ray path as function of the lens parameters and ray data is required. At the present time we work at the derivation of aberration coefficients from these formulae.

![Graph showing computation error of the wave aberration in four different orders of approximation]

**Fig.3** Computation error of the wave aberration in four different orders of approximation

### 7. REFERENCES