Seidel aberration coefficients for radial gradient-index lenses

F. Bociort* and J. Kross
Institute of Optics, Technical University of Berlin, Strasse des 17 Juni 125, D-10623 Berlin, Germany

Received January 25, 1994; revised manuscript received May 2, 1994; accepted May 2, 1994
Short and accurate analytic formulas for the Seidel coefficients of gradient-index lenses with arbitrary radial refractive-index distributions have been obtained. Starting from analytic ray-tracing formulas, we have developed a technique for decomposing the two components of the transverse aberration of an arbitrary skew ray in surface and inhomogeneous transfer contributions and a technique for shortening the large expressions for the third-order terms resulting from the transfer contributions. Unlike previously known derivation methods, our method delivers simple algebraic expressions for all Seidel coefficients.

1. INTRODUCTION
In the case of homogeneous optical systems, Seidel aberration coefficients have long been known to be a valuable tool for lens design. Even if the Seidel aberrations yield a description of the image quality that is valid strictly only for modest aperture and angular coverage, the Seidel coefficients are useful for analyzing the effects of changing various lens parameters on the aberrations of the system and for locating adequate starting values of the system data for subsequent optimization with ray tracing.

For gradient-index lenses, attempts to develop convenient methods for calculation of the Seidel coefficients have been made ever since the potential of the gradients for improving the performances of optical systems has been recognized.1-8 For instance, Sands has shown that, for general rotationally symmetric gradients, in addition to the ordinary surface contributions the Seidel coefficients of gradient-index lenses consist of inhomogeneous surface contributions at both end faces and of inhomogeneous transfer contributions.1 For axial gradients the expressions of the Seidel coefficients were obtained by the same author.2 For radial gradients, however, although the inhomogeneous surface contributions are given by simple aspheric-like terms (Section 2 below), the currently known methods for accurate computation of the transfer contributions are highly complex, a simple expression being published only for the Petzval curvature.3

In this paper we develop a method for deriving Seidel aberration coefficients for radial gradient-index (RGRIN) lenses directly from ray-tracing formulas and obtain simple formulas for the remaining transfer contributions, as well.

Previously a Lagrangian method suggested by Buchdahl5 was employed for the derivation of the transfer contributions.1 However, in view of the large number of symbolic calculations, we preferred to develop a new derivation method based on ray tracing, because it permits direct control of the intermediate steps of the calculation by comparison with numerical ray-tracing data and thus provides additional safety.

In an earlier paper7 we derived algebraic formulas for tracing rays in RGRIN lenses (Section 3, present paper). Relying on these formulas, we first describe a technique that permits the decomposition of the two components of the transverse aberration of an arbitrary skew ray in contributions from refraction at each surface and transfer through each inhomogeneous medium of the system (Section 4). From the resulting algebraic expressions the Seidel coefficients are obtained by collection of the coefficients of all terms having third degree in aperture and field coordinates (Section 5). At this stage the Seidel coefficients are given by large expressions. In Section 6 we show that these expressions can be substantially shortened by use of a heuristic symmetrization technique such that simple closed formulas are finally obtained for all Seidel coefficients of the RGRIN lenses, as in the case of homogeneous lenses. Finally, an independent test by comparison with ray tracing shows the correctness of all Seidel formulas given in this paper.

Consider an arbitrary ray in an optical system consisting of homogeneous and RGRIN lenses. Let \( (\tau_x, \tau_y) \) be the normalized Cartesian ray coordinates in the object plane and \( (\sigma_x, \sigma_y) \) be those of the intersection point of the ray with the entrance-pupil plane. These coordinates are normalized to unity such that the ray coordinates of the marginal ray are \( \sigma_x = 1, \sigma_y = \tau_x = \tau_y = 0 \), and those of the chief ray are \( \tau_x = 1, \sigma_x = \sigma_y = \tau_y = 0 \). In this paper, for any integer \( p, O(p) \) denotes any quantity containing only terms of total order \( \geq p \) in \( (\sigma_x, \sigma_y) \) and \( (\tau_x, \tau_y) \).

Assume that the refractive-index distribution of each RGRIN lens is given by a power-series expansion with respect to the distance to the optical axis \( r \). Each radial refractive-index distribution with a nonzero quadratic term can be written as

\[
n^2(r) = n_0^2(1 - kr^2 + N_4k^4r^4) + O(6),
\]

where \( n_0 \) is the refractive index on the optical axis. Only terms of order \( \leq 4 \) determining the Seidel coefficients are considered in Eq. (1). It will be seen below that this particular form for the second- and fourth-order terms is convenient for writing the formulas for both ray tracing and the Seidel coefficients as simply as possible. The

special case \( k = 0 \) (the so-called shallow gradients) is discussed in Section 7. It follows from the eikonal theory that in every rotationally symmetric system the two components \( \Phi_{1,3} \) and \( \Phi_{1,1,2} \) of the third-order transverse aberration of the ray are related to the total Seidel aberration coefficients of the system by

\[
\Phi_{1,3} = \left[ \Gamma_1(\sigma_x^2 + \sigma_y^2) + 2\Gamma_3(\sigma_x \tau_x + \sigma_y \tau_y) \right. \\
+ \left. (\Gamma_3 + P)\left(\tau_x^2 + \tau_y^2\right)\right] \tau_x + \left[ \Gamma_4(\sigma_x^2 + \sigma_y^2) \\
+ 2\Gamma_3(\sigma_x \tau_x + \sigma_y \tau_y) + \Gamma_4(\tau_x^2 + \tau_y^2)\right] \tau_x,
\]

\[
\Phi_{1,1,2} = \left[ \Gamma_1(\sigma_x^2 + \sigma_y^2) + 2\Gamma_3(\sigma_x \tau_x + \sigma_y \tau_y) \right. \\
+ \left. (\Gamma_3 + P)\left(\tau_x^2 + \tau_y^2\right)\right] \tau_y + \left[ \Gamma_4(\sigma_x^2 + \sigma_y^2) \\
+ 2\Gamma_3(\sigma_x \tau_x + \sigma_y \tau_y) + \Gamma_4(\tau_x^2 + \tau_y^2)\right] \tau_y.
\] (2)

The coefficients for spherical aberration, \( \Gamma_1 \); coma, \( \Gamma_2 \); astigmatism, \( \Gamma_3 \); Petzval curvature, \( P \); and distortion, \( \Gamma_4 \), are obtained by a summing up of the corresponding ordinary contributions \( S_p \) and \( P_S \) and inhomogeneous surface contributions \( S_p^* \) over all surfaces and the inhomogeneous transfer contributions \( T_p \) and \( P_T \) over all gradient media. Subsequently the sums are divided by the refractive index \( n_Q \) and the marginal ray angle \( u_Q \) at the image plane \( Q \):

\[
\Gamma_p = -\frac{1}{2n_Q u_Q} \left[ \sum_{\text{surfaces}} \left( S_p + S_p^* \right) + \sum_{\text{GRIN media}} \left( T_p \right) \right],
\]

\[
P = -\frac{1}{2n_Q u_Q} \left( \sum_{\text{surfaces}} P_p + \sum_{\text{GRIN media}} P_T \right).
\] (3)

In this paper we discuss in detail only the derivation of the transfer contributions. The main formulas giving the surface contributions are merely summarized below, and we refer the reader to the literature for their derivation.\(^{1,9}\)

2. SURFACE CONTRIBUTIONS TO THE SEIDEL COEFFICIENTS

The surface contributions to the Seidel coefficients can be calculated from the paraxial marginal and chief ray data at the lens surfaces and the paraxial refraction invariants,

\[
n_o i = n_o k p - n_o m u , \\
n_o j = n_o k m - n_o w .
\] (4)

The marginal and chief ray heights are denoted \( h \) and \( m \), and the corresponding marginal and chief ray angles are denoted \( u \) and \( w \) (German notation). The sign convention adopted here for the angles \( u \) and \( w \), which is that their signs are the opposites of those of the corresponding direction cosines. The variables \( p, i, \) and \( j \) are the surface curvature and the incidence angles for the marginal and the chief rays, respectively, and \( H \) is the paraxial system invariant:

\[
H = m n_o k - h n_o w .
\] (5)

In what follows, \( \Delta ( \cdot ) \) denotes the difference between the values after and before refraction or transfer of the quantity in the brackets.

The ordinary surface contributions are identical to those for a homogeneous spherical lens. With this notation these contributions are given by

\[
S_1 = (n_0 i)^2 h \Delta (u_0 n_0), \\
S_2 = n_0 i m n_0 \Delta (u_0 n_0), \\
S_3 = (n_0 j)^2 h \Delta (u_0 n_0), \\
P_S = -\rho H^2 \Delta (1/n_0), \\
S_4 = (n_0 j)^2 m \Delta (u_0 n_0) + n_0 j H \Delta (u_0 n_0) .
\] (6)

For the derivation of Eqs. (6) in a similar form see, e.g., Refs. 8 and 9. (Note, however, the opposite sign convention for \( u \) and \( w \) used there.)

Considering the general case in which the media on both sides of the surface are gradient media, Sands's results for the inhomogeneous surface contributions can be rewritten as\(^1\)

\[
S_1^* = -2h^4 \rho \Delta (m_0 n_0), \\
S_2^* = -2h^2 m \rho \Delta (m_0 n_0), \\
S_3^* = -2h^2 m^2 \rho \Delta (m_0 n_0), \\
S_4^* = -2h m^3 \rho \Delta (m_0 n_0).
\] (7)

At this point we note the complete symmetry existing between the quantities before and after refraction in Eqs. (6) and (7). The angles and the refractive-index data changing at refraction appear in these equations either through the refraction invariants or as differences given by the \( \Delta ( \cdot ) \) operators.

3. RAY-TRACING FORMULAS

Before starting the derivation of the transfer contributions, we briefly discuss the ray propagation in a GRIN medium.

In Ref. 7 we developed an approximate analytic method for solving the differential equations describing the ray path inside an GRIN lens. The resulting ray-tracing formulas are power series for the ray position and direction and are closely related to the aberration series. In the highest order of approximation calculated, the results are exact for an order in aperture and field coordinates as high as 7. However, only terms of the solutions of order \( \leq 3 \) are needed for calculating the Seidel coefficients. The necessary third-order ray-tracing formulas are given below.

Previously, third- and fifth-order analytic ray-tracing formulas had also been given by Marchand.\(^{10,11}\) Although Marchand's formulas have a form that seemingly is different from ours, a comparison on the basis of a numerical example\(^1\) yielded very close results.

Let the GRIN medium having an axial thickness \( d \) be situated between two curved surfaces. The ray-tracing formulas give the ray position and direction at the surface following the GRIN medium as functions of the ray parameters at the surface preceding it. The ray coordinates \( (x, y) \), the corresponding optical direction cosines \( (\xi, \eta) \), and any functions of \( x, y, \xi, \) and \( \eta \) are denoted as follows: before transfer (but after refraction at the first
where an overdot indicates \( \frac{d}{dt} \). For \( t \) given by Eq. (10), \( \phi_c \) and \( \phi_s \) are developed in Ref. 7 into power series with respect to the transfer invariant

\[
\mu = \frac{1}{2}(1 - \xi^2/n_0^2).
\]

It will be shown in Ref. 7, below that for determining the Seidel coefficients only the first two terms must be considered for \( \phi_c \) and \( \phi_s \):

\[
\phi_c = \phi_{c0} + \mu N_4 \phi_{c11} + \ldots, \\
\phi_s = \phi_{s0} + \mu N_4 \phi_{s11} + \ldots.
\]

With the calculation method presented in Ref. 7, the coefficients and their derivatives are given by

\[
\phi_{c0} = \cos t, \quad \phi_{s0} = -\sin t, \\
\phi_{c11} = -\frac{1}{8} a \cos 3t - \frac{1}{8} b \sin 3t - \frac{1}{2} bt \cos t \\
+ \frac{1}{8} a \cos t + \frac{1}{2} (a + 2) t \sin t + \frac{7}{8} b \sin t, \\
\phi_{s11} = -\frac{3}{8} b \cos 3t + \frac{3}{8} a \sin 3t + \frac{1}{2} (a + 2) t \cos t \\
+ \frac{3}{8} b \cos t + \frac{1}{2} bt \sin t + \frac{1}{8} (3a + 8) \sin t, \\
\phi_{s11} = -\frac{1}{8} b \cos 3t - \frac{1}{8} a \sin 3t + \frac{1}{2} (a - 2) t \cos t \\
- \frac{1}{8} b \cos t + \frac{1}{2} bt \sin t - \frac{1}{8} (a - 8) \sin t, \\
\phi_{s11} = -\frac{3}{8} a \cos 3t - \frac{3}{8} b \sin 3t + \frac{1}{2} bt \cos t \\
+ \frac{3}{8} a \cos t - \frac{1}{2} (a - 2) t \sin t + \frac{5}{8} b \sin t,
\]

with the abbreviations

\[
a = \mu g^2(x^2 + y^2) - 1, \quad b = \frac{g}{n_0 \mu}(x \xi + y \eta).
\]

For negative RGRIN media, i.e., \( k < 0 \), an analogous formalism is valid. However, since Eqs. (10)–(16) can be regarded as analytical functions, it is more convenient to allow \( g = k^{1/2} \) to become imaginary for \( k < 0 \) and to use the same ray-tracing formulas for both positive and negative RGRIN media. In the latter case several intermediate results have complex values, but the final results are always real. Moreover, it will turn out in Section 6 below that the expressions for the Seidel coefficients resulting from Eqs. (10)–(16) contain only \( g^2 = k \), and therefore no complex quantities arise for \( k < 0 \).

For rays propagating through the optical system close to the optical axis, i.e., when \( x, y, \xi, \) and \( \eta \) are small, the paraxial approximation holds. In this case, at each refraction or transfer only the linear terms have to be considered in the expressions giving the final values of the ray data as functions of the initial ones. If linearization is also performed in the opening formulas where the
ray position and direction are given through \((\sigma_x, \sigma_y)\) and \((\tau_x, \tau_y)\), \(x, y, \xi,\) and \(\eta\) are given by linear combinations of the aperture and field coordinates at each surface of the system. It can be shown that the coefficients are the height and slope of the marginal and chief rays at that surface. In what follows we denote with a tilde the paraxial approximation for \(x, y, \xi,\) and \(\eta\) and for any function of these variables. Consequently, we have, before transfer,

\[
\tilde{x} = m\tau_x + h\sigma_x, \quad \tilde{\xi} = -n_0\omega\tau_x - n_0\omega\sigma_x, \\
\tilde{y} = m\tau_y + h\sigma_y, \quad \tilde{\eta} = -n_0\omega\tau_y - n_0\omega\sigma_y. \tag{17}
\]

We note here that \(\tilde{x} = x + O(3)\). The same holds for the other ray parameters.

In order to find the paraxial approximation for the transfer equations in a RGRIN medium, we have to linearize Eqs. (10)-(16). Since for the optical direction cosines we have

\[
\xi^2 + \eta^2 + \xi^2 = n^2(r^2), \tag{18}
\]

it follows from Eq. (1) that

\[
\xi^2 = n_0^2[1 - (gr)^2 + N_4(gr)^4] - \xi^2 - \eta^2, \tag{19}
\]

and consequently \(\tilde{\xi} = n_0\). The paraxial approximation for Eq. (10) then reads as

\[
i = gd. \tag{20}
\]

Using Eq. (19), we have for invariant (12)

\[
\tilde{\mu} = \frac{1}{2}\left[\frac{1}{n_0^2}(\xi^2 + \eta^2) + g^2(x^2 + \tilde{y}^2)\right]. \tag{21}
\]

Because of Eq. (21), \(\mu\) is of \(O(2)\), and therefore Eqs. (15) and (16) do not contribute to the paraxial transfer formulas. From Eqs. (11), (13), (14), and (20) we obtain

\[
\tilde{x}' = \tilde{x}\cos gd + \tilde{\xi}/(n_0 g)\sin gd, \\
\tilde{\xi}' = \tilde{\xi}\cos gd - n_0 g\tilde{x}\sin gd. \tag{22}
\]

Similar equations hold, of course, for the paraxial approximations of \(y\) and \(\eta\).

We can now derive the transfer equations for the paraxial marginal and chief rays. For the marginal ray it follows from Eqs. (17) and from the definition of the normalized ray coordinates that

\[
\tilde{x} = h, \quad \tilde{x}' = h', \quad \tilde{\xi} = -n_0 u, \quad \tilde{\xi}' = -n_0 u', \tag{23}
\]

and therefore

\[
u' = u\cos gd + h\sin gd, \\
h' = -u/g\sin gd + h\cos gd. \tag{24}
\]

Similarly, for the chief ray we obtain

\[
u' = u\cos gd + mg\sin gd, \\
m' = -u/g\sin gd + m\cos gd. \tag{25}
\]

For \(k < 0\), \(g\) is imaginary, as noted above. Since Eqs. (24) and (25) are needed for numerical computations of the Seidel coefficients, a separate treatment of this case is convenient for avoidance of programming with complex numbers. As is well known, trigonometric functions with imaginary arguments can be replaced by the corresponding hyperbolic functions:

\[
sin ix = i\sinh x, \\
cos ix = \cosh x. \tag{26}
\]

Inserting \(g = (-k)^{1/2}\), i.e., \(g = -ig\), into Eqs. (24) and (25) yields

\[
u' = u\cosh gd + h\sinh gd, \\
h' = -u/g\sinh gd + h\cosh gd, \\
w' = w\cosh gd - mg\sinh gd, \\
m' = -w/g\sinh gd + m\cosh gd. \tag{27}
\]

For shallow gradients \(g\) tends to zero. For small values of \(g\) we have \(\sin gd = gd\) and \(\cos gd = 1\). Therefore in this case Eqs. (24) and (25) have the same form as for a homogeneous medium,

\[
u' = u, \\
h' = h - ud, \\
w' = w, \\
m' = m - wd. \tag{28}
\]

We can verify by direct substitution, using Eqs. (24) and (25), that \(H\) given by Eq. (5) is a paraxial invariant also at transfer through the RGRIN medium. Inserting Eqs. (17) into Eq. (21) yields additional paraxial transfer invariants. Since, at the transfer of an arbitrary ray,

\[
\tilde{\mu} = \frac{1}{2}\left[(kh^2 + u^2)(\sigma_x^2 + \sigma_y^2) + 2(khm + wu) \times (\sigma_x^2 + \sigma_y^2)(\tau_x^2 + \tau_y^2)\right] \tag{29}
\]

remains unchanged, it follows that

\[
e_1 = kh^2 + u^2, \\
e_2 = khm + wu, \\
e_3 = km^2 + w^2 \tag{30}
\]

are also paraxial transfer invariants. However, these four invariants are not independent. It can easily be proved that

\[
e_1e_2 = e_2^2 + \frac{k}{n_0^2}H^2. \tag{31}
\]

Invariants (30) were first obtained by Hopkins. 12
4. SURFACE AND TRANSFER CONTRIBUTIONS TO TRANSVERSE ABERRATION

We now turn to the determination of the transfer contributions of RGRIN lenses to the Seidel coefficients of the system. As a first step, in this section we derive a technique that, given the ray position and direction at each surface, permits the decomposition of the transverse aberration vector of an arbitrary finite ray in contributions from refraction at the surfaces and from transfer through the inhomogeneous media of the system. This technique is based on the idea of the quasi-invariant, first introduced by Buchdahl. Since the direct use of ray-tracing formulas requires several changes to Buchdahl's original definition, the idea of the quasi-invariant will be discussed in detail in this section.

We denote the object plane by $P$ and the paraxial image plane by $Q$. The media at these two planes are considered to be homogeneous. Consider now a finite ray traveling through the system. Using the notation of the previous sections, we define the components of the transverse aberration vector of the ray at the image plane, as usual, as

$$\Xi_x = x_Q - x_Q, \quad \Xi_y = y_Q - y_Q. \quad (32)$$

In the paraxial approximation a system invariant similar to Eq. (5) can be formed with the marginal ray and the projection on the $x-z$ plane of the chosen finite ray. Omitting indices numbering surfaces and media, one can write this invariant as

$$\lambda_z = n_0 u_x + h_\xi. \quad (33)$$

[The plus appears in Eq. (33) because of the sign convention used. See also Eqs. (23).] We now denote

$$\Lambda_{xP} = n_0 u_x + h_\xi, \quad \Lambda_{xQ} = n_0 u_x x_Q, \quad (34)$$

Because of the invariance of Eq. (33) at the object and image plane, we have

$$\lambda_{xQ} = \lambda_{xP} = \Lambda_{xP}. \quad (35)$$

Therefore we can write

$$n_0 u_Q x_Q - x_Q = n_0 u_Q x_Q - n_0 u_Q x_Q = n_0 u_Q x_Q - n_0 u_P x_P, \quad (36)$$

$$\Xi_z = \lambda_{xQ} - \lambda_{xP} = \Lambda_{xQ} - \Lambda_{xP}. \quad (37)$$

We can accomplish the required decomposition of Eq. (37) by defining a quantity $\Lambda_x$ having Eq. (33) as its paraxial approximation. Since the paraxial approximation of $\Lambda_x$ is an invariant, $\Lambda_x$ itself will be called, following Buchdahl, a quasi-invariant. We seek $\Lambda_x$ by

$$\Lambda_x = n_0 F(u) x + F(h) f(\xi). \quad (38)$$

where the unknown functions $F$ and $f$ still have to be determined. These functions will depend also on the coordinates of the finite ray, such that, according to Eq. (33),

$$F(u) = u, \quad F(h) = h, \quad f(\xi) = \xi. \quad (39)$$

Consider now the variation of Eq. (38) at each refraction or transfer in the system, starting from the object plane and ending with the image plane, and sum up all the resulting terms. We have

$$\Lambda_{xQ} - \Lambda_{xP} = \sum \Delta \Lambda_x. \quad (40)$$

As is known from aberration theory, transfer through homogeneous media does not contribute to the aberration coefficients. Therefore the unknown functions $F$ and $f$ can be determined by requiring that $\Delta \Lambda_x$ vanish at transfer through a homogeneous medium.

Consider first the case of the transfer between two planes. It follows from Eqs. (34) that at the object and paraxial image plane we must have $F(u) = u$, so we require that

$$F(u) = u, \quad F(h) = h \quad (41)$$

at any plane surface. It can easily be verified that the transfer contributions vanish for

$$f(\xi) = \frac{n}{\xi} \xi, \quad (42)$$

where $n$ is the refractive index of the homogeneous medium. If the planes are separated by distance $z$, we have

$$\Delta x = \frac{\xi}{\xi} z, \quad \Delta h = -u z, \quad (43)$$

and therefore

$$\Delta \Lambda_x = n u \Delta x + \frac{n \xi}{\xi} \Delta h = 0. \quad (44)$$

The generalization of Eq. (42) for inhomogeneous media is not unique. However, it seems natural simply to replace the constant refractive index by a varying one. For RGRIN media we thus have

$$f(\xi) = \frac{n(r^2)}{\xi} \xi, \quad (45)$$

with $n(r^2)$ given by Eq. (1).
5. THIRD-ORDER TERMS

In this section we collect from the variation of \( \Lambda_s \) at transfer through a RGRIN medium the third-order terms in \( (\sigma_x, \sigma_y) \) and \( (\tau_x, \tau_y) \), denoting their sum \( (\Delta \Lambda_s)_{3} \), and obtain therefrom raw expressions for the Seidel aberration coefficients.

Let us first separate in the expressions for \( x' \), \( y' \), \( \xi' \), and \( \eta' \) the terms of first and third order in \( x, y, \xi, \) and \( \eta \) from the higher-order terms:

\[
\begin{align*}
x' &= x'_1 + x'_3 + O(5), \\
y' &= y'_1 + y'_3 + O(5), \\
\xi' &= \xi'_1 + \xi'_3 + O(5), \\
\eta' &= \eta'_1 + \eta'_3 + O(5).
\end{align*}
\]

The expressions for the first- and third-order terms can be determined from Eqs. (10)-(16). If we write similarly

\[
\begin{align*}
\phi_x &= \phi_{x,0} + \phi_{x,2} + O(4), \\
\phi_y &= \phi_{y,0} + \phi_{y,2} + O(4),
\end{align*}
\]

it follows from Eqs. (11) that

\[
\begin{align*}
x'_{p+1} &= x \phi_{x,0} + \frac{x}{n_0 \bar{G}} \phi_{x,2}, \\
\xi'_{p+1} &= n_0 \phi_{x,0} + \xi \phi_{y,0}, \\
p &= 0, 2, \ldots.
\end{align*}
\]

Similar expressions can also be written for \( y' \) and \( \eta' \).

It will be seen below that, in order to simplify the derivation of \( (\Delta \Lambda_s)_{3} \) in the following calculations we can separate the terms of various orders in a way that differs...
from an usual power-series expansion in \( x, y, \xi, \) and \( \eta \) in two respects:

1. We disregard the dependence of the position of the planes \( A \) and \( A' \) on \( x, y, \xi, \) and \( \eta \) as given by Eq. (8) and therefore keep \( z \) and the barred marginal ray data constant.

2. We regard \( \mu \) as a quantity containing only second-order terms in \( x, y, \xi, \) and \( \eta \) (see Eq. (21)) and neglect the remaining terms of \( O(4) \).

Note first that \( t \) given by Eq. (10) also depends on \( x, y, \xi, \) and \( \eta \). Therefore it follows from Eq. (12) that

\[
\frac{t}{\xi} = \frac{n_0}{\xi} g z = g z / (1 - 2 \mu)^2 = g z (1 + \mu) + O(4). \tag{53}
\]

In Eqs. (14) we insert the Taylor-series expansions

\[
\cos t = \cos g z - \mu g z \sin g z + O(4),
\sin t = \sin g z + \mu g z \cos g z + O(4). \tag{54}
\]

Consequently it follows that

\[
\phi_{c,0} = \cos g z, \quad \phi_{s,0} = \sin g z, \quad \phi_{c,0} = -\sin g z, \quad \phi_{s,0} = \cos g z, \quad \phi_{c,2} = -\mu g z \sin g z + \mu_4 \phi_{c,11}, \quad \phi_{s,2} = -\mu g z \cos g z + \mu_4 \phi_{s,11}, \quad \phi_{c,2} = -\mu g z \sin g z + \mu_4 \phi_{c,11}, \quad \phi_{s,2} = -\mu g z \cos g z + \mu_4 \phi_{s,11}.
\]

Thus Eqs. (52) read for \( p = 1 \) as

\[
x_1' = x \cos g z + \frac{\xi}{n_0 g z} \sin g z, \quad \xi_1' = -n_0 g z \sin g z + \frac{\xi}{n_0 g z} \cos g z, \tag{56}
\]

and, for \( p = 3 \), after some algebra, as

\[
x_3 = \frac{\xi'_{11}}{n_0} + \mu_4 \phi_{c,11} x_1 + \mu_4 \phi_{s,11} \xi_1, \quad \xi_3 = -\mu n_0 g^2 x_1 + \mu_4 n_0 g^2 \phi_{c,11} x_1 + \xi \phi_{s,11}. \tag{57}
\]

Similarly, we have

\[
\frac{n}{\xi} = \left[ \frac{n}{\xi} \right]_0 + \left[ \frac{n}{\xi} \right]_2 + O(4) \tag{58}
\]

or, from Eq. (18),

\[
\frac{n}{\xi} = \frac{n}{(n^2 - \xi^2 - \eta^2)^{1/2}} = \frac{1}{\left[ 1 - \frac{1}{n^2} (\xi^2 + \eta^2) \right]^{1/2}} = 1 + \frac{1}{2 n_0^2} (\xi^2 + \eta^2) + O(4). \tag{59}
\]

Before transfer the series expansion in \( x, y, \xi, \) and \( \eta \) of Eq. (48) is given by

\[
\Lambda_z = n_0 g z + \frac{\eta}{n_0^2} \Lambda \eta (\xi^2 + \eta^2) + O(5) \tag{60}
\]

and after transfer by

\[
\Lambda'_z = n_0 g z' + \frac{\eta}{n_0^2} \Lambda \eta' (\xi'_1^2 + \eta^2) + O(5) \tag{61}
\]

It can be verified by direct substitution of the transfer equations for the primed quantities that

\[
n_0 g z + \frac{\eta}{n_0^2} \Lambda \eta (\xi^2 + \eta^2) - \frac{1}{2 n_0^2} \Lambda \eta (\xi'_1^2 + \eta^2) + O(5).
\]

Consequently we have

\[
\Lambda'_z - \Lambda_z = n_0 g z' + \frac{1}{2 n_0^2} [\Lambda \eta (\xi^2 + \eta^2) - \Lambda \eta (\xi'_1^2 + \eta^2)] + O(5). \tag{62}
\]

Since Eq. (63) is already of third order in \( x, y, \xi, \) and \( \eta \), \( \Lambda \Lambda_2 \) is obtained as follows: in each quantity depending on \( x, y, \xi, \) and \( \eta \) we keep only the lowest-order terms in these variables and then replace \( x, y, \xi, \) and \( \eta \) with their paraxial approximations (Eqs. (17)). Recall that a quantity thus modified is denoted by a tilde over it. Thus the barred marginal ray data are replaced by the unbarred, and \( z \) by \( d \). (This is why we could keep \( z \) and the barred marginal ray data constant in the above derivation. The \( O(2) \) contributions stemming from these quantities cancel out because of Eq. (62).) Thus we obtain from Eq. (63)

\[
\Lambda \Lambda_2 = n_0 g z'_2 + \frac{1}{2 n_0^2} \Delta [\Lambda \eta (\xi^2 + \eta^2)], \tag{64}
\]

where we have from Eqs. (57)

\[
x'_3 = \mu \frac{d \xi'}{n_0} + \mu_4 \phi_{c,11} x'_1 + \xi \phi_{s,11}, \quad \xi'_3 = -\mu n_0 g^2 d z' + \mu_4 n_0 g^2 \phi_{c,11} x'_1 + \xi \phi_{s,11}. \tag{65}
\]

The substitution of Eqs. (15)–(17), (20)–(22), and (24) into Eqs. (64) and (65) yields an expression of the form

\[
\Lambda \Lambda_2 = [c_1 (\sigma^2 + \sigma^2) + c_2 (\sigma_2, \tau_2 + \sigma_2, \tau_2) + c_3 (\tau_2^2 + \tau_2^2)] \times \sigma_2 + [c_4 (\sigma^2 + \sigma^2) + c_5 (\sigma_2, \tau_2 + \sigma_2, \tau_2) + c_6 (\tau_2^2 + \tau_2^2)] \tau_2. \tag{66}
\]

Because of the large number of symbolic calculations, this substitution and the subsequent processing of the results can be done most conveniently by means of computer algebra software. (We used DERIVE and MATHEMATICA.) The expressions for the coefficients \( c_1 \rightarrow c_9 \) are of considerable length, and therefore we do not reproduce them here. As can be expected from Eqs. (15), (22), and (24),

\[
\Lambda \Lambda_2 = n_0 g z + \frac{1}{2 n_0^2} \Lambda \eta (\xi^2 + \eta^2) + O(5)
\]
these expressions contain sums of sines and cosines. The coefficients of the trigonometric functions are quantities of total order 4 in the (unprimed) marginal and chief ray data.

It was noted in Section 4 that the definition of $A_4$ is not unique. All definitions can lead to the Seidel aberration coefficients, but the derivation is more direct if decomposition (49) is such that all third-order contributions have the same structure as Eqs. (5). For the transfer contributions this condition can be written as

$$-2(\Delta \lambda_4)_3 = \left[ T_1(\sigma_{\tau}^2 + \sigma_{\tau}^2) + 2T_2(\sigma_{\tau}^2 + \sigma_{\tau}^2) \right] + (T_3 + T_6)\tau_1^2 \tau_2 + T_4(\sigma_{\tau}^2 + \sigma_{\tau}^2) \right] \tau_3. \quad (67)
$$

This is not the case for the original definition of $\lambda_4$ given in Ref. 6. Therefore in Ref. 1 terms in the transfer contributions of $\Delta \lambda_4$ that deviate from Eq. (67) are canceled by corresponding terms in the surface contributions. If condition (67) is satisfied, then only five of the six coefficients of Eq. (66) are linearly independent. Otherwise, the deviating terms would cause all six coefficients to be independent. In our case, however, Eq. (67) is satisfied because by comparing the expressions for $c_3$ and $c_4$ we have found that $c_3 = 2c_4$.

Thus raw expressions for the transfer contributions of the medium to the Seidel aberration coefficients result from comparing the coefficients of Eqs. (66) and (67):

$$T_1 = -2c_1, \quad T_2 = -2c_4 = -c_9, \quad T_3 = -c_9, \quad P_T = -2c_9 + c_8, \quad T_4 = -2c_8. \quad (68)
$$

In accordance with Eqs. (2) and (3), $T_1$ stands for the spherical aberration, $T_2$ for coma, $T_3$ for astigmatism, $P_T$ for Petzval curvature, and $T_4$ for distortion.

A simple expression is obtained at this stage only for the Petzval curvature,

$$P_T = kdh^2/\theta_0, \quad (69)$$

as first obtained by Moore and Sands. 3

6. SYMMETRIZATION OF THE SEIDEL ABERRATION FORMULAS

In order to find simple expressions for the remaining transfer coefficients, we apply a heuristic technique. As noted above, the coefficients $c_1$ to $c_8$ containing large sums of trigonometric functions are all expressed through marginal and chief ray data before transfer. Since we have no reason to prefer ray data before transfer to those after transfer, we expect that a suitable combination of terms that are completely symmetrical in primed and unprimed quantities can lead to shorter expressions. By expressing the terms also through ray data after transfer, using Eqs. (24) and (25), we try to eliminate the explicit appearance of the trigonometric functions in Eqs. (68). Therefore we set up several requirements and seek all possible terms satisfying all of them.

1. We conjecture that, as in the case of surface contributions (6) and (7), the transfer contributions can be expressed by sums of terms in which the ray data appear either through invariants or through differences between quantities after and before transfer. No trigonometric functions should appear.

2. All terms must be of total order four in $u$, $h$, $w$, $m$, $u'$, $h'$, $w'$, and $m'$. Here we note that all invariants, $H$ given by Eq. (5) and $e_1$, $e_2$, and $e_3$ given by Eqs. (30), are of second order.

3. All terms must have the dimension of length. This is because both $\Xi_4$ and $\Lambda_4$ have that dimension and because $(\sigma_{\tau}, \sigma_{\tau})$ and $(\tau, \tau)$ are dimensionless. The elementary quantities having the dimension of length are $d$, $H$, $h$, $m$, $h'$, and $m'$; the dimensionless ones are $e_1$, $e_2$, $e_3$, $u$, $w$, $u'$, and $w'$; and $k = g^2$ has the dimension of length$^{-2}$.

4. All terms must vanish when $d$ tends to zero.

5. No term should contain a factor $d$ at a power larger than 1. It can be observed that in Eqs. (15) (with $r$ replaced by $gd$), $d$ appears either in the argument of trigonometric functions or as a linear factor; no higher powers occur. Therefore, after insertion of Eqs. (15), Eqs. (64) and (65) also contain $d$ only as a linear factor.

6. All terms must be real for $k < 0$ if the same formulas are used for both positive and negative gradients.

We have found only a limited number of independent terms satisfying all the above requirements. These terms can be divided into three groups:

(i) Terms containing a product of two invariants:

$$kdH^2, de_\rho e_\rho, \quad p, q = 1, 2, 3.$$  

(ii) Terms containing an invariant and the difference of a second-order ray data product:

$$H\Delta(u^2), \quad H\Delta(uw), \quad H\Delta(w^2),$$

$$e_\rho \Delta(hu), \quad e_\rho \Delta(mu), \quad e_\rho \Delta(mw), \quad p = 1, 2, 3.$$  

Because of the invariance of $H$, $e_1$, $e_2$, and $e_3$, other terms of this type can be expressed through the above six terms. We have

$$e_\rho \Delta(hu) = e_\rho \Delta(mu), \quad Hk\Delta(h^2) = -H\Delta(u^2), \quad Hk\Delta(hm) = -H\Delta(uw), \quad Hk\Delta(m^2) = -H\Delta(w^2).$$  

(iii) Terms containing the difference of a fourth-order ray data product:

$$\Delta(hu^2), \quad \Delta(hw^2), \quad \Delta(huw^2), \quad \Delta(h^2), \quad \Delta(mu^2), \quad \Delta(muw^2), \quad \Delta(mw^2).$$  

7. Because of Eqs. (2) and (3), each of transfer coefficients (68) must be of the same order in marginal ray data (MRD) $h$, $u$, $h'$, and $u'$ and chief ray data (CRD) $m$, $w$, $m'$, and $w'$, as the corresponding surface coefficient in Eqs. (6) and (7). Therefore the transfer coefficients are of the form

$$T_1: \text{MRD}^4, \quad T_2: \text{MRD}^3\text{CRD}, \quad T_3: \text{MRD}^2\text{CRD}^2, \quad T_4: \text{MRD}\text{CRD}^3,$$

where the superscript stands for the corresponding total order.
For each transfer coefficient we select all possible terms having the required MRD and CRD orders and assume that all transfer coefficients can be expressed as linear combinations of these terms. We assume, e.g., that $T_1$ is of the form

$$T_1 = n_0[C_1d_1 \Delta(hu) + C_2 \Delta(hu^2)].$$  \hspace{0.5cm} (70)

Inserting Eqs. (24) and (30) into Eq. (70) and then Eq. (70) into the first of Eqs. (68), we find that

$$-n_0(C_1 - N_4 - 1) \left[ \frac{1}{4} hu(g^2 h^2 - u^4) \cos 4gd \right] + \frac{1}{16g} (g^4 h^4 - 6g^2 h^2 u^2 + u^4) \sin 4gd$$

$$+ n_0(2C_2 + C_3 + 4N_4 - 1) \left[ \frac{1}{4} hu(g^2 h^2 - u^2) \cos 2gd \right] + \frac{1}{8g} (g^4 h^4 - u^4) \sin 2gd$$

$$+ \frac{n_0 d}{4} (2C_2 + 5N_4) g^3 h^3 u$$

$$- \frac{n_0}{4} (2C_2 + 2C_3 + 3N_4 - 2) hu^3 = 0. \hspace{0.5cm} (71)$$

Assumption (70) means that all brackets containing unknown coefficients must simultaneously vanish:

$$C_3 - N_4 - 1 = 0,$$

$$2C_2 + C_3 + 4N_4 - 1 = 0,$$

$$2C_1 + 3N_4 - 2 = 0,$$

$$2C_2 + 5N_4 = 0,$$

$$2C_2 + 2C_3 + 3N_4 - 2 = 0. \hspace{0.5cm} (72)$$

The system of linear equations (72) has the solutions

$$C_1 = 1 - 3N_4/2, \hspace{0.5cm} C_2 = -5N_4/2, \hspace{0.5cm} C_3 = 1 + N_4. \hspace{0.5cm} (73)$$

and assumption (70) turns out to be correct. Short expressions for $T_2$, $T_3$, and $T_4$ are obtained by the same technique. Finally, together with Eq. (69), the transfer contributions to the Seidel coefficients read as

$$T_1 = n_0d_1e_1^2(1 - 3N_4/2) + n_0(1 + N_4) \Delta(hu^2)$$

$$- 5n_0N_4 \Delta(hu)/2,$$

$$T_2 = n_0d_1e_2(1 - 3N_4/2) + n_0(1 + N_4) \Delta(hu^2 \omega)$$

$$- 5n_0N_4 \Delta(hu)/2 - N_4 \Delta(hu^2),$$

$$T_3 = n_0d_3e_3(1 - 3N_4/2) + n_0(1 + N_4) \Delta(hu^2 \omega)$$

$$- 5n_0N_4 \Delta(hu)/2 - 2N_4 \Delta(hu \omega) - N_4 P_{r/2},$$

$$T_4 = n_0d_4e_4(1 - 3N_4/2) + n_0(1 + N_4) \Delta(hu^2)$$

$$- 5n_0N_4 \Delta(hu \omega)/2 - N_4 \Delta(hu^2)/2. \hspace{0.5cm} (74)$$

It can easily be verified that if $k$ tends to zero for any finite value of $N_4$, i.e., in the case of a homogeneous medium, all transfer contributions vanish.

7. SHALLOW GRADIENTS

If the series expansion for the refractive index does not contain a quadratic term (shallow gradient), then Eq. (1) must be replaced by

$$n^2(r^2) = n_0^2(1 + e r^4) + O(6). \hspace{0.5cm} (75)$$

In this case all inhomogeneous surface contributions (7) vanish, and the paraxial transfer equations for the marginal and chief rays are given by Eqs. (28). However, we can obtain the transfer contributions to the Seidel coefficients directly from Eqs. (69) and (74) by letting $h$ tend to zero and $N_4$ tend to infinity such that $k = k^2 N_4$ is kept constant. This derivation can also be conveniently performed by means of computer algebra. In this case after applying a symmetrization technique too, we obtain

$$P_7 = 0, \hspace{0.5cm} (76)$$

$$T_1 = - \frac{4}{5} d_{n_0}e(h^4 + h^2 h' + h' h^2 + h h'^2 + h'^2),$$

$$T_2 = - \frac{1}{5} d_{n_0}e(4h^3 h' + 3h^2 h' h + 2 h h'^2$$

$$+ h' h^2 h', + 2 h h'^2 h' + 3 h h'^2 h'^2 + 4 h'^4),$$

$$T_3 = - \frac{2}{15} d_{n_0}e(6h^2 h^2 + 3 h h^2 h'^2 + h' h^2 h'^2$$

$$+ 4 h h'^2 h' + 3 h h'^2 h'^2 + h^2 h'^2$$

$$+ 3 h h'^2 h'^2 + 6 h^2 m^2),$$

$$T_4 = - \frac{1}{15} d_{n_0}e(4 h^2 m^2 + 3 h m^2 + 2 h^2 m^2$$

$$+ 2 h m^2 + 3 h m^2 + 2 h^2 m^2$$

$$+ 6 h^2 m^2 + 3 h m^2 + 4 h m^2). \hspace{0.5cm} (77)$$

8. TEST OF THE RESULTS

The validity of the Seidel aberration formulas given in the present paper can be verified by means of an independent numerical test based on ray tracing.

For the transverse magnification equal to $-1$, we have found with the above formulas several cases of thick single RGRIN lenses for which all five Seidel coefficients are zero, irrespective of the stop position, e.g., for

$$n_0 = 1.637, \hspace{0.5cm} k = 0.1087, \hspace{0.5cm} N_4 = 0.694658, \hspace{0.5cm} P_{r/2} = -0.783032, \hspace{0.5cm} d = 9.169290.$$

The distance from the object and the paraxial image to the corresponding end surfaces is 0.11487. All data are normalized such that the effective focal length is equal to unity.

We use this example to test the correctness of the Seidel formulas. Consider in an optical system a ray propagating close to the optical axis but otherwise arbitrary. Because for such a ray the aberration series converges rapidly, the only significant contributions to $X_2$ and $X_3$ are the ones of the lowest order that do not vanish, and all higher-order contributions can be neglected. Therefore, if all initial ray data are increased by a factor $a$, $X_2$ and $X_3$ will increase nearly by the factor $a^4$, where $\gamma$ stands for the lowest order of the nonvanishing
Table 1. Confirmation by Ray Tracing of the Correctness of the Seidel Aberration Formulas

<table>
<thead>
<tr>
<th>Step</th>
<th>( \gamma_p = \xi_p )</th>
<th>( \Xi_x )</th>
<th>( \xi_x )</th>
<th>( \Xi_\gamma )</th>
<th>( \xi_\gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.01</td>
<td>9.1920 (10^{-11} )</td>
<td>—</td>
<td>5.2739 (10^{-11} )</td>
<td>—</td>
</tr>
<tr>
<td>1</td>
<td>0.02</td>
<td>2.9681 (10^{-9} )</td>
<td>32.29</td>
<td>1.7460 (10^{-9} )</td>
<td>33.00</td>
</tr>
<tr>
<td>2</td>
<td>0.04</td>
<td>5.3551 (10^{-8} )</td>
<td>32.12</td>
<td>5.6207 (10^{-8} )</td>
<td>32.29</td>
</tr>
<tr>
<td>3</td>
<td>0.08</td>
<td>3.0740 (10^{-6} )</td>
<td>32.23</td>
<td>1.8134 (10^{-6} )</td>
<td>32.26</td>
</tr>
<tr>
<td>4</td>
<td>0.16</td>
<td>1.0106 (10^{-4} )</td>
<td>32.87</td>
<td>5.5927 (10^{-5} )</td>
<td>32.82</td>
</tr>
</tbody>
</table>

contributions. Set \( a = 2 \). If the Seidel formulas are correct, then the lowest-order contributions are in this case the fifth-order contributions, and we expect \( c = 2^5 = 32 \). Otherwise, the lens would not really have all Seidel coefficients zero, and we would obtain \( c = 2^3 = 8 \).

Since we need for this test accurate transverse aberration values that vary over several orders of magnitude, an analytical ray-tracing method is more adequate for this purpose than a numerical one because it does not involve step-size changes and numerical error evaluations. Therefore to determine \( \Xi_x \) and \( \Xi_\gamma \), we used the analytical method described in Ref. 7. For not-too-large values of aperture and field, by comparison with numerical solutions and with the exact solutions known in some special cases, these ray-tracing formulas have been found to be highly accurate.

We first trace a skew ray having, for simplicity, at the object plane \( x_p = \eta_p = 0 \) and \( \gamma_p = \xi_p = 0.01 \). We then double the initial ray data in four steps (to ensure that the results are not accidental) and determine for both aberration components the ratio \( \xi \) between the current value and the value at the previous step. As shown in Table 1, in all cases the result is in the vicinity of \( 2^5 = 32 \), as is required for proving the correctness of the Seidel formulas. The same result is also obtained with different rays. A similar test has also been successfully performed for arbitrary lenses by subtraction from the total aberrations the third-order contributions with use of Eqs. (2) and (3).

For small values of \( k \) and large values of \( N \), the numerical results obtained with the general Seidel formulas have been found to approach those obtained with the shallow gradient formulas, thus providing a successful test for the latter formulas, too.

CONCLUSIONS

In the first stage of the derivation of the transfer contributions of RGRIN lenses to the Seidel coefficients of the system, the technique described in this paper produces large algebraic expressions that seem similar to those given earlier by Moore and Sands.\(^3\) For shortening these expressions we have developed a heuristic method based on symmetry requirements. This method turns out to be very powerful. As in the case of the surface contributions [Eqs. (6) and (7)], simple closed expressions can be obtained for all transfer contributions, both in the general case [Eqs. (69) and (74)] and in the special case of shallow gradients [Eqs. (76) and (77)]. We can compute all contributions by tracing the marginal and chief rays through the system, using paraxial transfer formulas [Eqs. (24) and (25) and (27) and (28)] and the definitions of several paraxial invariants [Eqs. (4), (5), and (30)].

ACKNOWLEDGMENT

This study was supported by a grant to F. Bociort from the German Academic Exchange Service.

Permanent address, Laser Department, Institute of Atomic Physics, R-76900 Bucharest, Romania.

REFERENCES