Chromatic paraxial aberration coefficients for radial gradient-index lenses

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A simple derivation of analytic expressions for the chromatic paraxial aberration coefficients of radial gradient-index lenses is presented. By decomposing the transverse chromatic aberration vector of an arbitrary paraxial ray in contributions from refraction at the surfaces and from transfer through the inhomogeneous media of the system, remarkably short formulas for the contributions of transfer through the gradient medium to the axial and lateral color coefficients are obtained. In the thin-lens approximation these expressions lead to well-known results for the total chromatic aberrations of a radial gradient-index lens. © 1996 Optical Society of America

1. INTRODUCTION

The chromatic paraxial aberration coefficients for axial and lateral color, i.e., the coefficients describing the change with wavelength of the position and size of the image, are known to be a valuable tool for investigating the possibilities and limitations of chromatic correction for a given system layout. Traditionally, the two chromatic paraxial aberration coefficients are analyzed together with the five Seidel aberration coefficients, and experience shows that only optical systems in which both types of coefficients can be reduced to acceptable values are capable of producing an image of high quality. As in the homogeneous case, for gradient-index lenses simple expressions for chromatic paraxial aberration coefficients, describing specific dispersion properties of the gradients, are of considerable interest for optical design with these components.

The variety of dispersion effects that occur in gradient-index materials is larger than in the homogeneous case, because each parameter determining the refractive-index distribution of the gradient medium may change with wavelength. Thus, for a radial gradient-index (RGRIN) lens having the refractive-index distribution

$$n^2(r^2) = n_0^2(1 - kr^2 + \ldots), \quad (1)$$

cromatic paraxial aberrations are due to the chromatic changes of both $n_0$, the refractive index on the optical axis, and $k$, the coefficient of the quadratic refractive-index term. Design examples have shown that the variation with wavelength of the quadratic coefficient (gradient dispersion) plays a major role in the ability of RGRIN lenses to reduce the number of elements in optical systems in comparison with homogeneous systems meeting the same specifications.\(^1\)\(^2\)

Studies of the chromatic properties of gradient-index materials have revealed remarkable properties of the gradient dispersion. For instance, in addition to the ordinary Abbe number, an Abbe number describing gradient dispersion becomes a parameter that can be used for correcting the chromatic aberrations of the system. For gradients fabricated by the ion-exchange method, investigations of various glasses and ion-exchange pairs have shown that a large range of gradient Abbe numbers can potentially be obtained, particularly if multielement exchange processes are utilized.\(^3\) Thus even negative gradient Abbe numbers are possible.

Unlike in the homogeneous case, because of gradient dispersion the ray transfer through the gradient medium also contributes to the chromatic aberrations of a gradient-index lens. Thus, for gradient-index lenses having an arbitrary rotationally symmetric refractive-index distribution, Sands showed that the total chromatic paraxial aberrations of the lens can be written as sums of surface and transfer contributions and that the expressions of the surface contributions to the two chromatic coefficients are precisely the same as in the case of a homogeneous lens.\(^4\)

Analytic expressions for the transfer contributions of axial and radial gradients to the two chromatic paraxial aberrations were also first given by Sands.\(^5\) For axial gradients, results equivalent to those of Sands were obtained by the present author by the method described below. For radial gradients, however, although Sands’s derivation is in principle correct, the results contain a computation error\(^6\) that, seemingly, remained uncorrected for more than two decades. At present, only results in the thin-lens approximation are used in the literature for the chromatic aberrations of RGRIN lenses.\(^2\)

In this paper the transfer contributions of RGRIN lenses to axial and lateral color coefficients will be obtained. After a brief discussion of the paraxial approximation in Section 2, a technique for decomposing the transverse chromatic aberration vector of an arbitrary paraxial ray in contributions from refraction at the surfaces and from transfer through the inhomogeneous media of the system—a necessary prerequisite for deriving the aberration coefficients—will be developed in Section 3. The analytic expressions for the two transfer coefficients will then be derived in Section 4. These expressions turn out to be equivalent to those of Sands.\(^5\)
after correction but are simpler than those of Sands. In the opinion of the author, the derivation method described here, which is closely related to that described earlier5 for the derivation of the Seidel aberration coefficients of RGRIN lenses, is also less intricate than Sands’s original derivation. Finally in Section 5 it will be shown that, in a properly defined thin-lens approximation for RGRIN lenses, the chromatic paraxial aberration coefficients derived in this study lead to well-known results for the total chromatic aberrations of a radial gradient-index lens. In what follows, the notation will be the same as that used in Ref. 8.

2. PARAXIAL APPROXIMATION

To derive the transfer contributions to the chromatic paraxial aberrations, it is necessary to discuss briefly paraxial ray propagation in general and for RGRIN media in particular.

Consider a rotationally symmetric optical system consisting of homogeneous and RGRIN lenses. We denote the object plane \(P\), the paraxial image plane \(Q\), and the entrance pupil plane \(EP\). We define an arbitrary ray through the system by its normalized coordinates in the object plane \((x_r, y_r)\), the field coordinates, and at the entrance pupil \((\sigma_s, \sigma_f)\), the aperture coordinates. Thus, if \(r_{EP}\) is the radius of the entrance pupil and \(r_P\) is the maximum object height, the Cartesian ray coordinates at the object plane are given by

\[
x_P = r_P x_r, \quad y_P = r_P y_r
\]

and at the entrance pupil plane by

\[
x_{EP} = r_{EP} \sigma_s, \quad y_{EP} = r_{EP} \sigma_f.
\]

At each surface the position and direction of the ray are fully determined by the \(x\) and \(y\) coordinates of its point of intersection with the surface and by the optical direction cosines \(\xi\) and \(\eta\), corresponding to \(x\) and \(y\). (The optical direction cosines are the direction cosines multiplied by the refractive index.) Because of rotational symmetry, the power series expansions of \(x\) and \(y\) with respect to \(\sigma_s\), \(\sigma_f\), \(\tau_s\), and \(\tau_f\) contain only terms having an odd total order. In what follows, it will suffice to consider \(x\), \(y\), \(\xi\), and \(\eta\) only in the paraxial approximation, which will be denoted by a tilde over the corresponding symbol.

The paraxial approximation holds for rays propagating through the optical system close to the optical axis, i.e., when the aperture and field coordinates are small. In this case, if only the lowest-order terms in the aperture and field coordinates are kept, at each surface of the system \(x\), \(y\), \(\xi\), and \(\eta\) are given by linear combinations of \(\sigma_s\), \(\sigma_f\), \(\tau_s\), and \(\tau_f\), and it can be shown that the coefficients are the height and slope of the paraxially traced marginal and chief rays at that surface:

\[
\tilde{x} = m x_r + h \sigma_x, \quad \tilde{\xi} = -n_0 u \tau_s - n_0 w \sigma_x,
\]

\[
\tilde{y} = m y_r + h \sigma_y, \quad \tilde{\eta} = -n_0 u \tau_f - n_0 w \sigma_y.
\]

Here the paraxial marginal and chief ray heights are denoted \(h\) and \(m\), respectively, and the corresponding marginal and chief rays slopes are denoted \(u\) and \(w\). (See Fig. 1.) The sign convention adopted for \(u\) and \(w\) is that their signs are the opposite of those of the corresponding direction cosines. [This is why in Eqs. (4) we have minus signs in the equations for \(\tilde{\xi}\) and \(\tilde{\eta}\)].

Explicit expressions for paraxial ray tracing through RGRIN media can be found from the solutions for refractive-index distribution (1) of the differential equations describing the ray propagation in inhomogeneous media.9,10 The following notation is adopted: quantities after transfer are denoted by a prime, whereas quantities before transfer are left unprimed.

Consider first the case \(k > 0\). Given the values of the coordinates and optical direction cosines before transfer, the corresponding values after transfer are

\[
\tilde{x}' = \tilde{x} \cos gd + \frac{\tilde{\xi}}{n_0 g} \sin gd,
\]

\[
\tilde{\xi}' = \tilde{\xi} \cos gd - n_0 g \tilde{x} \sin gd,
\]

where \(d\) is the thickness of the RGRIN medium and \(g = k^{1/2}\). Similar equations hold of course for the paraxial approximations of \(y\) and \(\eta\). By substituting into Eqs. (5)

\[
\tilde{x} = h, \quad \tilde{x}' = h',
\]

\[
\tilde{\xi} = -n_0 u, \quad \tilde{\xi}' = -n_0 u',
\]

we then have for the marginal ray

\[
w' = u \cos gd + h g \sin gd,
\]

\[
h' = -\frac{u}{g} \sin gd + h \cos gd.
\]

Similarly, for the chief ray we obtain

\[
w' = w \cos gd + m g \sin gd,
\]

\[
m' = -\frac{w}{g} \sin gd + m \cos gd.
\]

For \(k < 0\), when \(g\) is imaginary, inserting \(g = (-k)^{1/2}\), i.e., \(g = -i g\), into Eqs. (7) and (8) and using the relations between trigonometric functions with imaginary arguments and the corresponding hyperbolic functions yields

\[
w' = u \cos \tilde{g}d - h \tilde{g} \sinh \tilde{g}d,
\]

\[
h' = -\frac{u}{\tilde{g}} \sinh \tilde{g}d + h \cosh \tilde{g}d,
\]

\[
w' = w \cosh \tilde{g}d - m \tilde{g} \sinh \tilde{g}d,
\]

\[
m' = -\frac{w}{\tilde{g}} \sinh \tilde{g}d + m \cosh \tilde{g}d.
\]

It might be noted that the difference between the transfer formulas in the two cases \(k > 0\) and \(k < 0\) disappears if
the trigonometric and hyperbolic functions are expanded into power series. Setting
\[ E_1(kd^2) = \cos \frac{gd}{kd}, \quad E_2(kd^2) = \frac{\sin \frac{gd}{kd}}{kd}, \]  
we have, for example, for the marginal ray
\[ u' = uE_1(kd^2) + hkdE_2(kd^2), \]
\[ h' = udE_2(kd^2) + hE_1(kd^2). \]  
From the series expansions of Eqs. (10) it can be observed that \( E_1 \) and \( E_2 \) depend indeed only on \( g^2 = k \):
\[ E_1(kd^2) = 1 - \frac{1}{2!} kd^2 + \frac{1}{4!} (kd^2)^2 - \ldots, \]
\[ E_2(kd^2) = 1 - \frac{1}{3!} kd^2 + \frac{1}{5!} (kd^2)^2 - \ldots. \]  
Note that for \( k = 0 \) we have \( E_1 = E_2 = 1 \), and Eqs. (7) and (8) become the well-known transfer equations for a homogeneous medium. For instance, for the marginal ray we obtain
\[ u' = u, \quad h' = h - ud. \]  
One can verify by direct substitution, using Eqs. (7) and (8), that
\[ H = mn_0u - hn_0w; \]  
the well-known paraxial-system invariant in the case of homogeneous optical systems is an invariant for systems containing RGRIN media, as well. More generally, for rotationally symmetric systems, invariants similar to Eq. (14) can be obtained for any pair of paraxial rays. In what follows, a paraxial invariant formed with the marginal ray and the projection on the \( x-z \) plane of an arbitrary paraxial ray,
\[ \tilde{\lambda}_s = n_0ux + h\xi, \]  
will play a major role in the derivation of chromatic aberration coefficients. As was first shown by Hopkins,\(^9\) additional quantities that remain unchanged at transfer through the RGRIN medium are
\[ e_1 = kh^2 + u^2, \quad e_2 = km + uw, \quad e_3 = km^2 + w^2. \]  

3. CHROMATIC PARAXIAL QUASI-INARIANT

The first step for deriving chromatic paraxial aberration coefficients will be the decomposition of the transverse chromatic aberration vector of an arbitrary paraxial ray in contributions from refraction at the surfaces and from transfer through the inhomogeneous media of the system. In Section 4 the transfer contributions to the transverse chromatic ray aberration will then lead to the transfer contributions to the axial and lateral color coefficients of RGRIN lenses.

The decomposition method given below can be regarded as a simplified version of the method based on the idea of the quasi-invariant first introduced by Buchdahl\(^11\) and is also closely related to that used for the derivation of the Seidel aberration coefficients of RGRIN lenses.\(^8\) Since Buchdahl’s method, which is a powerful tool for computing monochromatic and chromatic aberration coefficients of arbitrary order, has in its original form many details that are not of relevance for the present purpose, for the convenience of the reader a simplified version of it will be discussed here in detail.

In a rotationally symmetric optical system consisting of homogeneous and RGRIN media, let \( Q \) be the paraxial image plane conjugated to the object plane \( P \) for some reference wavelength \( \lambda_0 \). Assume that the media at planes \( P \) and \( Q \) are homogeneous. Because of dispersion, paraxial rays starting from the same object point \((x_P, y_P)\) but having different wavelengths intersect plane \( Q \) at different points. For a ray of an arbitrary wavelength \( \lambda \), the departures
\[ \tilde{\Xi}_x = \hat{x}_Q - \hat{x}_Q, \quad \tilde{\Xi}_y = \hat{y}_Q - \hat{y}_Q \]  
define the components of the transverse chromatic paraxial aberration vector of the ray. For convenience of notation, each quantity at an arbitrary wavelength is denoted by the subscript \( \lambda \) at the symbol of the quantity. All quantities without this subscript are considered at the reference wavelength \( \lambda_0 \).

As mentioned above, we first have to write the chromatic aberration vector [Eqs. (17)] as the sum of contributions from all refractions at surfaces and transfers through inhomogeneous media. Consider therefore the paraxial invariant \( \tilde{\lambda}_x \) given by Eq. (15). If we note that the change at refraction or transfer of \( \tilde{\lambda}_x \) vanishes, the basic idea is to construct a quantity \( \tilde{\lambda}_{\lambda x} \), which reduces to \( \tilde{\lambda}_x \) for \( \lambda = \lambda_0 \), in such a way that the contributions to the total chromatic aberration vector [Eqs. (17)] of refraction or transfer are related to the corresponding changes of \( \tilde{\lambda}_{\lambda x} \). Following Buchdahl, \( \tilde{\lambda}_{\lambda x} \) will be called the chromatic paraxial quasi-invariant.

At planes \( P \) and \( Q \), where we have \( h = 0 \), the simplest guess for \( \tilde{\lambda}_{\lambda x} \) is
\[ \tilde{\lambda}_{\lambda x} = n_Pu_P\hat{x}_P, \quad \tilde{\lambda}_{\lambda x} = n_Qu_Q\hat{x}_Q. \]  
Consequently, we can write
\[ n_Qu_Q(\hat{x}_Q - \hat{x}_Q) = \tilde{\lambda}_{\lambda x} - \tilde{\lambda}_{\lambda x}. \]  
However, since \( \tilde{\lambda}_x \) is an invariant throughout the entire system and since at the object plane \( P \) we have \( \hat{x}_P = \hat{x}_P \), it follows that
\[ \tilde{\lambda}_{\lambda x} = \tilde{\lambda}_{\lambda x} = \tilde{\lambda}_{\lambda x}. \]  
We thus have
\[ n_Qu_Q\tilde{\Xi}_x = \tilde{\lambda}_{\lambda x} - \tilde{\lambda}_{\lambda x}. \]  
Assuming that \( \tilde{\lambda}_{\lambda x} \) can be properly defined at each surface of the system, if we sum up the variation of \( \tilde{\lambda}_{\lambda x} \) over all
refractions and transfers in the system, Eq. (21) becomes the desired decomposition for the first of Eqs. (17):

\[ n_u u_q \xi_{la} = \sum \Delta \lambda_{la} . \]  

(22)

For an arbitrary quantity \( X \), by \( \Delta X \) we denote the difference between the values of \( X \) after and before refraction or transfer.

The precise form of \( \lambda_{la} \) can be determined by observing that for homogeneous optical systems, aberrations are considered to stem only from refraction at surfaces. Thus the variation of \( \lambda_{la} \) at transfer through homogeneous media must vanish. Let \( n \) and \( n_l \) be the refractive indices at wavelengths \( \lambda_0 \) and \( \lambda \) and \( d \) be the thickness of the homogeneous medium. Since \( u \) and \( \xi \) remain unchanged, we have

\[ \Delta \xi = \xi_{la} d / n, \quad \Delta h = -ud . \]  

(23)

It can then be easily seen that \( \lambda_{la} \) written as

\[ \lambda_{la} = n(x) x + n_{l}(x) h \xi \]  

(24)

satisfies the required condition:

\[ \Delta \lambda_{la} = n(x) \Delta x + n_{l}(x) \Delta h = 0 . \]  

(25)

For RGRIN lenses a straightforward generalization of Eq. (24) is

\[ \lambda_{la} = n_0(x) x + n_{l0}(x) h \xi . \]  

(26)

Equations (26) and (22), where the sum now extends only over surfaces and RGRIN media, give the required decomposition of the first of Eqs. (17) in surface and transfer contributions. If we replace in these equations \( x \) and \( \xi \) with \( y \) and \( n \), respectively, a similar decomposition can be written for the second of Eqs. (17). However, for the purpose of this study it will suffice to consider only the \( x \) component of chromatic aberration (17) and corresponding quasi-invariant [Eq. (26)].

If we note that for the ray parameters at wavelength \( \lambda \), equations similar to Eqs. (4) can be written,

\[ x = m_\lambda \sigma x + h_\lambda \sigma x, \quad \xi = -n_0(x) w_x \tau x - n_{l0}(x) w_x \sigma x, \]  

(27)

it follows that \( \xi_{la} \) is a linear quantity in aperture and field coordinates. Thus we can write

\[ -n_u u_q \xi_{la} = -\sum \Delta \lambda_{la} = \Gamma_{l1} \sigma x + \Gamma_{l2} w_x . \]  

(28)

Coefficients \( \Gamma_{l1} \) and \( \Gamma_{l2} \) are the total chromatic paraxial aberration coefficients of the system. The minus sign in Eq. (28) appears to ensure, as shown below, the well-known relationships between \( \Gamma_{l1} \) and \( \Gamma_{l2} \) and the chromatic change in image position and height.\(^{12}\) By setting \( \sigma = 1 \) and \( \tau = 0 \) in Eqs. (27) and (28), we have at plane \( Q \)

\[ \Gamma_{l1} = -n_u u_q h_\lambda \sigma \]  

(29)

From the second of Eqs. (13) it follows that the marginal ray of wavelength \( \lambda \) intersects the optical axis at the distance

\[ \delta_{l} s = h_\lambda / u_\lambda Q \]  

(30)

from the plane \( Q \). In fact, Eq. (30) gives the axial displacement of the paraxial image position at wavelength \( \lambda \) with respect to that at \( \lambda_0 \). If we consider only lowest-order chromatic effects, in Eq. (30) \( u_\lambda \) can be replaced by \( u_\lambda \). It follows then from Eq. (29) that axial displacement (30) is related to coefficient \( \Gamma_{l1} \), called the total axial color coefficient, by

\[ \delta_{l} s = -\Gamma_{l1} / n_u u_\lambda Q . \]  

(31)

Similarly, setting \( \sigma = 0 \) and \( \tau = 1 \) gives

\[ \Gamma_{l2} = -n_u u_\lambda (u_\lambda - m_\lambda) . \]  

(32)

Note that according to Eq. (14) we have \( m_\lambda n_u u_\lambda = H \). If we denote the chromatic change of an arbitrary quantity \( X \) by

\[ \delta_{l} X = X_{l} - X , \]  

(33)

the relative chromatic change of the image height is related to \( \Gamma_{l2} \) by

\[ \delta_{l} m_\lambda / m_\lambda = -\Gamma_{l2} / H . \]  

(34)

Therefore \( \Gamma_{l2} \) is called the total lateral color coefficient.

### 4. CHROMATIC PARAXIAL ABERRATION COEFFICIENTS

In this section the analytic expressions for the transfer contributions of a RGRIN medium to axial and lateral color will be obtained.

Consider the variation at refraction at a surface or transfer through a RGRIN medium of the chromatic quasi-invariant \( \lambda_{la} \) given by Eq. (26). Since \( \lambda = \lambda_0 \), \( \lambda_{la} \) becomes invariant (15), whose variation vanishes, we obtain by subtraction

\[ \Delta \lambda_{la} = \Delta \left[ n_0(x) x + n_{l0}(x) h \xi - \hat{x} \right] \]  

(35)

Substituting Eqs. (27) and (4) into Eq. (35) and using the notation of Eq. (33) yields

\[ \Delta \lambda_{la} = \Delta [n_0(x) \delta_{l} x + \delta_{l} m_\lambda \tau x] - n_0 \delta_{l} h [\delta_{l} x + \delta_{l} w \tau x] . \]  

(36)

After regrouping terms, we have

\[ -\Delta \lambda_{la} = \Delta [n_0(x) \delta_{l} x - \delta_{l} h] \sigma x + \Delta [n_0(x) \delta_{l} w - \delta_{l} m_\lambda \tau x] , \]  

(37)
Thus, as in the case of Eq. (28), we can write for refraction at a surface,
\[ -\Delta \lambda_{x} = S_{x} \sigma_{x} + S_{x2} \tau_{x}, \] (38)
and for transfer through a RGRIN medium,
\[ -\Delta \lambda_{x} = T_{x} \sigma_{x} + T_{x2} \tau_{x}. \] (39)

It follows then that the total chromatic paraxial aberration coefficients can be written as sums over surfaces and RGRIN media,
\[ \Gamma_{xp} = \sum_{\text{surfaces}} S_{xp} + \sum_{\text{RGRIN media}} T_{xp}, \quad p = 1, 2, \] (40)
where \( S_{x1} \) and \( T_{x1} \) denote surface and transfer contributions, respectively, to axial color and \( S_{x2} \) and \( T_{x2} \) denote corresponding contributions to lateral color.

Equations (37)–(39) can be used as starting points for deriving analytic expressions for the surface and transfer contributions to axial and lateral color. However, in this section only the derivation of the transfer contributions will be described in detail. Since the expressions for the surface contributions \( S_{xp} \) are already known from the literature, their derivation by the present method, although simple, will not be reproduced here. With the present notation these expressions read
\[ S_{x1} = h n_{0} i \left( \frac{\delta_{i} n_{0}}{n_{0}} \right), \quad S_{x2} = h n_{0} j \left( \frac{\delta_{j} n_{0}}{n_{0}} \right). \] (41)
Here \( n_{0} i \) and \( n_{0} j \) are the paraxial refraction invariants
\[ n_{0} i = n_{0} h p - n_{0} u, \quad n_{0} j = n_{0} m p - n_{0} w, \] (42)
where \( p \) is the surface curvature and \( i, j \) are the incidence angles for the marginal and the chief rays, respectively (see Fig. 1). Note that Eqs. (41) can be obtained from the corresponding relations in the homogeneous case\(^{12}\) simply by replacing \( n \) with \( n_{0} \).

For deriving the expressions for \( T_{x1} \) and \( T_{x2} \), the same approximation will be introduced as in the case of Eqs. (41) and of similar formulas that are commonly used for computing chromatic paraxial aberrations for homogeneous optical systems: In the following derivations only the lowest-order chromatic effects will be considered; i.e., since we interpret \( \delta \) given by Eq. (33) as an operator, in all expressions only the first-order terms in \( \delta \) will be kept.

Obviously, the above approximation is strictly valid only for small values of the difference \( \lambda - \lambda_{0} \). This situation is similar to that of the Seidel aberration coefficients, which give an accurate description of the monochromatic aberrations only for systems having small aperture and field. However, since the ability to control the lowest-order monochromatic and chromatic aberration contributions turns out to be a necessary (although not a sufficient) condition for obtaining good image quality, Seidel and chromatic paraxial coefficients are used in practical applications also for values of aperture, field, and wavelength range that are larger than those for which a numerically accurate description of the system’s imagery by these formulas can be expected. Thus \( \lambda - \lambda_{0} \) is usually taken in the expressions of chromatic aberrations as the full wavelength range over which the system is to be used.

When we compare Eqs. (37) and (39), the contributions of transfer through a RGRIN medium to axial and lateral color can be written as
\[ T_{x1} = n_{0} \left( (h' \delta_{x} u' - u' \delta_{x} h') - (h \delta_{x} u - u \delta_{x} h) \right), \]
\[ T_{x2} = n_{0} \left( (h' \delta_{x} u' - u' \delta_{x} m') - (h \delta_{x} w - u \delta_{x} m) \right). \] (43)

Consider first the coefficient \( T_{x1} \). For determining the explicit form of the quantities \( \delta_{x} u' \) and \( \delta_{x} h' \), note that keeping only first-order terms in \( \delta \) means that \( \delta \) can be regarded as a differential operator. Thus differentiating transfer equations (7) shows that \( \delta_{x} u' \) and \( \delta_{x} h' \) can be written as sums of three groups of terms,
\[ \delta_{x} u' = A_{1} + A_{2} + A_{3}, \]
\[ \delta_{x} h' = B_{1} + B_{2} + B_{3}, \] (44)
where we have
\[ A_{1} = \delta_{x} u \cos gd + \delta_{x} h \sin gd, \]
\[ A_{2} = (-u \sin gd + h g \cos gd) d \delta_{x} g, \]
\[ A_{3} = h \sin gd \delta_{x} g, \] (45)
\[ B_{1} = \delta_{x} u \sin gd + \delta_{x} h \cos gd, \]
\[ B_{2} = \left( \frac{u}{g} \cos gd - h \sin gd \right) d \delta_{x} g, \]
\[ B_{3} = \frac{1}{g^{2}} u \sin gd \delta_{x} g. \] (46)

Correspondingly, the quantity in the square brackets of the first of Eqs. (43) can also be written as
\[ (h' \delta_{x} u' - u' \delta_{x} h') - (h \delta_{x} u - u \delta_{x} h) = C_{1} + C_{2} + C_{3}, \] (47)
with
\[ C_{1} = h' A_{1} - u' B_{1} - (h \delta_{x} u - u \delta_{x} h), \]
\[ C_{2} = h' A_{2} - u' B_{2}, \]
\[ C_{3} = h' A_{3} - u' B_{3}. \] (48)

By substituting for \( h', u', A_{1}, \) and \( B_{1} \), their expressions given by Eqs. (7), (45), and (46), it follows after an elementary calculation that \( C_{1} = 0 \). Observe also that, according to Eqs. (7), \( A_{2} \) and \( B_{2} \) can be written as
\[ A_{2} = h' d g \delta_{x} g, \quad B_{2} = -u' d \frac{\delta_{x} g}{g}. \] (49)

Consequently, for \( C_{2} \) we obtain
\[ C_{2} = (h'^{2} d g^{2} + u'^{2} d) \left( \frac{\delta_{x} g}{g} \right) = d e_{1} \frac{\delta_{x} g}{g}, \] (50)
where \( e_{1} \) is the first of the paraxial transfer invariants given by Eqs. (16). Finally, if we write \( A_{3} \) as
\[ A_{3} = (u - u \cos gd) \frac{\delta_{x} g}{g}, \] (51)
the expression for $C_3$ reads

$$C_3 = \frac{\delta l k}{g} \left[ h'u' - u \left( h' \cos gd + \frac{1}{g} u' \sin gd \right) \right]. \quad (52)$$

Replacing $d$ with $-d$ in Eqs. (7) allows $h$ and $u$ to be expressed in terms of $h'$ and $u'$. Thus we have

$$h' \cos gd + \frac{1}{g} u' \sin gd = h,$$  \hspace{1cm} (53)

and Eq. (52) becomes

$$C_3 = \frac{\delta l g}{g} (h'u' - hu). \quad (54)$$

Since $g = k^{1/2}$, we can write

$$\frac{\delta l g}{g} = \frac{1}{2} \frac{\delta l k}{k}. \quad (55)$$

and the expression for $T_{x1}$ reads

$$T_{x1} = \frac{n_0}{2} \frac{\delta l k}{k} [d e_1 + \Delta(hu)]. \quad (56)$$

Proceeding similarly, $T_{x2}$ turns out to be

$$T_{x2} = \frac{n_0}{2} \frac{\delta l k}{k} [d e_2 + \Delta(hw)]. \quad (57)$$

Equations (56) and (57) are the final expressions of the contributions of transfer through RGRIN media to axial and lateral color. As will be shown in Section 5, for practical purposes it is convenient to express $\delta l k$ in these expressions through the gradient Abbe number. Thus, as in the homogeneous case, for RGRIN lenses Eqs. (40) and (41) and Eqs. (56) and (57) give the two chromatic paraxial aberration coefficients in terms of data of the marginal and chief rays, which are traced through the system paraxially at reference wavelength $\lambda_0$.

Note also the remarkable simplicity of Eqs. (56) and (57). As in the case of the Seidel coefficients, these expressions are shorter than other possible equivalent expressions because complete symmetry exists between the quantities before and after transfer. Thus in Eqs. (56) and (57) the paraxial marginal- and chief-ray data appear either through transfer invariants (16) or as differences given by the $\Delta$ operators. Correcting the error in the results obtained earlier by Sands leads to expressions that are equivalent to Eqs. (56) and (57) but not as simple as the latter.

As an additional verification of correctness for Eqs. (56) and (57), for various lenses where the radial gradient contributes substantially to chromatic aberrations, numerical results obtained with Eqs. (40) and (41) and Eqs. (56) and (57) were compared with results obtained with Eqs. (29) and (32), which are generally valid for rotationally symmetrical optical systems. Recall that, whereas exact formulas (29) and (32), which are expressed in terms of rays traced at both $\lambda$ and $\lambda_0$, are valid for arbitrary values of the difference $\lambda - \lambda_0$, i.e., for arbitrary values of $\delta l$, in approximate expressions (41) and (56) and (57), only linear terms in $\delta l$ have been kept. In numerical tests, for small values of $\delta l n_0$ and $\delta l k$ it was observed that when $\delta l n_0$ and $\delta l k$ are increased by a factor $\gamma$ the difference between the exact and the approximate results increases nearly as $\gamma^2$. This confirms the fact that for transfer through a RGRIN medium the linear terms in $\delta l$ are correctly described by Eqs. (56) and (57).

5. THIN-LENS APPROXIMATION

For simple optical systems, thin-lens theory is a useful tool for obtaining additional insight into the possibilities and limitations of aberration correction. In this section it will be shown that in a suitably defined thin-lens approximation, the results derived above lead for the total chromatic aberrations of RGRIN lenses to the same expressions as those that are well-known from the literature.

For homogeneous optical systems, thin-lens formulas for paraxial quantities and Seidel and chromatic aberrations are obtained by setting the lens thickness equal to zero in the exact formulas. For RGRIN lenses this approach is not adequate, because in this way the contributions stemming from transfer through the gradient medium would be lost. To include the gradient medium contributions into the thin-lens theory, we note that a thin RGRIN lens situated in air ($n_p = n_q = 1$) can be regarded as a homogeneous thin lens of refractive index $n_0$ having the same curvatures in contact with a thin Wood lens. (A Wood lens is a RGRIN lens with plane end faces.)

Consider first a thin Wood lens having a thickness much smaller than its focal length

$$d \ll |f_w|. \quad (58)$$

As will be shown below, the above approximation is equivalent to the condition

$$|k| d^2 \ll 1. \quad (59)$$

With condition (59), Eqs. (12) become $E_1 = E_2 = 1$, and transfer equations (11) for the marginal ray read

$$u' = u + kdh,$$  \hspace{1cm} (60)

$$h' = h - ud. \quad (61)$$

Note that Eq. (61) is the same as in the case of a homogeneous medium of finite thickness. However, as in the latter case, the change of the ray height inside the thin RGRIN lens can be regarded as small, and Eq. (61) will be replaced with

$$\Delta h = 0. \quad (62)$$

Consider an axial object point situated at infinity (see Fig. 2). Since the end surfaces are plane, we have in the gradient medium at the first surface $u = 0$, and consequently

$$h/f_w = u_Q = n_0 u' = n_0 kdh. \quad (63)$$

The first equality in Eq. (63) is deduced from Fig. 2, the second one comes from refraction at the second end face, and the third one follows from Eq. (60). Thus the focal length of the Wood lens is

$$f_w = (n_0 kd)^{-1}. \quad (64)$$
and its power \( \phi_g = f_w^{-1} \) is given by
\[
\phi_g = n_0kd. \tag{65}
\]

[Condition (59) now follows immediately from condition (58).] The total power of an arbitrary thin RGRIN lens having end faces of curvatures \( \rho_1 \) and \( \rho_2 \) will then be the sum \( \phi = \phi_h + \phi_g \) of the powers of its homogeneous part
\[
\phi_h = (n_0 - 1)(\rho_1 - \rho_2) \tag{66}
\]
and of its gradient part [Eq. (65)].

To obtain the total chromatic paraxial aberration coefficients of a thin RGRIN lens, consider first Eq. (56). With Eqs. (16) and Eqs. (60)–(62) the first term in the square brackets reads
\[
de_1 = kh^2d + u^2d = h\Delta u - u\Delta h = h\Delta u.
\]
Thus we have
\[
de_1 + \Delta(hu) = 2h\Delta u = 2h^2\frac{\phi_g}{n_0}, \tag{67}
\]
and the transfer contribution of the RGRIN medium to axial color reads
\[
T_{\lambda 1} = \frac{\delta\lambda}{k} h^2\phi_g. \tag{68}
\]
The total axial color of the lens is the sum of transfer contribution (68) and the corresponding contributions of the two end faces:
\[
\Gamma_{\lambda 1} = S_{\lambda 1,1} + T_{\lambda 1} + S_{\lambda 1,2}. \tag{69}
\]
It follows from Eqs. (41) and (42) and Eqs. (65) and (66) as that the sum of the surface contributions can be rewritten as
\[
S_{\lambda 1,1} + S_{\lambda 1,2} = h \frac{\delta\lambda}{n_0} [(n_0i)_1 - (n_0i)_2], \tag{70}
\]
where we have
\[
(n_0i)_1 - (n_0i)_2 = n_0h\rho_1 - n_0u - (n_0h\rho_2 - n_0u')
= n_0h(\rho_1 - \rho_2) + n_0kdh
= \frac{n_0h}{n_0 - 1} \phi_h + h\phi_g. \tag{71}
\]
Since we can write
\[
\frac{\delta\lambda n_0}{n_0} + \frac{\delta\lambda}{k} = \frac{\delta\lambda(n_0k)}{n_0k}, \tag{72}
\]
the total axial color [Eq. (69)] of a thin RGRIN lens reads
\[
\Gamma_{\lambda 1} = h^2 \left[ \frac{\delta\lambda n_0}{n_0 - 1} \phi_h + \frac{\delta\lambda(n_0k)}{n_0k} \phi_g \right]. \tag{73}
\]
Proceeding similarly, for lateral color we obtain
\[
\Gamma_{\lambda 2} = h m \left[ \frac{\delta\lambda n_0}{n_0 - 1} \phi_h + \frac{\delta\lambda(n_0k)}{n_0k} \phi_g \right]. \tag{74}
\]
Thus if the stop is situated at the lens, we have \( m = 0 \), and lateral color vanishes.

For a thin lens it can be easily shown that Eq. (29), giving the total axial color of the lens for arbitrary values of the difference \( \lambda - \lambda_0 \), can be rewritten as
\[
\Gamma_{\lambda 1} = h^2(\phi_A - \phi). \tag{75}
\]
However, since we have
\[
\phi_A - \phi = \delta\lambda \phi = \delta\lambda \phi_h + \delta\lambda \phi_g
= (\rho_1 - \rho_2)\delta\lambda n_0 + d\delta\lambda(n_0k)
= \phi_h \frac{\delta\lambda n_0}{n_0 - 1} + \phi_g \frac{\delta\lambda(n_0k)}{n_0k}, \tag{76}
\]
it can be observed that Eq. (73) for the total axial color of the thin RGRIN lens, which has been derived above by assuming that \( \lambda - \lambda_0 \) is a small quantity, is in fact valid for arbitrary values of \( \lambda - \lambda_0 \).

In analogy to the Abbe number of homogeneous glasses, it is convenient to define for RGRIN glasses an Abbe number describing the change with wavelength of \( n_0 \),
\[
\frac{n_0 - 1}{\delta\lambda n_0} = \nu_h = \frac{n_{od} - 1}{n_{oh} - n_{oc}}, \tag{77}
\]
and an Abbe number describing the change with wavelength of \( n_0k \),
\[
\frac{n_0k}{\delta\lambda(n_0k)} = \nu_g = \frac{(n_0k)_{od} - (n_0k)_c}{(n_0k)_f - (n_0k)_c}, \tag{78}
\]
where \( n_0 \) and \( k \) are considered, as usual, at the wavelengths of 587.6 nm (helium d line), 656.3 nm, and 486.1 nm (hydrogen C and F lines). Substituting Eqs. (77) and (78) into Eq. (73) leads, for the axial color coefficient of the thin RGRIN lens, to the well-known expression\(^7\)
\[
\Gamma_{\lambda 1} = h^2 \left[ \phi_h \frac{\phi_h}{\nu_h} + \phi_g \frac{\phi_g}{\nu_g} \right]. \tag{79}
\]
In the literature the index of refraction of a RGRIN glass is often given by
\[
n(r) = N_{00} + N_{10}r^2 + \ldots. \tag{80}
\]
It can be easily seen that the coefficients of Eq. (80) are related to the coefficients of the refractive-index distribution used in this study [Eq. (1)] by
\[
N_{00} = n_0, \quad N_{10} = -n_0k/2. \tag{81}
\]
Consequently, the gradient Abbe number [Eq. (78)] can be written also as

$$\nu_g = \frac{N_{10d}}{N_{10p} - N_{10c}}. \quad (82)$$

For numerical computations of the surface and transfer contributions to the chromatic paraxial aberrations [Eqns. (41) and Eqns. (56) and (57)] it is convenient to express $\delta_{1n_0}$ and $\delta_{1k}$ also through the Abbe numbers [Eqns. (77) and (78)]. We find that

$$\frac{\delta_{1n_0}}{n_0} = \frac{n_0 - 1}{n_0 v_h}, \quad (83)$$

and, from Eq. (72), that

$$\frac{\delta_{1k}}{k} = \nu_g^{-1} - \frac{n_0 - 1}{n_0 v_h}. \quad (84)$$

Thus Eqns. (41), Eqns. (56) and (57), and Eqns. (83) and (84) provide a generalization of thin-lens expression (79) for RGRIN lenses having finite thickness.

### 6. SUMMARY

Analytic expressions have been obtained for the contributions of transfer through a radial gradient medium of arbitrary thickness to the axial and lateral color coefficients [Eqns. (56) and (57)]. The remarkable simplicity of these expressions is due to the complete symmetry existing between the quantities before and after transfer through the medium. Together with the expressions of the surface contributions [Eqns. (41)], Eqns. (56) and (57) give the contributions of a RGRIN lens to the total chromatic paraxial aberration coefficients of the system [Eqns. (40)]. As in the homogeneous case, we can compute all contributions by tracing the marginal and chief rays through the system, using paraxial transfer formulas [Eqns. (7)–(9) and Eqns. (13)] and the definitions of several paraxial invariants [Eqns. (16) and (42)]. In the formulas for the chromatic aberrations, ordinary and gradient dispersion can be expressed through the two Abbe numbers characterizing the RGRIN material [Eqns. (83) and (84)]. In the thin-lens approximation, the total axial color coefficient of a RGRIN lens reduces to Eq. (79), already widely used in the literature on gradient-index optics.

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### REFERENCES AND NOTES

6. When in Ref. 5 the second of Eqns. (21) was calculated according to Eq. (20), a coefficient 1/2 was lost. This minor omission, affecting one term in a sum for each chromatic transfer contribution, has, however, a harmful effect on the final results.