Aberration balance in error functions calculated analytically

Florian Bociort *

Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

ABSTRACT

If the monochromatic error function of optical systems is computed analytically by means of aberration coefficients, certain intermediate results of this computation can be used for the investigation of the aberration balance. For the analysis of local minima resulting from optimization, a new automatic technique has been developed that groups together monochromatic aberrations having a large mutual compensation of their effect on the error function. Criteria are proposed to evaluate the quality of aberration balance. Present experience suggests the existence of a correlation between the quality of the pattern of aberration balance and the value of the error function.

Keywords: lens design, aberrations, geometrical optics

1. INTRODUCTION

It is well known to optical designers that in optimized optical systems aberration balancing (i.e. the partial compensation of the effect of aberrations of a given order and type by aberrations of different orders and types) plays a major role in obtaining a high imaging quality. Either for the transverse ray aberration or for the wavefront aberration, the analysis of aberration balance by means of Seidel and higher order coefficients¹, or alternatively by means of Zernike or other types of orthogonal polynomials², has lead to a deeper insight into the process of aberration correction. However, in present-day optical design practice, aberration balancing is examined most often only in an intuitive way, e.g. by means of aberration plots.

In this work we introduce a new technique to examine aberration balance that is based on the approximate analytical computation of the error function by means of aberration coefficients. Analytic merit functions based on aberration coefficients have been first used by Robb as a possible alternative to optimization with merit functions based on ray tracing³. Since in many cases a considerable number of aberration terms is necessary for achieving the necessary accuracy, such a use may be impractical. For systems optimized as usual with ray tracing, the analytic technique introduced in this work will be used to identify various types of patterns of aberration balance and to evaluate their “quality”. The analysis of balance patterns can supplement the information gained from the visual inspection of aberration plots and can also provide some guiding information for automatic lens design techniques such as global optimization.

Progress in global optimization methods and the availability of fast computers have lead in the past few years to increasingly powerful lens design software that is capable to detect multiple local minima corresponding to given system specifications. However, there is still no known technique to determine whether the local minimum having the lowest value of the error function among those already detected is indeed the global minimum, especially if the number of optimization variables is large. While this continues to be a very difficult problem, the analysis of balance patterns may shed some new light on this issue: Present experience suggests that there is a correlation between the “quality” of the aberration balance pattern and the value of the error function and that for poor local minima the balance pattern itself may indicate potential for further improvement.

For simplicity, in this work we consider only rotationally symmetric systems having a monochromatic error function of the type “root-mean-square spot size referenced to the chief ray”. For systems that have been optimized with a polychromatic error function, but where the limiting aberrations are monochromatic, the monochromatic balance pattern can still provide

¹ Permanent address: National Institute for Laser, Plasma and Radiation Physics, Department of Lasers, P.O. Box MG-36, 76900 Bucharest, Romania
valuable information. Also, we assume that the aperture coordinates of the rays are defined in the entrance pupil plane and that the system has no vignetting.

2. MONOCHROMATIC ABERRATION COEFFICIENTS

The monochromatic error function of the system can be expressed through the coefficients of the power series expansion of the transverse aberration. We define a ray through the system by two vectors \( s = (s_x, s_y) \) and \( t = (t_x, t_y) \), having as their components the normalized aperture and field coordinates respectively. The normalized aperture coordinates are the Cartesian ray coordinates in the entrance pupil divided by the entrance pupil radius. If the object is oriented along the y-axis, then \( t_x = 0 \) and the field coordinate \( t_y \) is the object height of the ray divided by the maximal object height if the object distance is finite, and is the tangent of the ray angle in the object space divided by the tangent of the maximum field angle if the object is at infinity.

Because of rotational symmetry the series expansion of the transverse aberration vector \( X(s, t) \) with respect to the aperture and field coordinates has a specific form:

\[
X(s, t) = B_1(s, t) + B_3(s, t) + \ldots + B_{2k+1}(s, t) + \ldots, \quad k = 0, 1, 2, 3, \ldots
\]

where we have

\[
B_{2k+1}(s, t) = \sum_{i, j}^{k} (m_{jk} s_i + n_{jk} t_i) \left( s^2 \right)^{k-j} \left( t^2 \right)^j
\]

and

\[
s^2 = s_x^2 + s_y^2, \quad st = s_x t_x + s_y t_y, \quad t^2 = t_x^2 + t_y^2.
\]

The coefficients \( m_{jk} \) and \( n_{jk} \) are the monochromatic transverse aberration coefficients of the system. There are two linear coefficients \( k=0 \), six third-order coefficients \( k=1 \), twelve fifth-order \( k=2 \), twenty seventh-order coefficients \( k=3 \) etc. Not all of them are independent, however\(^4\). For instance, from the six third-order coefficients only five (the Seidel coefficients) are independent.

If the image plane is the paraxial image plane, then \( B_1(s, t) = 0 \) and the lowest-order non-vanishing term in the expansion (1) is \( B_3(s, t) \). In this case, the aberration coefficients appearing in \( B_3(s, t) \) are the well-known Seidel aberration coefficients (with some rearrangement for astigmatism and Petzval sum). In most practical designs however, the image plane is slightly shifted with respect to the paraxial one. This shift has two effects on the expansion (1-3). First, two linear terms appear because of the defocus. Second, it turns out that the coefficients \( b_{i,k-j,j,i} \) and \( b_{i,k-j,j,2} \) of order three and higher are in fact not the ordinary aberration coefficients of the object imagery, but linear combinations between the object coefficients and the corresponding pupil aberration coefficients.

In the examples that will be presented, it will be convenient to use for the aberration coefficients a more suggestive notation. A given aberration coefficient will be denoted by the factor of aperture and field coordinates that multiplies the coefficient in Eq. (2), written in a symbolic form:

\[
b_{i,k-j,j,i} \equiv s^i t^j \cdot 2^k \frac{k-i}{i-j} \cdot \left( s \cdot t \right)^{k-j} \cdot \left( s \cdot t \right)^{k-j}
\]

The symbols * denote multiplication and power raising, respectively. If an “exponent” is equal to unity, it will be omitted; if the “exponent” is equal to zero, the entire “factor” is omitted. For instance, \( s \) stands for linear defocus\(^{b_{i,0,0}}\).
\( s^3s2 \) denotes the third-order spherical aberration \( b_{s,3,0,0} \) and \( t^3t2 \) is the third-order distortion \( b_{t,0,0,1} \). The new notation shows more clearly the role of a given coefficient. For instance, it can be now easily seen that coefficients having in the new notation only \( s \)'s are spherical aberrations of various orders, while coefficients having only \( t \)'s are distortions of various orders.

Several decades ago, Buchdahl has developed a remarkable technique for computing high-order aberration coefficients\(^4\). Recently\(^5\) we have simplified Buchdahl’s method and have translated it into an efficient Computer Algebra algorithm that can generate analytic expressions for all aberration coefficients of orders 3, 5 and 7.

Since the length of the analytic expressions increases rapidly with each additional order, for computing coefficients of orders higher than 7 it was more convenient to use a more recent method derived by Andersen\(^6\,7\). If the object is at infinity, Andersen’s aberration expansion can be easily converted to the form (1-3). However, for finite conjugates we had to generalize Andersen’s method by introducing an additional transformation (to be published). Comparisons showed that the simplified Buchdahl method and the generalized Andersen method produce identical results. With the generalized Andersen method we can now compute all aberration coefficients up to order 21 (i.e. \( k=10 \) in Eqs(1-3)). However, since the total number of coefficients becomes large, for the analysis of aberration balance we use only the aberrations up to order 15, supplemented by the spherical aberration of order 17.

Finally, the convergence of the series expansion (1-3) has been investigated by comparing for various optical systems the transverse aberration computed with the series expansion with the corresponding ray-tracing result. For the approximation of order 15 the agreement is excellent in most cases, the only exceptions we have found being some wide-angle systems where the aberration series converges slowly and, if high accuracy is desired, even higher orders must be computed.

### 3. ANALYTICAL COMPUTATION OF THE ERROR FUNCTION

If computed as usual by tracing an adequately chosen set of rays through the system, the error function of the type “root-mean-square spot size referenced to the chief ray” is given by

\[
e = \sqrt{\sum w_i (r_i - r_{i,\text{chief}})^2 / \sum w_i} \tag{5}
\]

where, for each individual ray \( i \), \( w_i \) is the corresponding weighting factor and \( r_i - r_{i,\text{chief}} \) is the ray aberration in the image plane referenced to the chief ray. However, the ray aberration \( r_i - r_{i,\text{chief}} \) can also be expressed by the power series (1-3). Since the series expansion for the chief ray contains only distortion terms, the quantity \( r_i - r_{i,\text{chief}} \) is given by Eqs (1-3) where all distortion terms are set equal to zero. The analytical approximation of Eq. (5) up to some order \( 2k_{\text{max}} + 1 \) is then obtained by replacing the summation over various rays by an analytical integration over the circular aperture and a summation using weighting factors over the field. Thus, in this approach trigonometric ray tracing is avoided completely.

Let \( N \) be the total number of aberration coefficients up to the order \( 2k_{\text{max}} + 1 \). Let us now arrange all coefficients \( b_{s,k,i-j,j,i} \), \( b_{t,k,i-j,j,i} \) into a single one-dimensional array, having an index varying from \( i \) to \( N \). We can then write

\[
e = \sqrt{\sum_{i,j} a_i b_{i,j}^2} \tag{6}
\]

with

\[
E = \sum_{i,j} a_i b_{i,j}^2
\]

where we have used the notations \( b_i = b_{s,l,m,n} \) and \( b_j = b_{t,l',m',n'} \) and where \( u \) and \( u' \) are either \( s \) or \( t \). If the two coefficients are of orders \( 2k + 1 \) and \( 2k'+1 \) respectively, then we have \( l + m + n = k \) and \( l'+m'+n' = k' \).
The matrix $a_{ij}$ is symmetric, $a_{ij} = a_{ji}$, and its coefficients result from the integration itself. A short analysis shows that the matrix elements must be of the form $a_{ij} = a(\alpha, \beta, \gamma)$ where the numbers $\alpha, \beta, \gamma$ are functions of $u,l,m,n, u',l',m',n'$ given in Table 1.

Table 1. The set of numbers $\{\alpha, \beta, \gamma\}$ as function of $u,l,m,n, u',l',m',n'$. The vectors $u$ and $u'$ are either s or t.

<table>
<thead>
<tr>
<th>s</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[l+l'+1, m+m', n+n']$</td>
<td>$[l+l', m+m'+1, n+n']$</td>
</tr>
</tbody>
</table>

Let $s$ and $t$ be the lengths of the vectors $s$ and $t$ and let $\phi$ be the angle between them. If the system has $p_{\text{max}}$ field points $t_p$, each having a field weight $W_p$, then, by integrating over the aperture and summing over the field, we have

$$a(\alpha, \beta, \gamma) = \left(2 \pi \sum_{p=1}^{p_{\text{max}}} W_p\right)^{-1} \sum_{p=1}^{p_{\text{max}}} W_p \left[\int_0^{2\pi} (s')^\alpha \left(st_p \cos \phi\right) (t'_p)^\gamma \right] s ds d\phi$$  \hspace{1cm} (7)

After some elementary calculations we finally obtain

$$a_{ij} = a(\alpha, \beta, \gamma) = \frac{I(\beta) F(\beta + 2\gamma)}{2\alpha + \beta + 2}$$  \hspace{1cm} (8)

where for any positive integers $x$ and $\beta$ we use the abbreviations

$$F(x) = \frac{\sum_{p=1}^{p_{\text{max}}} W_p t'_p^x}{\sum_{p=1}^{p_{\text{max}}} W_p}$$  \hspace{1cm} (9)

$$I(\beta) = \pi^{-1} \int_0^{2\pi} (\cos \phi)^\beta d\phi$$  \hspace{1cm} (10)

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I(\beta)$</td>
<td>1/2</td>
<td>3/4</td>
<td>5/8</td>
<td>35/64</td>
<td>63/128</td>
<td>231/512</td>
<td>429/1024</td>
<td>6435/16384</td>
<td>12155/32768</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Values of $I(\beta)$ when $\beta$ is even. For all odd values of $\beta$, $I(\beta) = 0$.

By examining the notation (4) for the aberration coefficients, we see that the “total power” in both $s$ and $t$ is always the odd number $2k+1$. We separate now the aberrations into two groups: one has an odd total power in $s$ and an even total power in $t$, and will be called the $A$ group, the other has an odd total power in $t$ and an even one in $s$, called the $C$ group. Table 1 shows that $\beta=m+m'$ when $u = u'$ and that $\beta=m+m'+1$ otherwise. A closer inspection reveals that $\beta$ is odd when the two aberrations belong to different groups and that $\beta$ is even when they belong to the same group. Thus, in Eq.(6) the coefficient $a_{ij}$ is nonzero only in the terms where the aberrations $b_i$ and $b_j$ belong to the same group. Eq.(6) can therefore be rewritten as two distinct sums over the corresponding groups

$$E = \sum_{i,j=A} a_{ij} b_i b_j + \sum_{i,j=C} a_{ij} b_i b_j$$  \hspace{1cm} (11)

It turns out that the diagonal elements of the matrices $a_{ii}$ are always positive ($a_{ii} > 0$ for all $i$). Thus, any diagonal term $a_{ii} b_i^2$, which is the individual contribution of a given aberration $i$ to the error function (11), is also always positive. Its square root
would be the value of the spot size if all other aberrations were absent, and can therefore be called the “individual” spot size of aberration \( i \). However, the non-diagonal terms \( a_i b_j b_j \), which involve two different aberrations, can be either positive or negative. Due to their presence the optimization process can achieve aberration balance, i.e. the partial compensation of the effects of one aberration by other aberrations. The actual spot size \( e \) is therefore smaller than for instance the individual spot size \( e_i \) of the largest aberration in the system. Moreover, both individual sums in Eq.(11) are also always positive. Thus, Eq.(11) shows that aberration balancing in the error function can occur only among aberrations that belong to the same group. For instance, it is not possible to compensate the effect of coma (which belongs to the \( C \) group) by increasing spherical aberration or astigmatism (which both belong to the \( A \) group).

For various types of optical systems we have compared the analytical value of the error function (Eq.(11)) with that computed with ray tracing (Eq.(5)). When the order of approximation is high enough so that the aberration series for individual rays provides sufficient accuracy, the agreement between the analytical and numerical values of the error function is also very good (see Table 3).

<table>
<thead>
<tr>
<th>Type of system</th>
<th>F/#</th>
<th>Field (deg)</th>
<th>( e ) (analytic)</th>
<th>( e ) (ZEMAX)</th>
<th>relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double Gauss</td>
<td>3</td>
<td>14</td>
<td>0.010422</td>
<td>0.010421</td>
<td>1x10^{-4}</td>
</tr>
<tr>
<td>Cooke triplet</td>
<td>5</td>
<td>20</td>
<td>0.011071</td>
<td>0.011069</td>
<td>2x10^{-4}</td>
</tr>
<tr>
<td>Inverse telephoto</td>
<td>3.6</td>
<td>31</td>
<td>0.001397</td>
<td>0.001394</td>
<td>2x10^{-3}</td>
</tr>
</tbody>
</table>

Table 3. Comparison between analytical and numerical values of the monochromatic error function. For the analytical values an approximation of order 15 has been used. The numerical values are computed with the ZEMAX optical design program.

4. PATTERNS OF ABERRATION BALANCE

The splitting of the aberration list into the two groups \( A \) and \( C \) occurs for any optical system, whether optimized or not, and the partial compensation of a given aberration by other aberrations of the same group can be large or insignificantly small. In this work we suggest that, if the system has been previously optimized, we gain additional insight into the properties of aberration correction if we partition the two groups even further by identifying subgroups of aberrations having large mutual compensation.

The precise composition of these subgroups, that will be called balance groups, depends of course on how we define a “large” compensation. In this work, the balance groups will be defined as groups of aberrations having the property that the sum of all contributions to Eq. (11) involving only aberrations from within the group is smaller than each individual aberration contribution, i.e. for any aberration \( b_i \) from a group \( l \) we have

\[
E_i = \sum_{n\neq i} a_n b_n b_i < a_i b_i^2 , \quad i \in l
\]

The set of all balance groups will be called the pattern of aberration balance for the given design. The balance groups must not overlap, i.e. each aberration is a member of only one group. It must be emphasized at this point that the partition of aberrations into balance groups is a rather qualitative technique and does not mean that the total \( E \) given by Eq.(11) is strictly equal to the sum over all groups of the quantities given by Eq. (12). As shown in the remainder of this paper, we believe this qualitative technique to be useful because different local optima having the same specifications can have different patterns of balance and because the “quality” of a given pattern appears to be one of the factors determining the value of the error function.

Remarkably enough, if Eq.(12) is used for the detection of the balance pattern it turns out that the optimization process tends to group aberrations together in a specific way:

a) For each balance group \( l \) the optimization process tends to minimize the quantity
b) Balance groups tend to be composed in such a way that for all pairs of aberrations $i$ and $j$ within the group the quantities

$$R_y = 2\left(a_y - \sqrt{a_y a_p}\right)$$

are the smallest possible ones, in the sense that if $k$ is an arbitrary aberration outside the group, then $|R_{ik}| < |R_{ak}|$. The quantities (14) will be called residual coefficients.

The degree to which the properties a) and b) are satisfied will be called the “quality” of the balance pattern. The following analysis can help us to understand these two properties.

For a balance group consisting of $n_l$ aberrations it is always possible to write $E_l$ as a sum of $n_l$ squares,

$$E_l = \sum_{p=1}^{n_l} y_{lp}^2,$$

Here, the quantities $y_{lp}$ are linear combinations of the aberrations in the group $l$ and can be computed by standard techniques of linear algebra from the eigenvectors and eigenvalues of the matrix $a_{ij}$ for the given balance group. (The coefficients of $y_{lp}$ are the components of the $p$-th eigenvector, multiplied by the square root of the $p$-th eigenvalue.) A small $E_l$ means thus that all $y_{lp}$ must also be small.

If the balance in the group has a high quality, it turns out that we always have the remarkable property that the quantity $y_{l,1}$ (the one that is related to the largest eigenvalue of the matrix) has coefficients that are very close to those of $Z_l$. On the other hand, when $p$ increases the absolute magnitude of the numerical coefficients which multiply the aberrations in the quantities $y_{l,p}$ decreases rapidly. (This is because the magnitude of the eigenvalues decreases very rapidly with $p$.) Thus, the optimization has to spend most of the effort to achieve the condition $y_{l,1} = Z_l = 0$, while the quantities $y_{l,p}$ with $p>1$ tend to be automatically small and require much less optimization effort. An example is shown in Tab.4 in the case of a frequently encountered balance group that is composed of linear defocus, third and fifth-order spherical aberration, $\{s, s*s^2, s*s^2*2\}$.

<table>
<thead>
<tr>
<th>$Z_l$</th>
<th>$y_{l,p}$ for the group ${s, s<em>s^2, s</em>s^2*2}$</th>
<th>optimized system</th>
<th>non-optimized system</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_l$</td>
<td>$0.707107 s +0.5 s<em>s^2 +0.408248 s</em>s^2*2$</td>
<td>-0.0102167</td>
<td>-0.196092</td>
</tr>
<tr>
<td>$p=1$</td>
<td>$0.695186 s +0.493785 s<em>s^2 +0.384719 s</em>s^2*2$</td>
<td>-0.0114574</td>
<td>-0.194456</td>
</tr>
<tr>
<td>$p=2$</td>
<td>$-0.129194 s +0.0761696 s<em>s^2 +0.135691 s</em>s^2*2$</td>
<td>-0.00646165</td>
<td>-0.0266771</td>
</tr>
<tr>
<td>$p=3$</td>
<td>$0.0050668 s-0.0193589 s<em>s^2 +0.015691 s</em>s^2*2$</td>
<td>0.00385469</td>
<td>0.0089589</td>
</tr>
</tbody>
</table>

Table 4. Coefficients and values for an optimized and a non-optimized system of $Z_l$ and $y_{l,p}$, $p=1,2,3$ for the group $\{s, s*s^2, s*s^2*2\}$. Note that the corresponding coefficients of $y_{l,1}$ and $Z_l$ are very close to each other and that also the values of $y_{l,1}$ and $Z_l$ are nearly equal, both in the optimized case and in the non-optimized one. The values of $y_{l,p}$ for $p=2,3$ are small even before optimization, because the coefficients multiplying the aberrations are small.

The above analysis shows that the optimization brings $Z_l$ close to zero. We can now gain more insight if by using Eqs.(13) and (14) we write

$$E_l = Z_l^2 + \sum_{m=1}^{n} R_{lm} b_m b_n$$

(15)
It can be seen that, if in Eq.(15) the residual coefficients $R_{ij}$ are also small, then the optimization can achieve a low value for $E_l$ without necessarily enforcing very low values for the individual aberrations in the group. Thus, the residual coefficients show how well aberrations can balance each other. The balance in group $l$ is of high quality a) if $Z_l$ given by Eq.(13) is close to zero and b) if the aberrations that are grouped together have low mutual residual coefficients.

A computer program has been written that detects automatically the balance pattern of optimized designs. Since the total number $N$ of aberration coefficients used for the computation of the error function is large (e.g. if the maximal order is 15 then $N=240$) it is not practical to partition all of them in balance groups. Instead, the aberrations are first sorted according to magnitude of the diagonal terms $a_n b_{ij}^2$ and the pattern detection is limited to the subset of the largest $N'$ aberrations. In the examples given in the next section we have chosen $N'=15$.

The detection technique implemented at present has two stages. At the beginning, each aberration forms its own group. In the first stage, for each group the program searches among all aberrations outside the group for the one which, if appended to the group, leads to the smallest value of $E_{ab bl} = \sum_{n=m} a_n b_n$. If the value of $E_l$ in the presence of the new aberration is smaller than in its absence, then the new aberration becomes a member of the group and the process is repeated for the group in its new composition; if not, the group remains unchanged and the program moves to the next group. The first stage ends when no further grouping can reduce the values of $E_l$ for the groups that have been detected.

In the second stage, it is verified for each group $l$ and each aberration $b_i$ in $l$ whether condition (12) is satisfied. If yes, the composition of group $l$ is final; if not, then the aberration $b_i$ is removed from the group. However, we use presently the two-stage detection only for $A$-type aberrations. For $C$-type aberrations not all coefficients are independent and this fact interferes slightly with the detection process. In this case, up to now the best results have been obtained by omitting the second stage.

It is useful to display the detected balance pattern as a bar diagram showing the partition of the largest $N'$ aberrations in balance groups and the way property a) is satisfied for the individual groups. In Figs. 3-8, for each group $l$ the bars indicate the individual aberration contributions $\sqrt{a_n b_n}$ to Eq.(13) and the “total” $Z_l$. Note that the absolute magnitudes of the individual contributions (i.e. the lengths of the bars) are in fact the “individual” spot sizes $e_i$ of the corresponding aberrations.

5. EXAMPLES

A first example of a system revealing an exceptionally good aberration balance is the lithographic lens shown in Fig. 1. The corresponding balance pattern is shown in Fig.3. The largest aberrations are the fifth-order spherical ($s^5s^2$) and seventh-order spherical aberration ($s^7s^2$) which are members of the same group (group 1, the lowest in Fig.3), together with the third-order ($s^3s^2$) and some very high order spherical aberrations ($s^{15}s^2$, $s^{16}s^2$).

a) It can be observed that for group 1 the “total” $Z_1$ (lowest bar in Fig.3) is very small as compared to the large individual contributions. b) The analysis of the residual coefficients leads to the well known conclusion that a spherical aberration of a given order is best balanced by spherical aberrations of different orders and consequently the composition of group 1 is indeed the best possible one. Thus, properties a) and b) are satisfied to a high degree and therefore we expect a high degree of mutual compensation of the aberrations in the group. Indeed, due to the presence of the other aberrations, the RMS spot size $e$ of the lens turns out to be more than 40 times smaller than the “individual” spot size $e_i$ of the largest aberration. (In some of the other groups, small aberrations appear as being unbalanced, i.e. they appear alone in their group. In fact these aberrations are balanced by even smaller ones that do not appear among the largest $N'=15$ aberrations.)
A balance pattern where the balance groups have a quite different composition is shown in Fig. 4 for the system in Fig. 2. The design is a wide-angle inverted telephoto lens discussed by Laikin and has f/2.8 with a 35° (semi) field of view. Again, the largest aberrations are all members of group 1, but this time these are very high-order astigmatism-type aberrations of the form $t^i s^i t^i 2^i$, $i=2,3,4,5,6$. This balance pattern also has a high quality. It can be observed that for all six groups the “total” is small. Moreover, the analysis of the residual coefficients indicates that the partition of the largest 15 aberrations into balance groups is precisely the one for which property b) is satisfied best.

With the new technique of balance pattern analysis, various sets of local minima corresponding to the same specifications have been investigated. The multiple local minima have been obtained with the global optimization feature of the ZEMAX optical design program or simply by optimizing a starting configuration of plane parallel plates. (In the latter case a curvature solve on the last surface provided the correct focal length.)

Remarkably enough, in many cases of multiple local minima investigated so far, systems having a balance pattern with a good quality tend also to have lower values of the monochromatic error function. This happens because when properties a) and b) are satisfied to a high degree then the quantities $E_i$ for the $l$ groups in the pattern are small and consequently $E = e^2$ tends also to be small. However, as noted before, the total $E$ given by Eq.(11) is not strictly equal to the sum of $E_i$ over all $l$ groups. Therefore, the above analysis is relevant only for comparing minima whose error function and quality of the balance pattern differ substantially.

It must however be emphasized that the quality of the balance pattern is only one of the factors that determine the value of the error function. For instance, another very important factor is the number of active constraints. When during optimization a constraint on a variable becomes active (i.e. an inequality turns into an equality), the potential of that variable for further improvement of performance is lost. Therefore, most often the best local minima have not only a good balance pattern, but also the lowest number of active constraints.

As an example showing the correlation between the quality of the balance pattern and the value of the monochromatic error function, consider two local minima A1 and A2 resulting from polychromatic optimization, but where the limiting aberrations are monochromatic. Both systems consist of 6 elements and have f/3 and a (semi) field of 14°, but the monochromatic error function $e$ of A2 is 8 times larger than that of A1. The corresponding monochromatic balance patterns for A1 and A2 are shown in Figs. 5 and 6 respectively.

Consider first the system A1 which has a good aberration balance. Property a) is satisfied because in all groups of Fig. 5 the “totals” are small. In order to see how property b) is satisfied, we select the largest aberrations in the first three groups, and for each of them we compute the coefficient $R_{ij}$ with all other $N'-1$ aberrations and sort the results in an increasing order of magnitude. The results are shown in Tab. 5. In group 1, where the largest aberration is the third-order spherical aberration $s*s^2$, we see that for this aberration the lowest values of $R_{ij}$ are those for $s*s^2$, $s*s^2$ and $s$, (fifth-order spherical aberration, seventh-order spherical aberration, and linear defocus, respectively). These are precisely the other aberrations in group 1; none of the aberrations outside the group has a smaller residual coefficient than any aberration within the group. Similarly, in groups 2 and 3 which are both composed of two aberrations, the aberration having the lowest $R_{ij}$ with the largest
aberration in the group is precisely the second aberration in the corresponding groups. Thus, property b) is also well satisfied.

For A2 which has the larger error function, the balance pattern has a lower quality than that of A1. It can be seen in Fig. 6 that some aberrations are unbalanced (in groups 2 and 3 there is only one member in the group) and that for group 1 the “total” is large. The large defocus s balances both s*s2 and s*t2 (which is a linear combination of third-order astigmatism and Petzval curvature). However, the residual coefficients show that s*t2 could balance much better with the unbalanced aberration in group 3, s*t2^2, (R_y = 0.00908) than with defocus (R_y = 0.1447) and s*s2 (R_y = 0.1227). Thus, the partition of the aberrations in balance groups is not optimal. For this system, neither property a) nor property b) is well satisfied.

<table>
<thead>
<tr>
<th>group 1: m = s*s2</th>
<th>group 2: m = s*t2</th>
<th>group 3: m = s*st</th>
</tr>
</thead>
<tbody>
<tr>
<td>R_{mn} n</td>
<td>R_{mn} n</td>
<td>R_{mn} n</td>
</tr>
<tr>
<td>1</td>
<td>0.0082</td>
<td>s*s2^2</td>
</tr>
<tr>
<td>2</td>
<td>0.0202</td>
<td>s*s2^3</td>
</tr>
<tr>
<td>3</td>
<td>0.0404</td>
<td>s</td>
</tr>
<tr>
<td>4</td>
<td>0.0723</td>
<td>s<em>s2</em>t2</td>
</tr>
<tr>
<td>5</td>
<td>0.1227</td>
<td>s*t2</td>
</tr>
<tr>
<td>6</td>
<td>0.1582</td>
<td>t<em>st</em>t2</td>
</tr>
</tbody>
</table>

Table 5. Residual coefficients ordered according to their magnitude for the first three groups of the balance patterns of A1.

Together with the largest aberration, aberrations above the line are precisely those forming the corresponding groups.

The balance patterns in Figures 3, 4, and 5 are examples of high-quality aberration balance. Frequently however, even the best local minima resulting from global optimization have their limitations. In these cases the balance patterns are reasonably good, but not perfect. Figure 7 shows the monochromatic balance pattern for the best local minimum B1 resulting from the polychromatic optimization of a system consisting of 9 lenses, having f/2.8 and a field of 7°. The largest aberrations appear here in two groups, 1 and 3. Group 3, consisting of defocus and spherical aberrations of orders 3, 5 and 7, has a very good balance. However, in group 1 consisting of s*t2 and s*s2*t2 the total, still considerably smaller than the individual contributions, is not as small as in group 3 for instance. Moreover, the residual coefficients indicate that s*t2 would balance best with s*t2^2 (the unbalanced aberration in group 2), the residual coefficient being nearly half of that with s*s2*t2. However, s*s2*t2 is the second best balance possibility for s*t2, in any case much better than any other of the remaining possibilities.

While for B1 the balance pattern is not perfect, other local minima corresponding to the same specifications have balance patterns of considerably lower quality. The pattern shown in Fig.8 corresponds to a minimum B2 having a monochromatic error function e 3.25 times larger than that of B1. Now, s*t2 is a large unbalanced aberration (group 2) and the aberration s*t2^2 that would balance it best balances instead s*s2 in group 1. In the latter group, the balance is also a rather poor one: The total is only slightly smaller than the contribution of s*t2^2, and the residual coefficient of s*t2^2 with s*s2 is 15 times larger than that with s*t2.

For systems where the number of optimization variables is not too small, the analysis of balance patterns can provide some guiding information in a search for better local minima that is pursued either by means of traditional techniques or by global optimization. We have seen above that the composition of good balance groups can be predicted by analyzing the various

* The example in Tab.5 shows how the composition of good balance groups can be predicted by analyzing the residual coefficients of the most important aberrations of the system with the other N'-1 largest aberrations. However, the relative magnitude of the various aberrations may differ from a local minimum to another, and some good balance partners, that may be too small to appear in the set of N' aberrations of the given minimum, could be large enough for efficient balancing in other minima. Therefore, other good balance possibilities can sometimes be found by examining also the residual coefficients of the important aberrations with all aberrations up to some maximal order.
combinations of residual coefficients. Then, if for the best local minimum we have obtained at a given stage the quality of
the balance pattern is still unsatisfactory (i.e. the “totals” in the various groups are not small and the composition of the
groups differs considerably from the predicted good one) then there is a reasonable chance that a local optimum with a lower
error function exists and the search for it may be worth continuing.

For automatic techniques such as global optimization, it is often important to decide whether two given local minima can be
considered to be qualitatively distinct. If the values of the error function happen to be close, it may be difficult for the algo-
rum to make this decision. In such situations the automatic analysis of balance pattern can provide an answer: The two
minima are qualitatively distinct if their balance patterns are different.

In this paper, the detection of the pattern of aberration balance assumes an error function of the type RMS spot size, and
therefore the method described above works best if optimization is done with the same type of error function. However, if
further studies will confirm its utility in this particular case, the new technique could be adapted also for other types of error
functions. This new technique could supplement the visual aberration analysis by means of aberration plots.

6. CONCLUSIONS

The analytical computation of the error function by means of aberration coefficients enables the development of an auto-
matic technique for the investigation of aberration balancing. The technique for the detection of the pattern of aberration
balance (Eq.(12)) and the criteria for evaluating the quality of the balance pattern (Eqs. (13-14)) are using matrix elements
(Eqs.(7-10)) that have been first derived for the analytical computation of the error function (Eq.11). While further studies
for various types of optical systems are necessary to establish the practical utility of this new technique, the examples we
have investigated indicate that for a given local minimum the quality of the balance pattern is one of the factors determining
the value of the error function. If the aberrations balance in the wrong way, then the error function tends to be larger. A
balance pattern having a poor quality is therefore a hint that for the same specifications better local optima may exist.

7. ACKNOWLEDGEMENTS

The financial support of this work by the Dutch Technology Foundation (STW) is gratefully acknowledged. The author
also wishes to thank Professor Leo Beckmann for his source code of the Andersen coefficients and for his efforts to adapt
the code according to the needs of this research.

8. REFERENCES

1041, 1976
Research Institute for Applications of Computer Algebra, Amsterdam,1995, also available on the World Wide Web at
http://www.can.nl/AboutCAN/RIACA/papers/florian/florian.html
3263-3268,1981
8. M. Laikin, Lens design, Marcel Dekker, Inc., New York, pp.90-92,Fig.8-2, 1995