Green’s function representations for seismic interferometry
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Summary
Using Rayleigh’s reciprocity theorem and the principle of time-reversal invariance, we derive representations for the Green’s function of an inhomogeneous medium in terms of cross-correlations of wave fields at two observation points. First we consider the hypothetical situation in which the primary sources are located at a closed surface. We carefully analyze the contributions from inhomogeneities inside as well as outside this surface. Next we consider the more realistic situation of an inhomogeneous medium below a free surface and primary sources on an open surface. Finally we illustrate the method with a numerical example.

Introduction
Seismic interferometry is the process of generating new seismic responses by cross-correlating seismic observations at different receiver locations [Schuster (2001), Wapenaar et al. (2002)]. Previously we used a one-way reciprocity theorem to derive a representation of the reflection response in terms of the transmission response of an arbitrary inhomogeneous medium. In this paper we present a similar derivation, based on Rayleigh’s reciprocity theorem, which leads to Green’s function representations for seismic interferometry.

Rayleigh’s reciprocity theorem
A reciprocity theorem relates two independent acoustic states in one and the same domain [de Hoop (1988), Fokkema and van den Berg (1993)]. Consider an acoustic wave field, characterized by the acoustic pressure \( p(x,t) \) and the particle velocity \( v_i(x,t) \), where \( x = (x_1, x_2, x_3) \) denotes the Cartesian coordinate vector (as usual the \( x_3 \)-axis is pointing downward) and \( t \) denotes time. We define the temporal Fourier transform of a space- and time-dependent quantity \( p(x,t) \) as \( \hat{p}(x,\omega) = \int_{-\infty}^{\infty} \exp(-j\omega t)p(x,t)dt \), where \( j \) is the imaginary unit and \( \omega \) the angular frequency. In the space-frequency domain the acoustic pressure and particle velocity in a lossless arbitrary inhomogeneous acoustic medium obey the equation of motion \( j\omega \rho \partial_t \hat{v}_i + \partial_i \hat{p} = 0 \) and the stress-strain relation \( j\omega \kappa \hat{v}_i + \partial_i \hat{\sigma} = \hat{q} \), where \( \partial_i \) is the partial derivative in the \( x_i \)-direction (Einstein’s summation convention applies for repeated lower-case subscripts), \( \rho(x) \) the mass density of the medium, \( \kappa(x) \) its compressibility and \( \hat{q}(x,\omega) \) a source distribution in terms of volume injection rate density. We consider the ‘interaction quantity’ \( \partial_i \{ \hat{p}_A \hat{v}_{i,B} - \hat{v}_{i,A} \hat{p}_B \} \), where subscripts \( A \) and \( B \) are used to distinguish two independent acoustic states. Rayleigh’s reciprocity theorem is obtained by substituting the equation of motion and the stress-strain relation for states \( A \) and \( B \) into the interaction quantity, integrating the result over a spatial domain \( \mathbb{D} \) enclosed by \( \partial \mathbb{D} \) with outward pointing normal vector \( \mathbf{n} = (n_1, n_2, n_3) \) and applying the theorem of Gauss. This gives

\[
\int_{\mathbb{D}} \{ \hat{p}_A \hat{q}_B - \hat{q}_A \hat{p}_B \} d^3x = \oint_{\partial \mathbb{D}} \{ \hat{p}_A \hat{v}_{i,B} - \hat{v}_{i,A} \hat{p}_B \} n_i d^2x. \quad (1)
\]

We call this a reciprocity theorem of the convolution type since the products in the frequency domain (\( \hat{p}_A \hat{v}_{i,B} \) etc.) correspond to convolutions in the time domain.

Since the medium is assumed to be lossless, we can apply the principle of time-reversal invariance (BojarSKI, 1983). In the frequency domain time-reversal is replaced by complex conjugation. Hence, when \( \hat{p} \) and \( \hat{v}_i \) are a solution of the equation of motion and the stress-strain relation with source distribution \( \hat{q} \), then \( \hat{p}^\ast \) and \( -\hat{v}_i^\ast \) obey the same equations with source distribution \( -\hat{q}^\ast \) (the asterisk denotes complex conjugation). Making these substitutions for state \( A \) we obtain

\[
\int_{\mathbb{D}} \{ \hat{p}_A^\ast \hat{q}_B + \hat{q}_A^\ast \hat{p}_B \} d^3x = \oint_{\partial \mathbb{D}} \{ \hat{p}_A^\ast \hat{v}_{i,B} + \hat{v}_{i,A}^\ast \hat{p}_B \} n_i d^2x. \quad (2)
\]

We call this a reciprocity theorem of the correlation type since the products in the frequency domain (\( \hat{p}_A^\ast \hat{v}_{i,B} \) etc.) correspond to correlations in the time domain.

Note that for both theorems we assumed that the medium parameters in states \( A \) and \( B \) are identical. De Hoop (1988) and Fokkema and van den Berg (1993) discuss more general reciprocity theorems that account also for different medium parameters in the two states.

Acoustic Green’s function representations
We choose impulsive point sources of the volume injection type in both states, according to \( \hat{q}_A(x,\omega) = \delta(x-x_A) \) and \( \hat{q}_B(x,\omega) = \delta(x-x_B) \), with \( x_A \) and \( x_B \) both in \( \mathbb{D} \). The wave fields in states \( A \) and \( B \) can thus be expressed in terms of Green’s functions, according to

\[
\hat{p}_A(x,\omega) = \hat{G}(x,x_A,\omega), \quad (3)
\]

\[
\hat{v}_{i,A}(x,\omega) = -(j\omega \rho(x))^{-1} \partial_i \hat{G}(x,x_A,\omega), \quad (4)
\]

\[
\hat{p}_B(x,\omega) = \hat{G}(x,x_B,\omega), \quad (5)
\]

\[
\hat{v}_{i,B}(x,\omega) = -(j\omega \rho(x))^{-1} \partial_i \hat{G}(x,x_B,\omega). \quad (6)
\]

Substituting these equations into the reciprocity theorem of the convolution type (equation 1) gives

\[
\hat{G}(x_B,x_A,\omega) - \hat{G}(x_A,x_B,\omega) = \frac{-1}{j\omega \rho(x)} \left( \hat{G}(x,x_A,\omega) \partial_i \hat{G}(x,x_B,\omega) - (\partial_i \hat{G}(x,x_B,\omega)) \hat{G}(x,x_B,\omega) \right) n_i d^2x. \quad (7)
\]

We assume that the Green’s functions \( \hat{G}(x,x_A,\omega) \) and \( \hat{G}(x,x_B,\omega) \) are the Fourier transforms of causal time-domain Green’s functions. Hence, when \( \partial D \) is a spherical...
This is the well-known source-receiver reciprocity relation for the Green’s function.

Making the same substitutions into the reciprocity theorem of the correlation type (equation 2) gives

\[
\tilde{G}^\ast(x_B, x_A, \omega) + \tilde{G}(x_A, x_B, \omega) = \int_{\partial D} \frac{-1}{j\omega \rho(x)} \left( \tilde{G}^\ast(x, x_A, \omega) \partial_t \tilde{G}(x, x_B, \omega) - (\partial_t \tilde{G}^\ast(x, x_A, \omega)) \tilde{G}(x, x_B, \omega) \right) n_i d^2x.
\]

Again, the right-hand side is independent of the choice of \( \partial D \), as long as it encloses \( x_A \) and \( x_B \). Note, however, that since \( \tilde{G}^\ast(x_B, x_A, \omega) \) is the Fourier transform of an anticausal time-domain Green’s function, the radiation conditions are not fulfilled and hence the right-hand side of equation (9) does not vanish. Using source-receiver reciprocity of the Green’s functions gives

\[
2\Re\{\tilde{G}(x_A, x_B, \omega)\} = \int_{\partial D} \frac{-1}{j\omega \rho(x)} \left( \tilde{G}^\ast(x_A, x, \omega) \partial_t \tilde{G}(x_B, x, \omega) - (\partial_t \tilde{G}^\ast(x_A, x, \omega)) \tilde{G}(x_B, x, \omega) \right) n_i d^2x,
\]

where \( \Re \) denotes the real part. Equation (10) is the basis for seismic interferometry, as will be discussed in the following sections; van Manen et al. (2005) propose an efficient modelling scheme, based on an expression similar to equation (10).

The terms \( \tilde{G} \) and \( \partial_t \tilde{G} \) under the integral in the right-hand side of equation (10) represent responses of impulsive monopole and dipole sources at \( x \) on \( \partial D \). The products \( \tilde{G}^\ast \partial_t \tilde{G} \) etc. correspond to cross-correlations in the time domain. Hence, the right-hand side can be interpreted as the integral of the Fourier transform of cross-correlations of observations of wave fields at \( x_A \) and \( x_B \), respectively, due to impulsive sources at \( x \) on \( \partial D \); the integration takes place along the source coordinate \( x \), see Figure 1. The left-hand side of equation (10) is the Fourier transform of \( G(x_A, x_B, -t) + G(x_A, x_B, t) \), which is the superposition of the response at \( x_A \) due to an impulsive source at \( x_B \) and its time-reversed version. Since the Green’s function \( G(x, x_B, t) \) is causal, it can be obtained by taking the causal part of this superposition. Alternatively, in the frequency domain the imaginary part of \( \tilde{G}(x_A, x_B, \omega) \) can be obtained from the Hilbert transform of the real part.

Note that equation (10) is exact and applies to any inhomogeneous lossless acoustic medium. The choice of the integration boundary \( \partial D \) is arbitrary (as long as it encloses \( x_A \) and \( x_B \)) and the medium may be inhomogeneous inside as well as outside \( \partial D \). The reconstructed Green’s function \( \tilde{G}(x_A, x_B, \omega) \) contains, apart from the direct wave between \( x_B \) and \( x_A \), all scattering contributions (primaries and multiples) from inhomogeneities inside as well as outside \( \partial D \).

Modifications for seismic interferometry

Applying equation (10) for seismic interferometry requires that monopole and dipole responses are available for all source positions \( x \) on \( \partial D \). In this section we perform a number of manipulations which will make equation (10) more suited for application in seismic interferometry. We temporarily denote \( \tilde{G}(x_A, x, \omega) \) and \( \tilde{G}(x_B, x, \omega) \) by \( \tilde{G}_A \) and \( \tilde{G}_B \), respectively. Furthermore, we write

\[
\tilde{G}_A = \tilde{G}^\ast_A + \tilde{G}^\ast_A, \quad (11)
\]

\[
\tilde{G}_B = \tilde{G}^\ast_B + \tilde{G}^\ast_B, \quad (12)
\]

where the superscripts ‘in’ and ‘out’ refer to waves propagating inward and outward from the sources at \( x \) on \( \partial D \), see Figure 2. Substituting these expressions into equation
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(10) gives

\[ 2R\{\hat{G}(x_A, x_B, \omega)\} = \int_{\partial \mathcal{D}} i \omega \rho(x) \left\{ (\hat{G}^\text{in}_A + \hat{G}^\text{out}_A) (\partial_i \hat{G}^\text{in}_B + \partial_i \hat{G}^\text{out}_B) 
- (\partial_i \hat{G}^\text{in}_A + \partial_i \hat{G}^\text{out}_A) (\hat{G}^\text{in}_B + \hat{G}^\text{out}_B) \right\} n_i d^2 x. \]

In the high frequency regime, the derivatives of the Green’s functions can be approximated by multiplying each constituent (direct wave, scattered wave etc.) by $\frac{1}{c(x)} \cos \alpha(x)$, where $c(x)$ is the local propagation velocity and $\alpha(x)$ the local angle between the pertinent ray and the normal on $\partial \mathcal{D}$. The minus-sign applies to inward propagating waves and the plus-sign to outward propagating waves. The main contributions to the integral in equation (13) come from stationary points on $\partial \mathcal{D}$ [Snieder (2004), Schuster et al. (2004), Wapenaar et al. (2004), Snieder et al. (2005)]. At those points the absolute cosines of the ray angles for $\hat{G}_A$ and $\hat{G}_B$ are identical. This implies for example that the terms $\hat{G}^\text{in}_A \partial_i \hat{G}^\text{out}_B$ and $- (\partial_i \hat{G}^\text{in}_A) \hat{G}^\text{out}_B$ give equal contributions to the integral, whereas the contributions of $\hat{G}^\text{in}_B \partial_i \hat{G}^\text{out}_A$ and $- (\partial_i \hat{G}^\text{in}_B) \hat{G}^\text{out}_A$ cancel each other. Hence, we can rewrite equation (13) as

\[ 2R\{\hat{G}(x_A, x_B, \omega)\} = \int_{\partial \mathcal{D}} \frac{2}{i \omega \rho(x)} \left\{ (\partial_i \hat{G}^\text{in}_A) \hat{G}^\text{out}_B + (\partial_i \hat{G}^\text{out}_A) \hat{G}^\text{in}_B \right\} n_i d^2 x. \]

Since the inward and outward propagating waves cannot be separately measured at $x_A$ and $x_B$, we use equations (11) and (12) to rewrite equation (14) as

\[ 2R\{\hat{G}(x_A, x_B, \omega)\} + \text{‘ghost’} = \int_{\partial \mathcal{D}} \frac{2}{i \omega \rho(x)} (\partial_i \hat{G}^\text{in}_A) \hat{G}^\text{out}_B \hat{G}(x_A, x_B, \omega) n_i d^2 x, \]

where

\[ \text{‘ghost’} = \int_{\partial \mathcal{D}} \frac{2}{i \omega \rho(x)} \left\{ (\partial_i \hat{G}^\text{in}_A) \hat{G}^\text{out}_B + (\partial_i \hat{G}^\text{out}_A) \hat{G}^\text{in}_B \right\} n_i d^2 x. \]

The right-hand side of equation (15) contains only one correlation product and therefore it has a more manageable form than equation (10). However, the left-hand side of equation (15) contains a ghost term that adds spurious events to the reconstructed Green’s function $\hat{G}(x_A, x_B, \omega)$. According to equation (16) this ghost term contains correlation products of waves that propagate inward in one state and outward in the other. Note that when $\partial \mathcal{D}$ is an irregular surface (which is the case when the sources are randomly distributed), these correlation products are not integrated coherently in equation (16) and can therefore be ignored in equation (15). Hence, when the medium is inhomogeneous inside as well as outside $\partial \mathcal{D}$, the Green’s function $\hat{G}(x_A, x_B, \omega)$ can be accurately retrieved from equation (15) as long as $\partial \mathcal{D}$ is sufficiently irregular. This interesting phenomenon was first observed with numerical experiments by Draganov et al. (2003).

Next we assume that the medium outside $\partial \mathcal{D}$ is homogeneous. In this case the Green’s functions $\hat{G}^\text{out}_A$ and $\hat{G}^\text{out}_B$ are zero, see Figure 2, hence, the ghost term vanishes, independent of the shape of $\partial \mathcal{D}$. If we now assume that $\partial \mathcal{D}$ is a sphere with large enough radius then all rays are normal to $\partial \mathcal{D}$ (i.e., $\alpha \approx 0$), hence $\partial_i \hat{G}^\text{in}(x_A, x_B, \omega) \approx j \frac{\omega}{c} \hat{G}^\text{in}(x_A, x_B, \omega)$, or

\[ 2R\{\hat{G}(x_A, x_B, \omega)\} \approx \frac{2}{\rho c} \int_{\partial \mathcal{D}} \hat{G}^\text{in}(x_A, x_B, \omega) \hat{G}(x_B, x, \omega) d^2 x, \]

where $\rho$ and $c$ are the mass density and propagation velocity of the homogeneous medium outside $\partial \mathcal{D}$. Apart from the proportionality factor $2/\rho c$, this result was also obtained by Derode et al. (2003) purely by physical reasoning.

Note that for the derivation of each of the expressions (9), (10) and (13) – (17), we assumed that impulsive point sources were placed on the surface $\partial \mathcal{D}$. Our derivation also holds for uncorrelated noise sources on $\partial \mathcal{D}$ whose source-time function satisfies $s(x, t) \ast s(x', -t) = \delta(x - x') C(t)$, with $C(t)$ the autocorrelation of the noise. When the noise is distributed over the surface, the cross-correlation of the observations at $x_A$ and $x_B$ leads to a double surface integral. The delta function reduces this to the single surface integral in the theory presented here [Wapenaar et al. (2002), Snieder (2004)]. A further discussion is beyond the scope of this paper.

Configuration with a free surface

For the seismic situation we define the closed surface as $\partial \mathcal{D} = \partial \mathcal{D}_0 \cup \partial \mathcal{D}_1$, where $\partial \mathcal{D}_0$ is a part of the free surface and $\partial \mathcal{D}_1$ an arbitrarily shaped surface in the subsurface, see Figure 3. Since the acoustic pressure vanishes on $\partial \mathcal{D}_0$, the integral on the right-hand side of equations (2), (9), (10) and (13) – (17) needs only be evaluated over $\partial \mathcal{D}_1$. Hence, the Green’s function $\hat{G}(x_A, x_B, \omega)$ can be recovered by cross-correlating and integrating the responses of sources on $\partial \mathcal{D}_1$ in the subsurface only. The free surface acts as a mirror which obviates the need of evaluating the integral over a closed surface.
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Numerical example

We illustrate equation (15) with a 2-D example for the situation of a homogeneous embedding (hence, ‘ghost’ = 0) and a free surface at $x_3 = 0$ (hence, $\partial \Omega \rightarrow \partial \Omega_1$). We consider a single diffractor at $(x_1, x_3) = (0, 600)m$ in a homogeneous medium with propagation velocity $c = 2000 m/s$, see Figure 4, in which $C$ denotes the diffractor. Further, we define $x_A = (-500, 100)m$ and $x_B = (500, 100)m$, denoted by A and B in Figure 4. The surface $\partial \Omega_2$ is a semicircle with its center at the origin and a radius of 800 m. The solid arrows in Figure 4 denote the Green’s function $G(x_A, x_B, t)$. We model the Green’s functions in equation (15) with the Born approximation, which means that we consider first order scattering at C only. To be consistent with the Born approximation, in the cross-correlations we also consider only the zeroth and first order terms. Figure 5a shows the time-domain representation of the integrand of equation (15) (convolved with a wavelet with a central frequency of 50 Hz). Each trace corresponds to a fixed source position $x$ on $\partial \Omega_1$; the source position in polar coordinates is $(\phi, r = 800)$. The sum of all these traces (multiplied by $r d\phi$) is shown in Figure 5b. This result accurately matches the directly modelled wave field (not shown). This figure clearly shows that the main contributions come from Fresnel zones around the stationary points of the integrand.

Conclusions

Using Rayleigh’s reciprocity theorem and the principle of time-reversal invariance, we derived Green’s function representations for seismic interferometry. We carefully analyzed the contributions from inhomogeneities inside as well as outside the surface on which the primary sources are located. Next, by assuming no inhomogeneities outside this surface, we obtained a result similar to that of Derode et al (2003). Finally we considered the more realistic situation of an inhomogeneous medium below a free surface and illustrated seismic interferometry for this situation with a numerical example.

References