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Short Note

Synthesis of an inhomogeneous medium from its acoustic transmission response

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INTRODUCTION

In 1968 Claerbout showed that the reflection response of a horizontally layered medium can be synthesized from the autocorrelation of its transmission response. During a workshop on passive imaging methods at the 2002 SEG conference, Claerbout showed that this result can be obtained straightforwardly from the principle of conservation of acoustic power. In this paper I briefly review this derivation and show that the 3D generalization can be obtained along the same lines using a power reciprocity theorem. The resulting expression confirms Claerbout's conjecture that "by crosscorrelating noise traces recorded at two locations on the surface, we can construct the wave field that would be recorded at one of the locations if there was a source at the other."

1D DERIVATION USING POWER CONSERVATION

Let $D(\omega)$ and $U(\omega)$ represent downgoing and upgoing plane wavefields in a horizontally layered medium in the frequency domain (ω denotes the angular frequency). Then the net downgoing power flux is given by $D^*D - U^*U$, where the asterisk denotes complex conjugation (this assumes implicitly that D and U are flux-normalized downgoing and upgoing wavefields). When the medium is lossless and source free, this quantity is the same in each layer. Now consider the plane-wave reflection experiment depicted in Figure 1a. The incident downgoing wave just below the free surface is denoted by 1, which corresponds to an impulsive plane-wave source in the time domain. The total upcoming reflected wavefield (including internal as well as free surface multiples) is denoted by R [Claerbout denotes this as $-R$; I take the liberty to modify the notation to facilitate the comparison with the 3D derivation]. Because of the free surface there is a downgoing reflected wavefield, denoted by $-R$. Hence, just below the free surface the total down- and upgoing wavefields are given by

$$D = 1 - R, \quad U = R \tag{1}$$

(see Figure 1a), so the net downgoing power flux becomes

$$\begin{aligned} D^*D - U^*U &= (1 - R^*)(1 - R) - R^*R \\ &= 1 - R - R^*. \end{aligned} \tag{2}$$

At the lowest boundary in Figure 1a, the downgoing transmitted wavefield (including internal as well as free surface multiples) is denoted by T (Claerbout uses E, for escaping wave). The medium below this boundary is assumed to be homogeneous, so there is no upgoing wavefield. Hence, in the lower half-space the net downgoing power flux is given by

$$D^*D - U^*U = D^*D = T^*T. \tag{3}$$

Since the power flux $D^*D - U^*U$ is conserved, the right-hand side of equation (2) is identical to that of equation (3). Hence,

$$R(\omega) + R^*(\omega) = 1 - T^*(\omega)T(\omega). \tag{4}$$

Using reciprocity, the downgoing transmitted wavefield T below the lowest boundary is equal to the upgoing transmitted wavefield observed at the free surface (again denoted by T, see Figure 1b) as a result of an impulsive plane-wave source below the lowest boundary.

In the time domain equation (4) becomes

$$R(t) + R(-t) = \delta(t) - T(-t) * T(t), \tag{5}$$

where $*$ denotes convolution, and t is time. Since the reflection response $R(t)$ is causal, it is easily obtained by taking the causal part of $R(t) + R(-t)$.

Equation (5) states that the reflection response is obtained from the autocorrelation of the transmission response of an impulsive source in the subsurface. However, the autocorrelation

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does not change when the impulsive source is replaced by any source of which the autocorrelation is again an impulse. Hence, the reflection response $R(t)$ of an impulsive source at the surface can be obtained from the autocorrelation of the transmission response of a white noise source in the subsurface. Note that the depth of this source is immaterial since any time shift of the transmission response is annihilated by the autocorrelation. The only condition is that this source is below the lowest boundary of the layered medium.

3D DERIVATION USING POWER RECIPROCITY

The generalization of the derivation to the 3D situation is based on acoustic reciprocity. In general, an acoustic reciprocity theorem formulates a relation between two acoustic states that could occur in one and the same domain (de Hoop, 1988; Fokkema and van den Berg, 1993). One can distinguish between two-way and one-way reciprocity theorems of the convolution type and of the correlation type. In this paper I work with the correlation-type one-way reciprocity theorem [Wapenaar and Grimbergen (1996), their equation (81)], which is valid for nonevanescent waves in 3D inhomogeneous lossless media. The reason for choosing the correlation type is obvious: Claerbout's conjecture is about crosscorrelations. The one-way choice stems from the fact that the concepts of reflection and transmission apply to downgoing and upgoing wavefields (rather than to full wavefields). Let $\partial\mathcal{D}_0$ and $\partial\mathcal{D}_m$ represent two depth levels of fixed x_3 and let A and B denote two independent acoustic states (for example, two independent seismic experiments). When the medium between $\partial\mathcal{D}_0$ and $\partial\mathcal{D}_m$ is source free

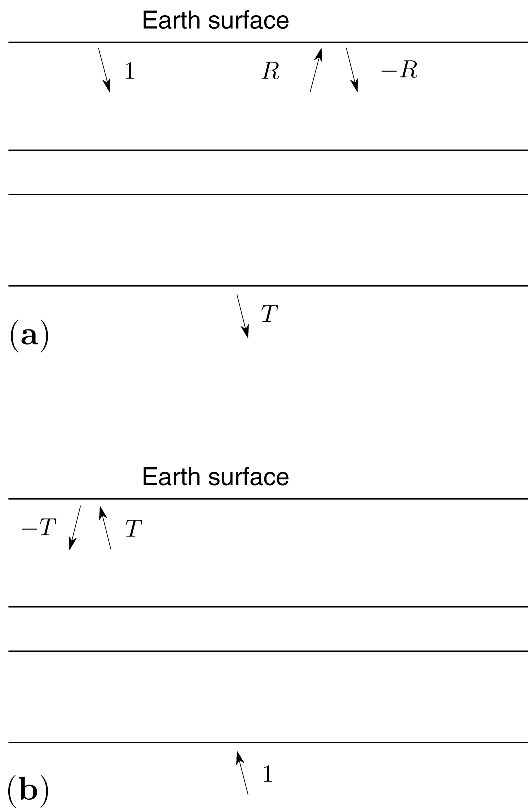

FIG. 1. (a) Plane-wave reflection response of a horizontally layered medium. (b) Plane-wave transmission response of the same horizontally layered medium.

and the medium parameters are chosen identically in states A and B, equation (81) of Wapenaar and Grimbergen (1996) simplifies to

$$\int_{\partial\mathcal{D}_0} \left\{ D_A^* D_B - U_A^* U_B \right\} \mathrm{d}^2\mathbf{x}_H = \int_{\partial\mathcal{D}_m} \left\{ D_A^* D_B - U_A^* U_B \right\} \mathrm{d}^2\mathbf{x}_H. \tag{6}$$

Here, $D(\mathbf{x}, \omega)$ and $U(\mathbf{x}, \omega)$ represent flux-normalized downgoing and upgoing wavefields; $\mathbf{x} = (x_1, x_2, x_3)$ denotes the Cartesian coordinate vector and $\mathbf{x}_H = (x_1, x_2)$ the horizontal coordinate vector. When states A and B are identical, equation (6) states that the net downgoing power flux $\int\{D^*D - U^*U\}\mathrm{d}^2\mathbf{x}_H$ is conserved. For this reason equation (6) is also called a power reciprocity theorem for downgoing and upgoing waves.

Now consider the point source reflection experiment depicted in Figure 2a. An impulsive point source is located at $\mathbf{x}_A$ at the free surface. The depth level $\partial\mathcal{D}_0$ is chosen just below this free surface. The incident downgoing wave just below the free surface is denoted by $\delta(\mathbf{x}_H - \mathbf{x}_{H,A})$, where $\mathbf{x}_{H,A} = (x_{1,A}, x_{2,A})$ denotes the horizontal coordinates of the source at $\mathbf{x}_A$. The total upcoming reflected wavefield (including internal as well as free surface multiples) is denoted by $R(\mathbf{x}, \mathbf{x}_A, \omega)$, where $\mathbf{x}$ denotes an arbitrary receiver point at $\partial\mathcal{D}_0$. Because of the free surface there is a downgoing reflected wavefield, denoted by $-R(\mathbf{x}, \mathbf{x}_A, \omega)$. Hence, just below the free surface (i.e., at $\partial\mathcal{D}_0$) the total down- and upgoing wavefields are given by

$$D_A(\mathbf{x}, \omega) = \delta(\mathbf{x}_H - \mathbf{x}_{H,A}) - R(\mathbf{x}, \mathbf{x}_A, \omega), \tag{7}$$

$$U_A(\mathbf{x}, \omega) = R(\mathbf{x}, \mathbf{x}_A, \omega) \tag{8}$$

(see Figure 2a). Next, consider a second independent reflection experiment with an impulsive point source at another point $\mathbf{x}_B$ at the free surface (not shown in Figure 2a). For this second experiment the total down- and upgoing wavefields at $\partial\mathcal{D}_0$ read

$$D_B(\mathbf{x}, \omega) = \delta(\mathbf{x}_H - \mathbf{x}_{H,B}) - R(\mathbf{x}, \mathbf{x}_B, \omega), \tag{9}$$

$$U_B(\mathbf{x}, \omega) = R(\mathbf{x}, \mathbf{x}_B, \omega). \tag{10}$$

Substituting equations (7)–(10) into equation (6) yields for its left-hand side

$$\int_{\partial\mathcal{D}_0} \left\{ D_A^* D_B - U_A^* U_B \right\} \mathrm{d}^2\mathbf{x}_H = \delta(\mathbf{x}_{H,A} - \mathbf{x}_{H,B}) - R(\mathbf{x}_A, \mathbf{x}_B, \omega) - R^*(\mathbf{x}_B, \mathbf{x}_A, \omega) \tag{11}$$

[compare with equation (2)]. At the lowest boundary $\partial\mathcal{D}_m$ in Figure 2a, the downgoing transmitted wavefields (including internal as well as free-surface multiples) for the two experiments are denoted by $T(\mathbf{x}, \mathbf{x}_A, \omega)$ and $T(\mathbf{x}, \mathbf{x}_B, \omega)$, respectively, where $\mathbf{x}$ now denotes an arbitrary observation point at $\partial\mathcal{D}_m$. The medium below this boundary is assumed to be homogeneous, so there are no upgoing wavefields. Hence, the right-hand side of equation (6) simplifies to

$$\int_{\partial\mathcal{D}_m} D_A^* D_B \,\mathrm{d}^2\mathbf{x}_H = \int_{\partial\mathcal{D}_m} T^*(\mathbf{x}, \mathbf{x}_A, \omega) T(\mathbf{x}, \mathbf{x}_B, \omega) \,\mathrm{d}^2\mathbf{x}_H \tag{12}$$

[compare with equation (3)]. As a result of the power reciprocity theorem (equation 6), the right-hand side of equation (11) is identical to that of equation (12). Hence, using source–receiver reciprocity for the reflection and transmission responses, these equations yield

$$R(\mathbf{x}_A, \mathbf{x}_B, \omega) + R^*(\mathbf{x}_A, \mathbf{x}_B, \omega) = \delta(\mathbf{x}_{H,A} - \mathbf{x}_{H,B}) - \int_{\partial\mathcal{D}_m} T^*(\mathbf{x}_A, \mathbf{x}, \omega) T(\mathbf{x}_B, \mathbf{x}, \omega)\, \mathrm{d}^2\mathbf{x}_H, \quad (13)$$

where $T(\mathbf{x}_A, \mathbf{x}, \omega)$ is the upgoing transmitted wavefield observed at $\mathbf{x}_A$ at the free surface as a result of an impulsive point source at position $\mathbf{x}$ at $\partial\mathcal{D}_m$ (see Figure 2b). $T(\mathbf{x}_B, \mathbf{x}, \omega)$ is the upgoing transmitted wavefield at $\mathbf{x}_B$ resulting from the same source (not shown in Figure 2b). For the special case of a horizontally layered medium, equation (4) is obtained by applying a spatial Fourier transform to all terms in equation (13) and selecting the normal incidence component.

Equation (13) almost proves Claerbout's conjecture, cited in the introduction. The term $T^*(\mathbf{x}_A, \mathbf{x}, \omega) T(\mathbf{x}_B, \mathbf{x}, \omega)$ represents the crosscorrelation of traces recorded at two locations ($\mathbf{x}_A$ and $\mathbf{x}_B$) on the surface for a source at $\mathbf{x}$ in the subsurface; the term $R(\mathbf{x}_A, \mathbf{x}_B, \omega)$ is the wavefield that would be recorded at one of the locations ($\mathbf{x}_A$) if there were a source at the other ($\mathbf{x}_B$). The main discrepancy with the conjecture is the integral in equation (13) over all possible source positions $\mathbf{x}$ at surface $\partial\mathcal{D}_m$. Equation (13) is useful when the transmission responses for the different source positions $\mathbf{x}$ at surface $\partial\mathcal{D}_m$ can be measured sequentially. However, when the sources at $\partial\mathcal{D}_m$ act simultaneously (for example, in the case of natural noise sources with long duration), the integral in equation (13) cannot be evaluated because the transmission responses are not available for all individual source positions $\mathbf{x}$. This is resolved when the sources for different source positions $\mathbf{x}$ on $\partial\mathcal{D}_m$ are mutually uncorrelated. To see this, apply an inverse Fourier transform to all terms in equation (13) and discretize the integral, according to

$$R(\mathbf{x}_A, \mathbf{x}_B, t) + R(\mathbf{x}_A, \mathbf{x}_B, -t) = \delta(\mathbf{x}_{H,A} - \mathbf{x}_{H,B})\delta(t) - \sum_i T(\mathbf{x}_A, \mathbf{x}_i, -t) * T(\mathbf{x}_B, \mathbf{x}_i, t), \quad (14)$$

where the sum is applied over all $\mathbf{x}_i$ at $\partial\mathcal{D}_m$ (the discretization intervals are included as a factor in the transmission responses). Let $N_i(t)$ and $N_j(t)$ be mutually uncorrelated white noise sources at $\mathbf{x}_i$ and $\mathbf{x}_j$; hence, $N_i(-t) * N_j(t) = \delta_{ij}\delta(t)$. Inserting these noise sources in the right-hand side of equation (14) yields

$$R(\mathbf{x}_A, \mathbf{x}_B, t) + R(\mathbf{x}_A, \mathbf{x}_B, -t) = \delta(\mathbf{x}_{H,A} - \mathbf{x}_{H,B})\delta(t) - \sum_i \sum_j T(\mathbf{x}_A, \mathbf{x}_i, -t) * N_i(-t) * T(\mathbf{x}_B, \mathbf{x}_j, t) * N_j(t) \quad (15)$$

or

$$R(\mathbf{x}_A, \mathbf{x}_B, t) + R(\mathbf{x}_A, \mathbf{x}_B, -t) = \delta(\mathbf{x}_{H,A} - \mathbf{x}_{H,B})\delta(t) - T_{\mathrm{obs}}(\mathbf{x}_A, -t) * T_{\mathrm{obs}}(\mathbf{x}_B, t) \quad (16)$$

[compare with equation (5)], with

$$T_{\mathrm{obs}}(\mathbf{x}_A, t) = \sum_i T(\mathbf{x}_A, \mathbf{x}_i, t) * N_i(t), \quad (17)$$

$$T_{\mathrm{obs}}(\mathbf{x}_B, t) = \sum_i T(\mathbf{x}_B, \mathbf{x}_i, t) * N_i(t). \quad (18)$$

Note that $T_{\mathrm{obs}}(\mathbf{x}_A, t)$ and $T_{\mathrm{obs}}(\mathbf{x}_B, t)$ may be seen as transmission responses, observed at $\mathbf{x}_A$ and $\mathbf{x}_B$ on $\partial\mathcal{D}_0$, from a distribution of uncorrelated noise sources at a number of positions $\mathbf{x}_i$ on $\partial\mathcal{D}_m$. The right-hand side of equation (16) describes the crosscorrelation of these observations. The impulsive point-source reflection response $R(\mathbf{x}_A, \mathbf{x}_B, t)$ is obtained by taking the causal part of the left-hand side of this equation. This finalizes the proof of Claerbout's conjecture.

Of course, in reality the noise sources will not be evenly distributed along a single surface $\partial\mathcal{D}_m$ (see Figure 3). However, the actual depth of the sources is almost immaterial, since the extra

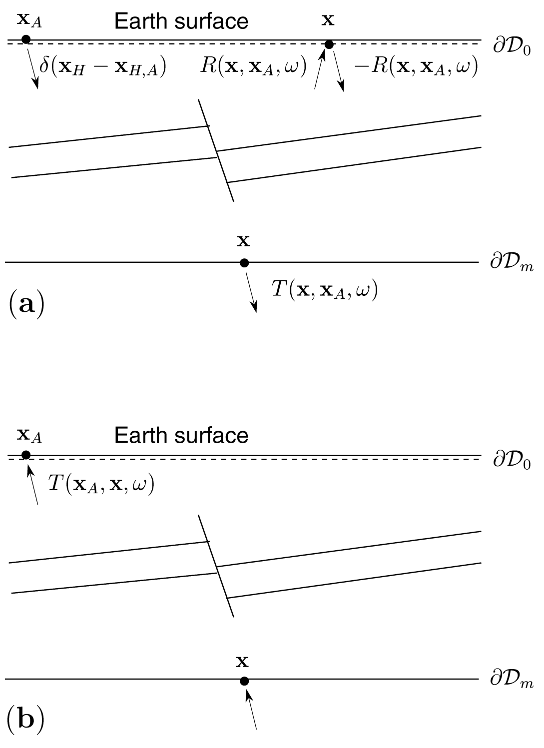

FIG. 2. (a) Point source reflection response of an inhomogeneous medium. (b) Point source transmission response of the same inhomogeneous medium.

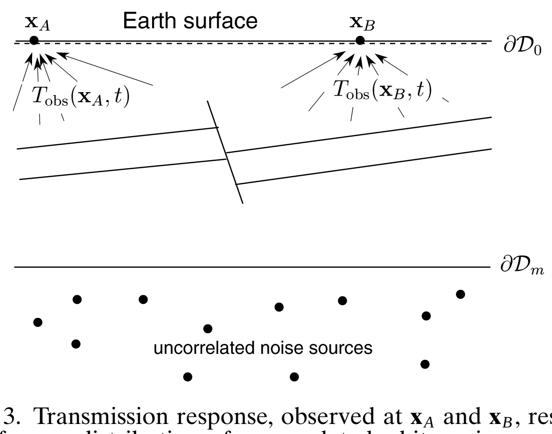

FIG. 3. Transmission response, observed at $\mathbf{x}_A$ and $\mathbf{x}_B$, resulting from a distribution of uncorrelated white noise sources. According to equation (16), their crosscorrelation yields the reflection response observed at $\mathbf{x}_A$ as if there were an impulsive point source at $\mathbf{x}_B$.

traveltime between the actual source depth and $\partial\mathcal{D}_m$ drops out in the correlation process. Of course, the accuracy degrades in the case of an irregular source distribution. Other effects, like crosstalk, nonwhiteness, etc., also degrade the accuracy of equation (16). Finally, note it was assumed that the half-space below $\partial\mathcal{D}_m$ is homogeneous. When this assumption is not fulfilled, extra events will appear in the transmission responses which will not be correctly mapped to the reflection response. However, the traveltimes of these ghost events depend on the source depth. So when the source depth is irregular, these ghost events do not contribute coherently to the integral and therefore are suppressed in the reflection response (Draganov et al., 2003).

CONCLUSIONS AND DISCUSSION

Claerbout (1968) shows that the reflection response of a horizontally layered medium can be synthesized from the autocorrelation of its transmission response. This result is most easily derived by applying the power conservation principle. In this paper I show that the 3D extension can be derived along the same lines by applying a power reciprocity theorem for downgoing and upgoing waves. The final expressions [equations (16)–(18)] show that by crosscorrelating noise traces recorded at two locations $\mathbf{x}_A$ and $\mathbf{x}_B$ on the surface, one can construct the wavefield that would be recorded at one of the locations if there were an impulsive source at the other. This is the basis for 3D acoustic daylight imaging, i.e., synthesizing an inhomogeneous medium from its acoustic transmission response, obtained from passive measurements at the surface of noise sources in the subsurface. Rickett and Claerbout (1999) demonstrate this principle convincingly with solar seismology. Some initial results with acoustic daylight imaging on earth are reported by Daneshvar et al. (1995). The method may be improved by using significantly longer noise recordings and by using geophone arrays that suppress surface waves.

In the derivation of equation (16) I have made no particular assumptions on the medium, except that it is lossless and sandwiched between a free surface and a homogeneous half-space. The sources are assumed to be mutually uncorrelated noise sources, distributed sufficiently dense to avoid spatial aliasing and covering enough aperture to account for the required range of propagation angles in the reconstructed impulsive reflection response (i.e., the Green's function). Another approach to reconstructing the Green's function from noise measurements is discussed by Lobkis and Weaver (2001). They show that the Green's function of a medium emerges by cross-correlating the recordings of two receivers in a diffuse field. Their assumptions are the complement of the assumptions described above: a single source is sufficient, but the medium parameters must have enough randomness for the wavefield (including its coda resulting from internal multiple scattering) to be diffuse. Their derivation shows similarities with that of acoustic time reversal in chaotic cavities (Draeger and Fink, 1999). Campillo and Paul (2003) use the approach of correlating diffuse fields to reconstruct surface-wave responses between two stations from recordings of distant earthquakes.

In practice, the conditions for passive seismics will most likely be a combination of the two situations described above, i.e., a limited number of uncorrelated noise sources and a limited amount of randomness of the medium parameters. Both conditions contribute to the accuracy of the reconstruction of the Green's fuction.

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