Steady-state displacements of a beam on an elastic half-space due to a uniformly moving constant load

H. A. DIETERMAN * and A. V. METRIKINE **

ABSTRACT. — The steady-state displacements of an Euler-Bernoulli beam on an elastic half-space due to a uniformly moving constant load are determined using the concept of the equivalent stiffness of the half-space. The displacements are calculated for four different cases of beam and half-space parameters. The displacements in each case are derived for five relevant load velocities. The lowest is small with respect to the shear (S)-wave velocity in the half-space, the subsequent velocities are near the lowest critical velocity, between the critical velocity and the Rayleigh (R)-wave velocity, between the R-wave and S-wave velocity and larger than S-wave velocity. The beam displacement under the load is also determined for each case for all load velocities. Near the critical velocity the effect of an external viscous damping along the beam on the displacement under the load is studied.

1. Introduction

Railway companies in Swiss (SBB), France (SNCF), Germany (DB), Holland (NS) and Great-Britain (BR) have measured substantial increases of the vertical movements in the track when trains move with a velocity in the order of the Rayleigh wave velocity of the subsoil. This problem already takes place in traditional track on soft soils, such as peat, at velocities as low as 140 km/h. Mostly speed limits of the trains have been ordered on these track parts to avoid the coupling between the train and the lowest critical velocity.

At high-speed track evasion of this phenomenon requests a substantial improvement of the quality of the supporting subsoil.

The problem of a moving load over a beam on an elastic half-space has first been studied by Filippov (1961) as a model for the train-subsoil interaction. It was shown that the critical velocity of the train is approximately equal to the R-wave velocity. Labra (1975) extended the results of Filippov taking into account the axial stresses in the beam showing that the critical velocity is decreasing with increasing axial compression. He confirmed the results found by Kerr (1972) who studied the beam on a Winkler
foundation. Both Filippov and Labra studied the load velocity range up to the R-wave velocity and did not derive the beam displacements due to the load.

A recent paper (Dieterman and Metrikine, 1966) investigated the critical velocities of a moving load over an Euler-Bernoulli beam in smooth contact with a half-space for all load velocities, using the concept of equivalent stiffness of the half-space. It showed a second critical velocity below the R-wave velocity.

In the present paper we calculate the steady-state displacements of the beam for subseismic, transeismic and superseismic speed ranges of the uniformly moving load, showing the amplifications of the displacements of the beam near the critical velocities. To study the effect of damping an external viscous friction has been placed along the beam.

The calculations are performed for two sets of half-space parameters, modelling soft and stiff subsoil and for two sets of beam parameters, modelling small and large track-ballast-embankment systems. Therefore four different cases are investigated.

In each case the beam displacements will be determined for five different load velocities. The lowest velocity is small with respect to the shear (S)-wave velocity in the half-space, the subsequent velocities are near the lowest critical velocity, between the critical velocity and the Rayleigh (R)-wave velocity, between the R-wave velocity and the S-wave velocity and larger than S-wave velocity but smaller than the compression (P)-wave velocity.

Also the beam displacement directly under the load is determined for each case for all load velocities. This displacement is equal to the maximum displacement for small load velocities only. For higher velocities, but smaller than the critical velocity, the maximum displacement shifts behind the load due to the external viscous damping of the beam. For velocities larger than the critical velocity the shift is more substantial due to the wave radiation. To show this phenomenon the maximum displacements before and behind the load are determined for one of the cases.

Near the critical velocity the effect of the external viscous damping along the beam on the displacement under the load is studied and shows an almost inverse proportionality with the amount of damping.

The results given in the paper are an extension to the paper of Kenney (1954), who calculated the beam displacements due to a uniformly moving constant load over an Euler-Bernoulli beam on a visco-elastic Winkler foundation.

2. Model and method of solution

We consider a constant load $P$ moving uniformly ($x = Vt, y = 0, z = 0$) along an Euler-Bernoulli beam resting on an elastic half-space as depicted in Figure 1.

It is assumed that the contact between the beam and the half-space is smooth, hence the shear stresses $\tau_{xz}$ and $\tau_{yz}$ are zero at the interface. It is further assumed that the normal stresses between the beam and the half-space are uniformly distributed over the width of the beam. Then, as it was shown in Dieterman and Metrikine (1996), the
steady-state displacement of the beam due to the moving load is given as (adding an external viscous friction to the beam)

\begin{equation}
W(x, t) = -\frac{P}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(ik(x-Vt))}{EI k^4 - mк^2 V^2 - ik \delta V + 2\pi \mu \chi(k V, k)} \, dk,
\end{equation}

where $EI$ is the bending stiffness of the beam, $m$ is the mass per unit length of the beam, $\delta$ is the viscosity coefficient, $\mu$ is the Lame constant of the half-space and $2\pi \mu \chi(\omega, k)$ is the equivalent stiffness of the half-space ($\omega$ and $k$ are the frequency and wave number of the waves in the beam).

The expression for $\chi$ according to Dieterman and Metrikine (1996) for $k > 0$ has been rewritten in a more convenient form for the numerical calculations as presented in the Appendix. The real part of $\chi$ is symmetrical and the imaginary part is asymmetrical with respect to $k$, i.e.

\begin{align*}
\text{Re } (\chi(KV, k)) &= \text{Re } (\chi(-KV, -k)), \\
\text{Im } (\chi(KV, k)) &= -\text{Im } (\chi(-KV, -k)).
\end{align*}

Using the symmetry of the equivalent stiffness and introducing the relative velocity $\nu = V/\alpha$, we rewrite (1) as

\begin{equation}
W = W^{\text{sym}} + W^{\text{asym}},
\end{equation}

where

\begin{align}
W^{\text{sym}}(\xi) &= -2A \int_{0}^{\infty} \frac{(k^4 - \alpha^2 \nu^2 k^2 + \beta^2 \text{Re } (\chi)) \cos(k \xi)}{(k^4 - \alpha^2 \nu^2 k^2 + \beta^2 \text{Re } (\chi))^2 + (k \nu \gamma - \beta^2 \text{Im } (\chi))^2} \, dk, \\
W^{\text{asym}}(\xi) &= 2A \int_{0}^{\infty} \frac{(k \nu \gamma - \beta^2 \text{Im } (\chi)) \sin(k \xi)}{(k^4 - \alpha^2 \nu^2 k^2 + \beta^2 \text{Re } (\chi))^2 + (k \nu \gamma - \beta^2 \text{Im } (\chi))^2} \, dk.
\end{align}

Here $A = P/2\pi EI$, $\alpha^2 = mc^2/EI$, $\beta^2 = 2\pi \mu / EI$, $\gamma = \delta c/IE$, and $\chi = \chi(\nu, k)$ and $\xi = x - Vt$ is the distance from the moving load.

Expressions (3) and (4) describe the symmetrical $W^{\text{sym}}$ and asymmetrical $W^{\text{asym}}$ parts of the beam displacement with respect to the location of the moving load.
3. Results

The calculations have been performed for two sets of half-space parameters (stiff and soft) and two sets of beam parameters (small and large). These parameters are shown in Table I and Table II. Parameters of the half-space are denoted as follows: $E$ is Young's modulus, $\rho$ is the density, $\mu$ is the Lamé constant, $c_l$, $c_t$, $c_R$ are the velocities of compressional, shear and Rayleigh waves respectively). The values of the parameters are chosen to describe stiff and soft subsoil of railroad tracks.

<table>
<thead>
<tr>
<th>Table I</th>
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<tr>
<td>$E$ (N/m²)</td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td>1. Stiff</td>
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<tr>
<td>2. Soft</td>
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<th>Table II</th>
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<tr>
<td>$m$ (kg/m)</td>
</tr>
<tr>
<td>-------------</td>
</tr>
<tr>
<td>1. Small</td>
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<tr>
<td>2. Large</td>
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The beam is actually an equivalent beam the parameters of which include the properties of the ballast, track and embankment. The Poisson ratio of the half-space is taken 0.3.

Parameters $\alpha^2$, $\beta^2$, $\gamma$ and $\alpha$ are given in Table III. The value of $\gamma$ is taken to be about 10% of the critical viscosity. This is defined as the viscosity at which there is no amplification of the beam displacement for $V = V^{cr}$ with respect to the quasistatic case ($V \ll c_t$).

<table>
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<th>Table III</th>
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<td>Half-space-Beam</td>
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<td>-----------------</td>
</tr>
<tr>
<td>1. Stiff-Small</td>
</tr>
<tr>
<td>2. Stiff-Large</td>
</tr>
<tr>
<td>3. Soft-Small</td>
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<tr>
<td>4. Soft-Large</td>
</tr>
</tbody>
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In the Figure 2 the displacement of the beam under the load is depicted as a function of the relative load velocity $V/c_t$ for all cases considered.

Analysing the figures the following conclusions can be drawn.

1. The largest displacement under the load takes place when the half-space is soft and the beam is small;

2. The difference between $V^{cr}$ and $c_R$ is very small for the soft half-space and is about 10% for the stiff half-space. This difference is increasing as the beam becomes larger.
3. The displacement of the beam remains finite for \( V^{cr} \) (the external viscosity is not equal to zero) and becomes infinite when the velocity of the load tends to the Rayleigh wave velocity in the half-space (the vertical dotted lines in the figures);

When the velocity of the load is larger than the Rayleigh wave velocity there is an interval of the load velocities for which the displacement under the load is directed upwardly (qualitatively the same result was obtained in Lansing (1966) for the case of a point load moving along a half-space). This interval is wider for a stiff half-space.

It is important to note that the displacement under the load is not always equal to the maximum displacement of the beam. This is seen in Figure 3a where the beam displacement under the load (curve 1), maximum displacement before (curve 2) and behind (curve 3) the load are depicted for the “stiff-small” case for \( V < c_R \). The physical

EUROPEAN JOURNAL OF MECHANICS, 20 SOLIDS, VOL. 16, NO 2, 1997
reasons for this phenomenon are the viscosity of the beam for subcritical velocities and the wave radiation and the viscosity for supercritical load velocities.

To illustrate the influence of the beam viscosity, the beam displacement under the load is shown in Figure 3b for different values of the viscosity in the interval of the load velocities \([0; c_R]\). The “stiff-small” case is elaborated. It shows the well-known influence of the viscosity on the amplitude of the resonance vibrations.

Fig. 3. – (a) Displacement under the load (1), maximum displacement before (2) and behind (3) the load; (b) displacement under the load for different viscosities.

In Figures 4-7 the relative beam displacement \((W*2\pi10^8/P)\) shapes versus relative velocity \(V/c_t\) for the load at the origin of the reference, moving with the load velocity, are shown for the “stiff-small”, “stiff-large”, “soft-small” and “soft-large” cases respectively. The Figure 4a-7a are related to the quasistatic case \(V = 0.2c_t\). The Figures 4b-7b are related to a load velocity slightly smaller than \(V^{cr}\). The Figures 4c-7c are related to a load velocity located between \(V^{cr}\) and the Rayleigh wave velocity. The figures 4d-7d are related to a load velocity slightly larger than the Rayleigh wave velocity. In the figures 4e-7e the load velocity is located between the shear and the compressional wave velocities in the half-space \((V = 1.6c_t)\).

4. Conclusions

In this paper the steady-state wave forms in an Euler-Bernoulli beam in smooth contact with an elastic half-space due to a uniformly moving load have been derived. The beam has additionally an external viscous type of damping (about 10% of the critical).

For load velocities smaller than the lowest critical velocities the wave forms travelling with the load are almost symmetrical. The small asymmetry seen at velocities slightly before the lowest critical velocity is due to the viscous damping. As the load velocity approaches the critical velocity the maximum displacement grows to values which are
Fig. 4. - Displacement shapes versus velocity. Stiff-Small case.
Fig. 5. – Displacement shapes versus velocity. Stiff-Large case.
Fig. 6. – Displacement shapes versus velocity. Soft-Small case.
Fig. 7. - Displacement shapes versus velocity. Soft-Large case.
substantially larger than at speeds relative small with respect to the S-wave velocity. The maximum displacement at the critical velocity is increasing with a decrease of the viscous damping in the system and a decrease of the elasticity modulus. Hereby we have to keep in mind that no radiation occurs at subcritical velocities, so the viscous damping models the frictional losses in the subsoil.

The second critical load velocity is located at R-wave velocity showing a infinite (steady-state) amplification. This is explained by the zero equivalent stiffness of the half-space at this velocity.

The shift of the lowest critical velocity to a relative lower value for the large beam on the stiffer half-space is not in line with expectations. However this is explained by the smooth contact model and the increase of the beam mass.

The model used in this paper should be improved to a non-smooth contact to get a better understanding of the influence of the contact between beam and half-space on the critical velocities.

APPENDIX

\[ \chi(k V, k) = -c_t^2 \frac{a k}{V^2 I_n}, \]

where

\[ n = \begin{cases} 
1, & \text{when } V < c_R; \\
2, & \text{when } c_R < V < c_t; \\
3, & \text{when } c_t < V < c_l; \\
4, & \text{when } c_l < V,
\end{cases} \]

\[ I_1 = S_1 - 2 G_1 - 2 G_2, \quad I_2 = S_2 - 2 G_1 - 2 G_2 + i S_3, \]

\[ I_3 = S_2 - 2 G_1 - 2 G_3 + i S_3 - 2 G_4, \quad I_4 = S_2 - 2 G_5 + i S_3 - 2 G_6 - 2 G_7, \]

\[ S_1 = S(\exp(-ak \sqrt{1 - \beta_R^2}) - 1), \quad S_2 = S(\cos(ak \sqrt{\beta_R^2} - 1) - 1), \]

\[ S_3 = S \sin(ak \sqrt{\beta_R^2} - 1), \]

\[ S = \frac{\bar{R}_t}{2(\beta_R^2 - 1)(2(2 \beta_R^2 - \beta_t^2) - (\bar{R}_t \beta_R^2/\bar{R}_t + \bar{R}_t \beta_t^2/\bar{R}_t + 2 \bar{R}_t \bar{R}_t))}, \]

\[ G_1 = -\int_{\sqrt{1-\beta_t^2}}^{\infty} \frac{Q_t^+ (\exp(-ak \eta) - 1)}{(2(1-\eta^2) - \beta_t^2)^2 + 4(1-\eta^2) Q_t^+ Q_t^-} d\eta, \]

\[ G_2 = -\int_{\sqrt{1-\beta_t^2}}^{\infty} \frac{4(1-\eta^2)(Q_t^-)^2 Q_t^+ (\exp(-ak \eta) - 1)}{(2(1-\eta^2) - \beta_t^2)^4 + 16(1-\eta^2)(Q_t^+)^2 (Q_t^-)^2} \frac{d\eta}{\eta}, \]
\[ G_3 = - \int_0^{\sqrt{1 - \eta^2}} \frac{4 (1 - \eta^2) (Q_{11}^+)^2 Q_{11}^+ (\exp(-ak\eta) - 1)}{(2(1 - \eta^2) - \beta_1^2) + 16(1 - \eta^2)(Q_{11}^+)^2 \eta} \, d\eta \]
\[ G_4 = \int_0^{\sqrt{1 - \xi^2}} \frac{4(1 + \xi^2)(P_{11}^+)^2 P_{11}^- (\exp(iak\xi) - 1)}{(2(1 - \xi^2) - \beta_1^2) + 16(1 - \xi^2)(P_{11}^-)^2 \xi} \, d\xi \]
\[ G_5 = - \int_0^{\infty} \frac{Q_{11}^+ (\exp(-ak\eta) - 1)}{(2(1 - \eta^2) - \beta_1^2) + 4(1 - \eta^2)Q_{11}^+ Q_{11}^+} \eta \, d\eta \]
\[ G_6 = \int_0^{\sqrt{1 - \xi^2}} \frac{P_{11}^- (\exp(iak\xi) - 1)}{(2(1 - \xi^2) - \beta_1^2) + 4(1 + \xi^2)P_{11}^- P_{11}^-} \xi \, d\xi \]
\[ G_7 = \int_{\sqrt{1 - \xi^2}}^{\infty} \frac{4(1 + \xi^2)(P_{11}^+)^2 P_{11}^- (\exp(iak\xi) - 1)}{(2(1 + \xi^2) - \beta_1^2) + 16(1 + \xi^2)(P_{11}^-)^2 \xi} \, d\xi \]
in which
\[ \vec{R}_{1,t} = \sqrt{\beta_1^2 - \beta_{1,t}^2}, \quad Q_{11}^\pm = \sqrt{\pm \beta_{1,t}^2 \pm \eta^2 \mp 1}, \quad P_{11}^\pm = \sqrt{\mp \beta_{1,t}^2 \mp \xi^2 \pm 1}, \]
c_R, c_t and c_l are the velocities of the Rayleigh, shear and compressional waves in the half-space respectively, 2a is the beam width.

The expressions above are the results (27)-(30) derived in Dieterman and Metrikine (1996) which have been rewritten into a more convenient form for the numerical calculations. The contribution of \( S_0 \) (see Dieterman and Metrikine, 1996) has been included into the integrals. This is possible since \( S_0 \) is the contribution of the pole \( \xi = 0 \) in the integral (25) in Dieterman and Metrikine (1996) and we can cancel it by adding and subtracting "1" into the numerator of the integral (25) and then using the equality \( \int_{-\infty}^{\infty} \frac{\vec{R}_{1,t}(\xi) \, d\xi}{\Delta(\xi)} = 0 \).

REFERENCES


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