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A systematic approach to *n*D orientation representation

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Abstract

In this paper, we present new insights in methods to solve the orientation representation problem in arbitrary dimensions. The gradient structure tensor is one of the most used descriptors for local structure in multi-dimensional images. We will relate its properties to the double angle method (2D) and the Knutsson mapping. We present a general scheme to reduce the dimensionality of the Knutsson mapping and derive some properties of these reduced mappings.

Keywords: Gradient structure tensor; Knutsson mapping; Continuous orientation representation; Local structure

1. Introduction

We define simple neighborhoods in images as areas that are shift invariant in at least one direction and not shift invariant in at least one other direction. Such areas play a key role in the description of local structure. The aforementioned shift directions can be determined up to point inversion. Therefore, a pair of opposite directions is designated by a single orientation.

The first order intensity variation is the gradient. A collection of local gradients is needed to compute a dominant orientation. The accompanying intensity variations and that of orthogonal directions can be used to describe lines, surfaces and edges as well as texture. A characterization of simple neighborhoods is by the dominant orientation [1,2,7,10].

Orientation is direction up to point inversion, therefore, leaving room for ambiguity in representation. Direction is described by the full angle, which is in 2D characterized by $\vartheta \in [0, 2\pi]$, and in general by $n-1$ angles in nD. Direction can also be represented by vectors, for example $v = (1, 2)$ or $w = (-1, -2)$. These vectors point in opposite directions but have the same orientation.

Representing orientation by vectors (direction information) leads to troublesome descriptions, in the sense that it is discontinuous. Representing a line in 2D by its angle with respect to a fixed coordinate axis and a plane in 3D by its normal vector is, therefore, not a suitable representation. In Fig. 1, a test and a real 2D image are shown on the left and the orientation fields on the right. We clearly see two jumps or discontinuities. They cannot be removed, for example, by phase unwrapping [5,14,20]. Phase unwrapping can only successfully be applied to discontinuities that form non-intersecting closed lines.

A consistent definition of direction is only possible in a global frame work, whereas most image operators are bound to a local neighborhood. The heart of the problem is sketched in Fig. 2. The support of the operator may have a size as indicated by the circle. The scale is local, whereas the structure has a global connection. The content of the upper and lower window are the same, although the outward pointing normal vector changes continuously along the line. So we are left with two identical windows and an estimated normal orientation with a discontinuity.

The gradient vectors in a local neighborhood need to be combined to obtain an estimate of the local orientation. A simple averaging of gradient vectors fails, on lines in 2D (planes in 3D, etc.) because vectors from opposite sides of the line point in opposite directions and will cancel each other. Thus, we need a suitable continuous representation of gradient vectors to average the structure inside a local window.

Furthermore, a discontinuous representation is very often not suitable for further processing. Most image

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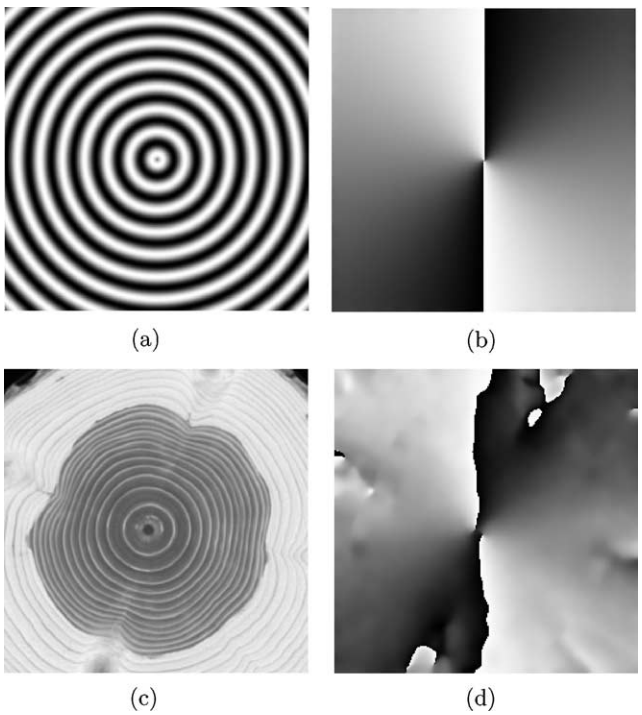

Fig. 1. (a) Image with concentric circles forming an oriented texture, (b) the local orientation field $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (dark, light) with a discontinuity mod π, (c) a cross section through a CT image of a tree trunk and (d) the orientation field.

operators give incorrect response to apparent discontinuities. Therefore, the approach should be as follows: obtain the gradient vectors, map them to a continuous representation, carry out the averaging (or apply another filter). The interpretation of the results of the filtering operation on the new representation is then—in general—not straightforward. A well-known tool to analyze local structure is the Gradient Structure Tensor $\bar{G}$ (GST) [1,2,6–8,10,15]. It is defined as

$$\bar{G} := \overline{\nabla I \nabla I^{\mathrm{t}}} \quad \text{with } G := \nabla I \nabla I^{\mathrm{t}}, \tag{1}$$

where I is a grey-value image and the over-lining stands for averaging the elements inside a local neighborhood.

A physical interpretation of the GST can be given in the terms of friction [19]. Imagine the grey-value surface of a 2D image as a washboard, where the friction is proportional to absolute gradient strength. The GST is then a measure for the local mean squared friction $o^{\mathrm{t}}\bar{G}o$ along orientation o when rubbing over the surface. An eigensystem analysis of the GST yields the orientation with the least and the highest friction; along and perpendicular to the ribs of the washboard.

The structure tensor is also similar to the covariance matrix $C = \langle X^2 \rangle - \langle X^2 \rangle$. In statistical pattern recognition, the covariance matrix is used to describe a set of points (here the points are generated by the endpoints of the gradient vectors) [3,9]. The relation is given by

$$C = \bar{G} - \overline{\nabla I_i \nabla I_j}, \qquad 1 \le i, j \le n. \tag{2}$$

The covariance matrix and the structure tensor are identical if the average (expectation) per element is zero, $\langle X \rangle = 0$, i.e. on lines and planes. The orientation estimation of the GST is very robust to noise. An accurate estimate for the orientation can be obtained for signal-to-noise ratios close to 0 dB (dependent on the window size) [11,21].

The GST is also related to a well-known quantity in physics, the inertia tensor [18] by the following relation

$$\mathbf{J} = \mathrm{tr}(\mathbf{G})\mathbf{I} - \mathbf{G} \quad \text{or} \quad J_{ij} = \mathrm{tr}(G_{ij})\delta_{ij} - G_{ij}. \tag{3}$$

One way to see the connection is that the GST estimates the inertia of the gradient vector endpoints, which are translated to the a common origin.

The GST overcomes the problem in averaging orientation by mapping the local gradient ∇I via a dyadic product

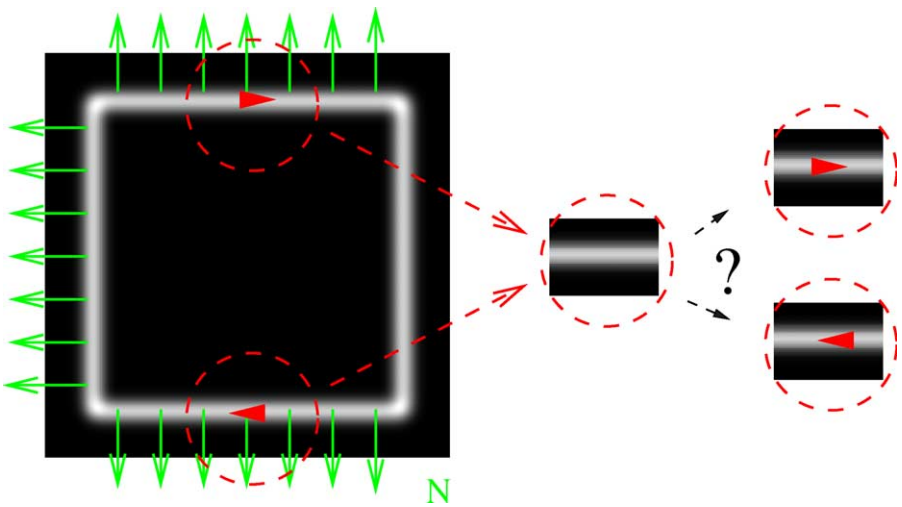

Fig. 2. Local analysis and the problem of direction versus orientation information. The normal vector in the two dashed windows point in different direction whereas the content of the windows is the same. Global knowledge is needed to resolve direction information.

to a continuous representation, which allows filtering; averaging with a weight function. The outcome cannot be interpreted directly but first an eigenvalue analysis of $\bar{G}$ has to be done, where the ratios of the eigenvalues characterize local structure [6,8], i.e. the local dimensionality. Due to the non-linearity of the structure tensor, applying arbitrary linear filters to the tensor result may produce an unexpected outcome.

The GST clearly treats gradients (x) pointing in opposite direction $(-x)$ equally with respect to direction and magnitude $G : \mathbb{R}^n \ni x \rightarrow xx^t \in \mathbb{R}^{n\times n}$. These two properties are the necessary conditions for sensible averaging of the tensor elements. In other words, rotation of the image yields an equally rotated result of the tensor space. In other circumstances, it may be desirable to preserve absolute differences in orientation in the mapping resolving the orientation problem. The GST has $n(n+1)/2$ independent components. Limiting the number of elements needed reduces the memory requirements for higher dimensional images. We will look for a minimal set of elements to describe local orientation satisfying three conditions that are outlined below.

2. Requirements of a continuous distance preserving mapping

Knutsson proposed the following three properties for a continuous distance preserving representation of orientation: Uniqueness, Uniform Stretch and Polar Separability [12,13]. Let $x \in \mathbb{R}^n$:

- *Uniqueness*. Antipodal vectors should be mapped onto one point, this removes the phase jump, e.g. opposite gradient vectors are treated equally $M(x) = M(-x)$.
- *Polar separability*. The norm of the mapped vector should be rotation invariant; information carried by the magnitude of the original vector does not normally depend on the angle. $\|M(x)\| = f(\|x\|)$, where f is an arbitrary function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$.
- *Uniform stretch*. The mapping should carry implicitly information about the distances in the original space that is rotation invariant and scales linearly with the angle between two hyper planes $\|\delta M(x)\| = c\|\delta x\|$ for $\|x\| = \text{const}$.

3. The mapping

A mapping that fulfills the above requirements is $M : \mathbb{R}^n \rightarrow \mathbb{R}^{n\times n}$

$$M(x) = \frac{xx^t}{\|x\|}. \tag{4}$$

It was introduced by Knutsson in this form in 1989 [13]. From the construction, it is clear that M is symmetric and has only $n(n+1)/2$ independent components. The mapping is slightly different from the structure tensor G. The latter does not meet all of the above requirements. The uniqueness requirement is met by G, also the polar separability as $\|G(x)\| = \|x\|^2$, but the uniform stretch property is not met as $\|\delta G(x)\| = c\|x\|\|\delta x\|$. The structure tensor is, therefore, no distance preserving mapping.

The mappings M and G are non-linear mappings, thus the outcome of filtering operations applied to the elements of M or G is subject to interpretation. For G, the smoothing and an eigenvalue analysis of $\bar{G}$ is now standard. In general, statements are possible about the properties laid down in the requirements, the norm of a variational vector $\|\delta M(x)\|$ and the norm of a mapped vector $\|M(x)\|$. The uniform stretch property can be utilized to compute different properties than local structure by applying other filters than blurring filters [16,17].

The Knutsson mapping M is a scaled version of G. Where G has only one non-zero eigenvalue $\|x\|^2$ with eigenvector x, M has the same eigenvector to the scaled eigenvalue $\|x\|$.

For the norm in the uniformity and the polar separability Knutsson chose the Fröbenius norm

$$\|M\|^2 := \sum_{ij} m_{ij}^2. \tag{5}$$

If the norm is rotation invariant, as given here by the polar separability requirement, then

$$\|M\|^2 = \text{tr}(M^t M) = \sum_n \lambda_n^2, \tag{6}$$

where λ_n are the eigenvalues of M. Here the Fröbenius norm is equal to the often used spectral norm $\|M\|_{\text{spec}} := \sqrt{\text{largest eigenvalue of}(M^t M)}$ as M has only one non-zero eigenvalue $\|x\|$.

At this point, we notice that we can further reduce the number of independent components of M; and G for that matter. The polar separability requirement guarantees for both mappings a rotation invariant norm and

$$\|M\|^2 = \lambda_{\max}^2 = \lambda_1^2 = \text{tr}(M)^2 = \text{const}. \tag{7}$$

So there is another restriction imposed on the sum of the diagonal elements of M. Therefore, from the n diagonal elements only $n-1$ linear combinations are necessary; the mappings M and G have $n(n+1)/2 - 1$ independent components. This can be important as the fewer dimensions to process, the less memory is needed.

3.1. The mapping in 2D

In the following, we identify $M(x) = x_i x_j/\|x\|$ as an ordered n-tuple. For the 2D case, the mapping M reads $x \in \mathbb{R}^2$

$$M(x) = \frac{1}{\|x\|}(x_1^2, x_1x_2, x_2x_1, x_2^2), \tag{8}$$

or in polar coordinates $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$

$$M(x) = r(\cos^2\varphi, \sin\varphi\cos\varphi, \cos\varphi\sin\varphi, \sin^2\varphi). \quad (9)$$

From the above consideration, we know that there are only $(n(n+1)/2) - 1 = 2$ independent components. Linear combinations of the diagonal elements yield only one component carrying information

$$\cos^2\varphi + \sin^2\varphi = 1 \quad (10)$$

$$\cos^2\varphi - \sin^2\varphi = \cos 2\varphi. \quad (11)$$

To scale all elements evenly, we take twice the off-diagonal element m_{12}, $2\sin\varphi\cos\varphi = \sin 2\varphi$. Summing up we get a reduced set

$$M_{r2D}(x) = \frac{1}{\|x\|}(x_1^2 - x_2^2, 2x_1x_2) = r(\cos 2\varphi, \sin 2\varphi). \quad (12)$$

The Knutsson mapping M, reduced to M_{r2D} in 2D, is equivalent to the well known *double angle method* [1,8]: $r(\cos\varphi, \sin\varphi) \rightarrow r(\cos 2\varphi, \sin 2\varphi)$. Note that the double angle method cannot be generalized to higher dimensions in a straightforward manner. The functions $r(\cos 2\varphi, \sin 2\varphi)$ are the circular harmonics [4]. They form a basis of the polynomials of second degree in 2D, so the reduced set M_{r2D} (Eq. (12)) is indeed minimal.

Already in 1987, Bigün and Granlund proposed an orientation operator for 2D images using these linear combinations of the components of the structure tensor to describe the orientation φ [1]

$$\tan 2\varphi = \frac{2\overline{\partial_x I \partial_y I}}{\overline{(\partial_x I)^2} - \overline{(\partial_y I)^2}}, \quad (13)$$

where the overhead bar denotes averaging in a local neighborhood.

In Fig. 3, we show that the gradient $(\partial_x I, \partial_y I)$ is not continuous whereas the reduced set is, and therefore averaging of these elements are allowed (i.e. it does not yield false responses to the apparent discontinuities).

3.2. The mapping in 3D

Again we want to reduce the dimensionality of the general mapping M. For the 3D case, we apply the successful recipe from 2D; taking twice the off-diagonal elements and cyclic combinations of the diagonal elements

$$\frac{1}{\|x\|}(\underbrace{x_1^2 - x_2^2, 2x_1x_2}_{\text{again 2D}}, 2x_1x_3, 2x_2x_3, x_2^2 - x_3^2, x_3^2 - x_1^2). \quad (14)$$

We have six components, from the above considerations, we know that there are only $(n(n+1)/2) - 1 = 5$ independent components. Due to the restriction on the trace, a combination of the cyclic diagonal elements m_1, m_5, m_6 is suitable to reduce the set by one. We choose $m_6 - m_5$

$$M_{r3D}(x) = \frac{1}{\|x\|}\left(x_1^2 - x_2^2, 2x_1x_2, 2x_1x_3, 2x_2x_3, \frac{1}{\sqrt{3}}(2x_3^2 - x_1^2 - x_2^2)\right). \quad (15)$$

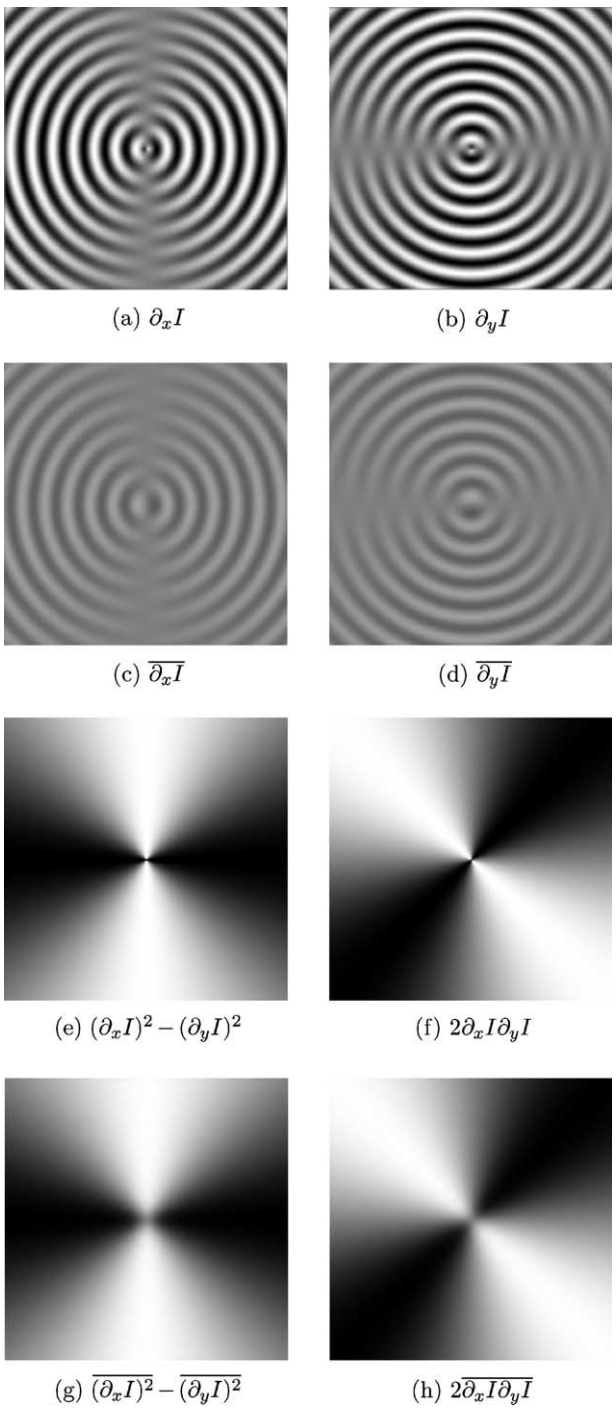

Fig. 3. Illustration along with the mapping (Eq. (13)), (a,b) the derivatives of the image in Fig. 1a), (c,d) smoothing of the derivatives with a Gaussian filter, (e,f) the reduced mapping M_{r2D}, (g,h) smoothing of the reduced mapping with a Gaussian filter. For easier educational insight elements (e–h) are normalized by the gradient magnitude.

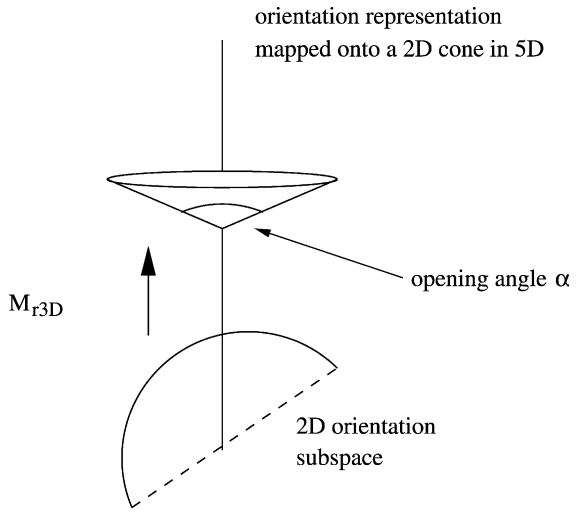

Fig. 4. A 2D orientation half plane in 3D is mapped via M_{3rD} onto a 2D cone in 5D with opening angle $\alpha = 120°$ (slope of the cone $1/\sqrt{3}$ and $\tan 30° = 1/\sqrt{3}$).

For a 2D subspace ($x_3 = 0$), this reads

$$M_{r3D}(x) = \frac{1}{\|x\|}\left(x_1^2 - x_2^2, 2x_1x_2, 0, 0, \frac{-\|x\|^2}{\sqrt{3}}\right), \tag{16}$$

where we recognize the 2D mapping again. A 2D orientation half plane in 3D is mapped onto a 2D cone in 5D with opening angle $\alpha = 120°$ (slope of the cone $1/\sqrt{3}$ and $\tan 30° = 1/\sqrt{3}$) as depicted in Fig. 4.

3.2.1. Knutsson 1985

In 1985, Knutsson introduced a mapping fulfilling the above requirements suitable for 3D [12]. This function $M_K : \mathbb{R}^3 \rightarrow \mathbb{R}^5$ is written in spherical polar coordinates $r \in \mathbb{R}^+$, $\phi \in [-\pi, \pi]$ and $\theta \in [0, \pi]$

$$M_K(r, \theta, \phi) \rightarrow r(s, t, u, v, w), \tag{17}$$

$$s = \sin^2\theta \cos 2\phi,$$

$$t = \sin^2\theta \sin 2\phi,$$

$$u = \sin 2\theta \cos\phi,$$

$$v = \sin 2\theta \sin\phi,$$

$$w = \sqrt{3}(\cos^2\theta - \frac{1}{3}).$$

Knutsson gave no derivation for his mapping M_K and said it was heuristically derived. We can derive M_K in a systematical way from the general mapping M (Eq. (4)).

This function M_K is in fact not different from M. M_K is the reduced set M_{r3D} (Eq. (15)) in spherical polar coordinates: $x_1 = r\cos\phi\sin\theta$, $x_2 = r\sin\phi\sin\theta$, $x_3 = r\cos\theta$. The extension from 2D to 3D is now no longer heuristic, as we presented its derivation. Extension to higher dimensions has become straightforward.

3.2.2. Another way to M_K

The extension to 3D (Eq. (17)) of the 2D solution, the double angle method (Eq. (12)), can also be derived by analogy transfer. The double angle method maps the tangent to the circular harmonics. Now let's have a look at the spherical harmonics [4]

$$Y_l^m(\theta, \phi) = \begin{cases} P_l^m(\cos\theta)\cos(m\phi), & \text{for } m = 0, 1, 2, \ldots, l \\ P_l^{|m|}(\cos\theta)\sin(|m|\phi), & \text{for } m = -1, \ldots, -l, \end{cases} \tag{18}$$

where P_l^m are the associated Legendre polynomials.

The spherical harmonics of second degree Y_2^m are (up to a scaling factor) identical to the components of the mapping M_K

$$s = 1/3Y_2^2,$$

$$t = 8Y_2^{-2},$$

$$u = 2/3Y_2^1,$$

$$v = 4Y_2^{-1},$$

$$w = 2/\sqrt{3}Y_2^0.$$

The spherical harmonics form a basis of the polynomials of second degree in 3D, so the set of components of M_{r3D} (Eq. (15)) is indeed minimal.

4. Properties of the mappings

In Table 1, we summarize some properties of the mappings presented here.

The angle ψ of two vectors $x, y \in \mathbb{R}^n$ in the original space can be related to the angle α that the mapped vectors $M(x), M(y) \in \mathbb{R}^{n\times n}$ will form in the mapped space M. Knutsson has done this only for M_K [12,13]. For the general mapping M (Eq. (4)) the deduction is

$$\cos\psi = \frac{x\cdot y}{\|x\|\cdot\|y\|}, \tag{19}$$

$$\cos\alpha = \frac{M(x)\cdot M(y)}{\|M(x)\|\cdot\|M(y)\|} = \frac{\sum_{ij} x_i x_j y_i y_j}{\|x\|^2\cdot\|y\|^2} = \frac{(x\cdot y)^2}{\|x\|^2\cdot\|y\|^2} \tag{20}$$

$$\Rightarrow \cos\alpha = \cos^2\psi. \tag{21}$$

In Fig. 5, we plot the angle transfer functions for M, M_K, M_{r2D} between the angle of two vectors in one space and the angle of the vectors in the mapped space. We observe

Table 1
Summary of the properties of the different mappings

	Dimension	Uniform stretch	Polar separability	Angle
G	n	–	$\|x\|^2$	–
M	n	$c = \sqrt{2}$	$\|x\|$	$\cos\alpha = \cos^2\psi$
M_{r2D}	2	$c = 2$	$\|x\|$	$\cos\alpha = \cos 2\psi$
M_K	3	$c = 2$	$\frac{2}{\sqrt{3}}\|x\|$	$\cos\alpha = \frac{3}{4}(\cos 2\psi + \frac{1}{3})$

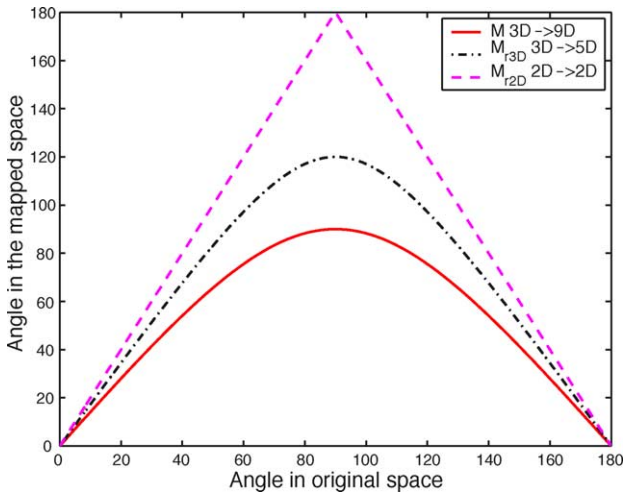

Fig. 5. Transfer function of the angle depicting the relation between the angle in the original space and the corresponding angle after mapping the vectors. The maximal angle in the mapped space is the opening angle of the mapped cone of a 2D half plane, compare Fig. 4.

the following properties

$$M : n\mathrm{D} \rightarrow n^2\mathrm{D}\psi \in [0, 180°] \rightarrow \alpha \in [0, 90°] \tag{22}$$

$$M_\mathrm{K} : 3\mathrm{D} \rightarrow 5\mathrm{D}\psi \in [0, 180°] \rightarrow \alpha \in [0, 120°] \tag{23}$$

$$M_{r2\mathrm{D}} : 2\mathrm{D} \rightarrow 2\mathrm{D}\psi \in [0, 180°] \rightarrow \alpha \in [0, 180°]. \tag{24}$$

Both reduced mappings $M_{r2\mathrm{D}}$ and $M_{r3\mathrm{D}} \equiv M_\mathrm{K}$ have the same uniformity factor of two, but the maximal angle in the mapped space is different. The 2D mapping maps the 2D orientation half plane onto the full 2D plane, which is a cone with opening angle 180°, whereas $M_{r3\mathrm{D}}$ maps the half plane to a cone with opening angle 120° (Fig. 4). Flatting of the 3D cone to a 2D plane is possible. The stretching is canceled by the polar separability factor $2/\sqrt{3}$ to one, thus flattening of $M_{r3\mathrm{D}}$ is identical to use $M_{r2\mathrm{D}}$ directly.

Furthermore, Knutsson pointed out that the sum of two non-zero mapped vectors (by M or M_K) is always greater than zero because the maximal angle they can make is smaller than 180°. He interpreted this as a consequence of the fact that there is no fully symmetric representation[1] of two 3D lines [12], but there is one of three 3D lines, i.e. three perpendicular lines. If we map three perpendicular lines

$$z\text{-axis} : \quad M_\mathrm{K}(r, 0, \phi) = r(0, 0, 0, 0, 2/\sqrt{3}),$$

$$y\text{-axis} : \quad M_\mathrm{K}(r, \pi/2, \pi/2) = r(1, 0, 0, 0, -1/\sqrt{3}),$$

$$x\text{-axis} : \quad M_\mathrm{K}(r, \pi/2, 0) = r(-1, 0, 0, 0, -1/\sqrt{3}),$$

then the sum of these three mapped vectors is zero. Here, we see the possibility to generate, from two mapped vectors (by M_K) in 5D, a third vector that is perpendicular to the other two and that is in the image of M_K. To find a vector in 5D that is perpendicular to two other vectors is not difficult, it is, however, more complicated to find the one which lies in the subspace of the image of M_K. In 3D, the outer product is used to obtain a perpendicular vector from two other, in 5D the sum of those three must be zero

$$3\mathrm{D} : \quad z = x \times y, \tag{25}$$

$$5\mathrm{D} : \quad M_\mathrm{K}(z) = -[M_\mathrm{K}(x) + M_\mathrm{K}(y)]. \tag{26}$$

[1] Fully symmetric in the sense that reflections map the lines onto each other.

5. Conclusions

We have shown in which manner specific solutions (2D [1,10] and 3D [12,13]) for a continuous orientation representation are connected through a general framework for arbitrary dimensions. In a general manner, we can reduce the dimensionality of the Knutsson mapping (Eq. (4)) and of the GST to a minimal set necessary to describe orientation in multi-dimensional images. The relation with the circular and spherical harmonics has become clear, as a minimal set of necessary components in 2D and 3D. Furthermore, the difference between the structure tensor and the Knutsson mapping are discussed, the first being not distance preserving which is important when applying other than averaging filters. The uniqueness and polar separability are properties of both mappings.

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