
The area of the triangle $ABC$ equals half the area of the parallelogram spanned by the vectors $b - a = \langle 0, 2, -2 \rangle$ and $c - a = \langle 2, -2, 0 \rangle$. Hence:

$$\text{area}(ABC) = \frac{1}{2} |(b - a) \times (c - a)| = \frac{1}{2} |(-4, -4, -4)| = 2 |\langle 1, 1, 1 \rangle| = 2\sqrt{3}. $$


Suppose that $y = \arctan(2x)$, then we have: $\tan y = 2x$ and $y \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$. Suppose that $x > 0$ and draw a right angled triangle for which the orthogonal sides have lengths $2x$ and $1$. Then the length of the other side (hypotenuse) is $\sqrt{1 + 4x^2}$ (according to the theorem of Pythagoras). Then we have:

$$\sin(\arctan(2x)) = \frac{2x}{\sqrt{1 + 4x^2}}.$$ 

For $x < 0$ we have: $\tan(-y) = -2x$. Then we do the same with $2x$ replaced by $-2x$ and find that

$$\sin(\arctan(2x)) = -\sin(-y) = -\frac{-2x}{\sqrt{1 + 4x^2}} = \frac{2x}{\sqrt{1 + 4x^2}}.$$ 

It is clear that the formula also holds for $x = 0$.


An equation of the tangent line at the point $(-3, 1)$ has the form $y - 1 = r(x + 3)$, where $r$ denotes the slope. Now we have $r = \frac{dy}{dx}\bigg|_{(-3,1)}$. Using implicit differentiation we obtain:

$$4(x^2 + y^2) \left( 2x + 2y \frac{dy}{dx} \right) = 50x - 50y \frac{dy}{dx}.$$ 

For $x = -3$ and $y = 1$ we find that

$$-240 + 80 \frac{dy}{dx} = -150 - 50 \frac{dy}{dx} \implies r = \frac{dy}{dx}\bigg|_{(-3,1)} = \frac{90}{130} = \frac{9}{13}.$$ 

Finally we have: $13y - 13 = 9x + 27$ or equivalently $13y - 9x = 40$.


Using integration by parts we find:

$$\int_{1}^{e} \frac{\ln x}{x^2} \, dx = -\int_{1}^{e} \frac{\ln x}{x} \, dx \bigg|_{1}^{e} + \int_{1}^{e} \frac{1}{x} \, d\ln x = -\frac{1}{e} + 1 - \frac{2}{e}. $$


Using the substitution $e^x = t$ we easily obtain:

$$\int \frac{e^x}{1 + e^{2x}} \, dx = \int \frac{dx}{1 + e^{2x}} = \int \frac{dt}{1 + t^2} = \arctan t + C = \arctan (e^x) + C.$$ 


We have:

$$\int_0^\infty \frac{dx}{1 + x^2} = \lim_{A \to \infty} \arctan A = \frac{\pi}{2} \quad \text{and} \quad \int_{-\infty}^0 \frac{dx}{1 + x^2} = \lim_{B \to -\infty} (-\arctan B) = \frac{\pi}{2}.$$ 

Hence:

$$\int_{-\infty}^\infty \frac{dx}{1 + x^2} = \int_0^0 + \int_{-\infty}^0 \frac{dx}{1 + x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$ 

Further we have

$$\int_0^\infty xe^{-x^2} \, dx = \lim_{C \to \infty} \left[ -\frac{1}{2} e^{-x^2} \right]_0^C = \frac{1}{2}.$$ 


Let $y(t)$ denote the amount of salt (in g) in the tank at time $t$ (in min). Then we have:

$$y(0) = 50 \text{ (initial condition)} \quad \text{and} \quad \frac{dy}{dt} = 10 - 4 \cdot \frac{y}{100 - 2t} = 10 - \frac{2y}{50 - t}.$$ 

This differential equation is linear. We look for an integrating factor $I(t)$:

$$y'(t) + \frac{2}{50 - t}y(t) = 10 \quad \implies \quad I(t)y'(t) + \frac{2}{50 - t}I(t)y(t) = 10I(t).$$ 

For $I(t)$ we must have: $I'(t) = -\frac{2}{50 - t}I(t)$. Hence: $I(t) = e^{-2\ln(50-t)} = (50 - t)^{-2}$ (for instance). Then we have:

$$\frac{d}{dt} \left[ (50 - t)^{-2}y(t) \right] = 10(50 - t)^{-2} \quad \implies \quad (50 - t)^{-2}y(t) = 10(50 - t)^{-1} + C$$

and therefore: $y(t) = 10(50-t) + C(50-t)^{-2}$. With $y(0) = 50$ we have: $500 + 2500 C = 50$ or equivalently $C = -\frac{450}{2500} = -\frac{9}{50}$. Hence

$$y(t) = 10(50-t) - \frac{9}{50} (50-t)^2 \quad \implies \quad y(40) = 100 - \frac{9}{50} \cdot 100 = 100 - 18 = 82.$$ 


The differential equation is separable. Note that $y = -1$ is a solution. For $y \neq -1$ we have:

$$\frac{dy}{y + 1} = (1 + t^2) \, dt \quad \implies \quad \ln |y + 1| = t + \frac{1}{3} t^3 + K \quad \implies \quad y(t) + 1 = \pm e^K \cdot e^{t^3/3}.$$ 

Hence: $y(t) = C e^{t^3/3} - 1$. From $y(0) = 1$ we have: $1 = C - 1$ or equivalently $C = 2$.

9. E. See: Stewart, Appendix H.

In the polar form we have: $\sqrt{3} + i = 2 e^{\frac{i}{3} \pi}$ and $-1 + i\sqrt{3} = 2 e^{\frac{2}{3} \pi i}$. Then we have:

$$\frac{(\sqrt{3} + i)^6}{(-1 + i\sqrt{3})^5} = \frac{(2 e^{\frac{i}{3} \pi})^6}{(2 e^{\frac{2}{3} \pi i})^5} = 2 e^{(1 - \frac{5}{3}) \pi i} = 2 e^{-\frac{2}{3} \pi i} = 2 e^{\frac{2}{3} \pi i} = 2 \left( \frac{1}{2} - \frac{1}{2} i\sqrt{3} \right) = 1 - i\sqrt{3}.$$
