
The auxiliary equation is $4r^2 + 1 = 0 \iff r = \pm \frac{1}{2}i$. Therefore, the general solution is:

$$y(x) = c_1 \cos \left( \frac{1}{2}x \right) + c_2 \sin \left( \frac{1}{2}x \right)$$

with $c_1, c_2 \in \mathbb{R}$. Now we have:

$$\begin{cases} 
  y(0) = 1 : & c_1 = 1 \\
  y(\pi) = 1 : & c_2 = 1 
\end{cases} \implies y(x) = \cos \left( \frac{1}{2}x \right) + \sin \left( \frac{1}{2}x \right).$$

2. See: Stewart, § 17.2.

The auxiliary equation is $r^2 + 5r + 4 = 0 \iff (r + 1)(r + 4) = 0$, hence: $r = -1$ or $r = -4$.

Therefore, the general solution of the complementary equation is:

$$y_h(x) = c_1 e^{-x} + c_2 e^{-4x}$$

with $c_1, c_2 \in \mathbb{R}$. For a particular solution we now take:

$$y_p(x) = Axe^{-x} + Be^{-4x}.$$ 

Then we have:

$$y_p'(x) = A(1 - x)e^{-x} + 2Be^{-4x}, \quad y_p''(x) = A(x - 2)e^{-x} + 4Be^{-4x}.$$ 

Substitution now leads to:

$$A(x - 2 + 5 - 5x + 4x)e^{-x} + (4 + 10 + 4)Be^{-4x} = 6e^{-x} + 3e^{2x} \implies A = 2 \quad \text{and} \quad B = \frac{1}{6}.$$ 

Hence: $y_p(x) = 2xe^{-x} + \frac{1}{6}e^{2x}$. Therefore, the general solution is:

$$y(x) = y_p(x) + y_h(x) = 2xe^{-x} + \frac{1}{6}e^{2x} + c_1 e^{-x} + c_2 e^{-4x}, \quad c_1, c_2 \in \mathbb{R}.$$ 

3. See: Stewart, § 11.3 and § 11.9.

(a) Compare with exercise 6 of § 11.3.

The series $\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{(2n + 1)^2} \right| = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2}$ is convergent, since

$$\int_1^{\infty} \frac{dx}{x^2} = \left[ -\frac{1}{x} \right]_1^{\infty} = 1$$

is convergent. This implies that $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^2}$ is absolutely convergent.

(b) This is a combination of example 1 and example 7 of § 11.9.

Since $\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$ we conclude that

$$\frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{for} \quad |x^2| < 1 \quad \text{or equivalently} \quad |x| < 1.$$ 

Integration now leads to

$$\arctan(x) = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} x^{2n+1} \quad \text{for} \quad |x| < 1.$$ 

Now take $x = 0$ to obtain that $0 = \arctan(0) = C + 0$ which implies that $C = 0.$
(c) Now we have
\[
\lim_{x \to 0} \frac{\arctan(x)}{x} = \lim_{x \to 0} \frac{x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots}{x} = \lim_{x \to 0} (1 - \frac{1}{3}x^2 + \ldots) = 1 + 0 = 1.
\]

(d) Compare with exercise 28 of § 11.9.
Using the Taylor series we obtain
\[
\int_0^1 \frac{\arctan(x)}{x} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^1 x^{2n} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left[ \frac{x^{2n+1}}{2n+1} \right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = G.
\]

(a) We have:
\[
\sqrt{1 + x^3} = (1 + x^3)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{\binom{1}{n}}{n!} x^{3n} = \left( \frac{1}{0} \right) + \left( \frac{1}{0} \right) x^3 + \left( \frac{1}{0} \right) x^6 + \left( \frac{1}{0} \right) x^9 + \ldots
\]
\[
= 1 + \frac{1}{2} x^3 - \frac{1}{8} x^6 + \gamma x^9 + \ldots,
\]
where \( \gamma = \left( \frac{1}{2} \right) = \frac{1}{2}. \)

(b) Using the first three non-zero terms of this series we obtain
\[
\int_0^{\frac{1}{2}} \sqrt{1 + x^3} \, dx \approx \int_0^{\frac{1}{2}} \left( 1 + \frac{1}{2} x^3 - \frac{1}{8} x^6 \right) \, dx = \left[ x + \frac{1}{8} x^4 - \frac{1}{56} x^7 \right]_0^{\frac{1}{2}} = \frac{3639}{7168} \approx 0.507673.
\]

(c) Using Taylor series we find that
\[
\lim_{x \to 0} \frac{x \sin(x^2)}{1 - \sqrt{1 + x^3}} = \lim_{x \to 0} \frac{x \left( x^2 - \frac{1}{2} x^6 + \ldots \right)}{1 - \left( 1 + \frac{1}{2} x^3 - \frac{1}{8} x^6 + \ldots \right)} = \lim_{x \to 0} \frac{1 - \frac{1}{6} x^4 + \ldots}{1 + \frac{1}{8} x^3 + \ldots} = -2.
\]

5. See: Stewart, § 13.3.
The (arc) length of the curve is: \( L = \int_0^\pi |r'(t)| \, dt. \)
Now we have:
\[
r'(t) = (-\sin(t) + \sin(t) + t \cos(t), t^2, \cos(t) - \cos(t) + t \sin(t))
\]
\[
= (t \cos(t), t^2, t \sin(t)) = t(\cos(t), t, \sin(t)).
\]
Hence: \( |r'(t)| = t \sqrt{1 + t^2}. \) This implies that
\[
L = \int_0^\pi |r'(t)| \, dt = \int_0^\pi t \sqrt{1 + t^2} \, dt = \left[ \frac{1}{3} (1 + t^2)^{\frac{3}{2}} \right]_0^\pi = \frac{1}{3} [(1 + \pi^2)^{\frac{3}{2}} - 1].
\]

An equation of the tangent plane is \( z - 1 = f_x(2,3)(x-2) + f_y(2,3)(y-3) \) (note that \( f(2,3) = 1 \)), where
\[
f_x(x, y) = \ln(xy - 5) + x \cdot \frac{1}{xy - 5} \cdot y \quad \Rightarrow \quad f_x(2,3) = 6
\]
and
\[
f_y(x, y) = x \cdot \frac{1}{xy - 5} \cdot x \quad \Rightarrow \quad f_y(2,3) = 4.
\]
Hence, we have: \( z = 1 + 6(x-2) + 4(y-3). \)