Cauchy's residue theorem

Cauchy's residue theorem is a consequence of Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi\, i} \oint_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz,$$

where f is an analytic function and $\mathcal{C}$ is a simple closed contour in the complex plane enclosing the point z_0 with positive orientation which means that it is traversed counterclockwise.

Since the integrand is analytic except for $z = z_0$, the integral is equal to the same integral with $\mathcal{C}$ replaced by a small circle inside the contour $\mathcal{C}$ with center z_0. This implies that we have with $z = z_0 + r\,\mathrm{e}^{i\theta}$ and $0 \leq \theta \leq 2\pi$

$$\begin{aligned} \oint_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz &= \lim_{r\downarrow 0} \int_0^{2\pi} \frac{f(z_0 + r\,\mathrm{e}^{i\theta})}{r\,\mathrm{e}^{i\theta}}\, i\, r\,\mathrm{e}^{i\theta}\, d\theta \\ &= i\, f(z_0) \int_0^{2\pi} d\theta = 2\pi\, i\, f(z_0), \end{aligned}$$

which proves Cauchy's integral formula.

This formula can be iterated to

$$f^{(n)}(z_0) = \frac{n!}{2\pi\, i} \oint_{\mathcal{C}} \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \quad n \in \mathbb{N}.$$

If f is an analytic function except for an isolated singularity at $z = z_0$, then f has a Laurent series representation of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Then the coefficient a_{-1} is called the *residue* of f at $z = z_0$. We use the notation

$$\mathrm{Res}_f(z_0) = a_{-1}.$$

Now we have Cauchy's residue theorem

Theorem. *If $\mathcal{C}$ is a simple closed, positively oriented contour in the complex plane and f is analytic except for some points $z_1, z_2, \ldots, z_n$ inside the contour $\mathcal{C}$, then*

$$\oint_{\mathcal{C}} f(z)\, dz = 2\pi\, i \sum_{k=1}^{n} \mathrm{Res}_f(z_k).$$

If f has a removable singularity at $z = z_0$, then the residue is equal to zero. If f has a single *pole* at $z = z_0$, then

$$\mathrm{Res}_f(z_0) = \lim_{z\to z_0} (z - z_0) f(z)$$

and if f has a pole of order k at $z = z_0$, then

$$\mathrm{Res}_f(z_0) = \frac{1}{(k-1)!} \lim_{z\to z_0} \frac{d^k}{dz^k} \left\{ (z - z_0)^k f(z) \right\}, \quad k \in \{1, 2, 3, \ldots\}.$$

Cauchy's residue theorem can be used to compute real integrals by applying an appropriate contour in the complex plane.